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ON THE UNIQUENESS OF THE SOLUTION FOR THE BOUNDARY VALUE PROBLEM OF INFINITE ORDER

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The aim of this paper is to study the Dirichlet problem for the elliptic equations of infinite order with arbitrary nonlinearities:

$$\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x,u,\ldots, D^{\alpha}u) = h(x),\ x \in \Omega. \tag{0.1}$$

As an example of such equations we have the equation

$$\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^{\alpha}(\varphi_{\alpha}(D^{\alpha}u(x))) = h(x),$$

where $\varphi_{\alpha}: \mathbf{R}^1 \to \mathbf{R}^1$ are continuous, odd, nondecreasing functions and $\varphi_{\alpha}(+\infty) = +\infty$. For t sufficiently large the functions $\varphi_{\alpha}(.)$ may behave, crudely speaking, as polynomials (see [1] and [2]), as exponentials or logarithms, and similar such functions [3,4].

In the study of boundary value problems for equations of infinite order a decisive role is played by the nontriviality of the corresponding «energy» function spaces: this role is in our view, of independent interest. In [3,4] the «energy» function spaces for the Dirichlet problems of infinite order are Sobolev-Orlicz classes of infinite order $\overset{o}{W}{}^{\infty} \mathcal{L}\{\varphi_{\alpha}, \Omega\}$, which are nonlinear spaces in general case. The uniqueness of solution for Dirichlet problems is proved under the condition that the corresponding class $\overset{o}{W}{}^{\infty} \mathcal{L}\{\varphi_{\alpha}, \Omega\}$ is a linear space

In this paper we shall introduce Sobolev-Orlicz spaces of infinite order, which contain Sobolev-Orlicz classes $\overset{o}{W}{}^{\infty} \mathcal{L}\{\varphi_{\alpha}, \Omega\}$ as subsets, and establish the uniqueness of solution of the Dirichlet problems of infinite order without the condition V in [3] assuring providing the linearity of the corresponding « energy » space.

The paper consists of three sections. The first one is devoted to the definition of Sobolev-Orlicz spaces of infinite order and their principal properties. We shall verify that criterions of nontriviality for Sobolev-Orlicz classes of infinite order [3, 4, 5] are still valid for our new spaces in three of most commontly encountered cases in analysis: a bounded region $\Omega \subset \mathbb{R}^n$, full Euclidean space $\mathbb{R}^n$ and the torus $T^n = S^1 \times \ldots \times S^1$, where S^1 is the unit circle. We present effective methods of testing the nontriviality of these spaces. The dual space to the Sobolev-Orlicz space is described in §2. In §3 we establish the existence and uniqueness of solution of Dirichlet problems for the equations (0. 1).

§1. THE SOBOLEV-ORLICZ SPACES

$$L\mathring{W}^{\infty}\{\varphi_\alpha, \Omega\},\ LW^{\infty}\{\varphi_\alpha, T^n\},\ LW^{\infty}\{\varphi_\alpha, R^n\}.\quad 1)$$

Let G be a domain in $\mathbb{R}^n$ and let $\varphi(t)$ be an N-function, i.e. a real-valued, continuous convex and even function, where $\varphi(t) > O$ for $t > O$, $\varphi(t)/t \to O$ as $t \to O$ and $\varphi(t)/t \to \infty$ as $t \to \infty$. We shall refer to the function $\overline{\varphi}(t) = \sup\{ts - \varphi(s),\ s \in \mathbb{R}^1\}$ as the N-function complementary to $\varphi(t)$. We use the anisotropic Sobolev–Orlicz space of order N with the norm,

$$\| u \|_N = \sum_{|\alpha|=0}^{N} \| D^\alpha u \|_{(\varphi_\alpha)}, \tag{1, 1}$$

where $\| \cdot \|_{(\varphi)}$ is defined by

$$\| u \|_{(\varphi)} = \inf_k \{ k > 0 : \int_G \varphi(u/k)dx \leq 1\},$$

which is called the Luxenburg norm [6]. This space denoted by $W^N L\{\varphi_\alpha, G\}$, is a Banach space with the norm (1. 1). In the space $W^N L\{\varphi_\alpha, G\}$ we also introduce the norm

$$\| u \|_{(N)} = \inf_k \{ k > O : \sum_{|\alpha|=0}^{N} \int_G \varphi_\alpha(D^\alpha u/k)dx \leq 1 \}. \tag{1, 2}$$

1) We distinguish these spaces from the spaces $\mathring{W}^{\infty}L\{\varphi_\alpha, \Omega\}$, $W^{\infty}L\{\varphi_\alpha, T^n\}$, $W^{\infty}L\{\varphi_\alpha, \mathbb{R}^n\}$ considered in [3, 4, 5], which are their subspaces.

LEMMA 1. 1. *The norms (1. 1) and (1.2) are equivalent. Moreover*

$$\|u\|_{(N)} \leq \|u\|_N \leq M \|u\|_{(N)},$$

where $M = \sum_{i=o}^{N} S(i)$, $S(i) = Card \{\alpha = (\alpha_1, \ldots, \alpha_n), |\alpha| = i\}$.

Proof. Indeed, we have

$$\{k > 0 : \sum_{|\alpha|=0}^{M} \int_G \varphi_\alpha\left(\frac{D^\alpha u}{k}\right) dx \leq 1\} \subset \{k > 0 : \int_G \varphi_\alpha\left(\frac{D^\alpha u}{k}\right) dx \leq 1\},$$

i.e.

$$\|D^\alpha u\|_{(\varphi_\alpha)} \leq \|u\|_{(N)}, \forall \alpha, |\alpha| \leq N.$$

Hence,

$$\|u\|_N = \sum_{|\alpha|=0}^{N} \|D^\alpha u\|_{(\varphi_\alpha)} \leq M. \|u\|_{(N)}.$$

On the other hand, for all $|\alpha| \leqslant N$ $\varepsilon_\alpha > 0$ can be chosen to be arbitrarily small enough. Setting

$$K = \sum_{|\alpha|=0}^{N} (\|D^\alpha u\|_{(\varphi_\alpha)} + \varepsilon_\alpha) \geqslant K_\alpha = \|D^\alpha u\|_{(\varphi_\alpha)} + \varepsilon_\alpha > 0,$$

we have

$$\sum_{|\alpha|=0}^{N} \int_G \varphi_\alpha\left(\frac{D^\alpha u}{k}\right) dx \leq \sum_{|\alpha|=0}^{N} \int_G \frac{k_\alpha}{k} \varphi_\alpha\left(\frac{D^\alpha u}{k_\alpha}\right) dx \leq \sum_{|\alpha|=0}^{N} \frac{k_\alpha}{k} = 1.$$

It follows that

$$\|u\|_{(N)} \leq K = \|u\|_N + \sum_{|\alpha|=0}^{N} \varepsilon_\alpha;$$

where $\sum_{|\alpha|=0}^{N} \varepsilon_\alpha$ can be arbitrarily small. Therefore $\|u\|_{(N)} \leq \|u\|_N$.

The lemma is proved.

1.1. The space $L\overset{0}{W}{}^\infty\{\varphi_\alpha, \Omega\}$. Let Ω be a bounded domain in R^n and denote its boundary by $\partial\Omega$. Further, let $\overset{0}{W}{}^N E\{\varphi_\alpha, \Omega\}$ be the closure of $C_0^\infty(\Omega)$ in $W^N L\{\varphi_\alpha, \Omega\}$ by either the norm (1.1) or the norm (1.2). To define $\overset{0}{W}{}^N L\{\varphi_\alpha, \Omega\}$ we may use the $\sigma(\Pi L\varphi_\alpha, \Pi E\overline{\varphi_\alpha})$ topology, i. e. $W^N L\{\varphi_\alpha \Omega\}$ is the closure of $C_0^\infty(\Omega)$ by the topology $\sigma(\Pi L\varphi_\alpha, \Pi E\varphi_\alpha)$ [9].

We consider now the following space of function $u(x)$

$$L\overset{0}{W}{}^\infty\{\varphi_\alpha, \Omega\} \equiv \{u(x) \in C_0^\infty(\Omega), \|u\|_{(\infty)} < +\infty\},$$

where

$$\| u \|_{(\infty)} = \inf_k \{k > 0, \sum_{|\alpha|=0}^{\infty} \int_\Omega \varphi_\alpha\left(\frac{D^\alpha u}{k}\right)dx \leqslant 1\}. \qquad (1.3)$$

$L\overset{0}{W}{}^{\infty}\{\varphi_\alpha; \Omega\}$ is called a Sobolev-Orlicz space of infinite order on the bounded domain Ω. We recall the definition of Sobolev-Orlicz classes of infinite order [3, 4]:

$$\mathcal{L}\overset{0}{W}{}^{\infty}\{\varphi_\alpha, \Omega\} \equiv \{u(x) \in C_0^\infty(\Omega)\},$$
$$\overset{\infty}{\rho}(u) = \sum_{|\alpha|=0}^{\infty} \int_\Omega \varphi_\alpha(D^\alpha u(x))dx < +\infty\}.$$

LEMMA 1. 2. *The space* $L\overset{0}{W}{}^{\infty}\{\varphi_\alpha, \Omega\}$ *is a linear hull of the class* $\mathcal{L}\overset{0}{W}{}^{\infty}\{\varphi_\alpha, \Omega\}$.

Proof. Let $u \in \mathcal{L}\overset{0}{W}{}^{\infty}\{\varphi_\alpha, \Omega\}$ and $\overset{\infty}{\rho}(u) = K$. It is sufficient to consider the case $K > 1$. We have

$$\sum_{|\alpha|=0}^{\infty} \int_\Omega \varphi_\alpha\left(\frac{D^\alpha u}{K}\right)dx \leqslant \frac{1}{K}\overset{\infty}{\rho}(u) = 1.$$

Hence, $\| u \|_{(\infty)} \leqslant K$, i. e. $u \in L\overset{0}{W}{}^{\infty}\{\varphi_\alpha, \Omega\}$. This proves the lemma.

LEMMA 1. 3. *For any* $u(x) \in L\overset{0}{W}{}^{\infty}\{\varphi_\alpha, \Omega\}$

$$\lim_{N\to\infty} \| u \|_{(N)} = \| u \|_{(\infty)}.$$

Proof. It suffices to consider the case

$$\lim_{N\to\infty} \| u \|_{(N)} = M_1 > 0.$$

For any non-negative integer N, we have

$$\inf \{k > 0: \sum_{|\alpha|=0}^{N} \int_\Omega \varphi_\alpha\left(\frac{D^\alpha u}{k}\right)dx \leqslant 1\} \leqslant M_1.$$

This implies

$$\sum_{|\alpha|=0}^{N} \int_\Omega \varphi_\alpha\left(\frac{D^\alpha u}{M_1}\right)dx \leqslant 1, \forall N = 0,1, \ldots$$

Hence

$$\|u\|_{(\infty)} \leqslant M_1. \qquad (1.4)$$

Otherwise, if $0 < \| u \|_{(\infty)} = M_2$, by the definition of the norm $\|.\|_{(\infty)}$ we have: $\forall \varepsilon > 0, \exists k_0 > 0: M_2 \leqslant k_0 \leqslant M_2 + \varepsilon$ such that

$$\sum_{|\alpha|=0}^{\infty} \int_\Omega \varphi_\alpha\left(\frac{D^\alpha u}{k_0}\right) dx \leqslant 1.$$

Thus, for every non-negative integer N,

$$\sum_{|\alpha|=o}^{N} \int_{\Omega} \varphi_\alpha \left(\frac{D^\alpha u}{k_o}\right) dx \leqslant 1 ,$$

This shows that $\|u\|_{(N)} \leqslant k_o < M_2 + \varepsilon$ and, hence

$$\lim_{N \to \infty} \|u\|_{(N)} \leqslant M_2 + \varepsilon .$$

Since ε is an arbitrary non-negative number, we have

$$\lim_{N \to \infty} \|u\|_{(N)} \leqslant M_2 . \tag{1.5}$$

The assertion of the Lemma 1.3 follows from (1.4) and (1.5).

THEOREM 1.4. *The space* $L\overset{o\infty}{W} \{\varphi_\alpha , \Omega\}$ *is a Banach space that is the limit of the monotone decreasing sequence of Sobolev-Orlicz space* $\overset{oN}{W} L \{\varphi_\alpha , \Omega\}$, $N = 0,1,...$ *In addition,*

$$lim_{N \to \infty} \|u\|_{(N)} = \|u\|_{(\infty)} .$$

Proof. Obviously, we have

$$\overset{\circ}{W}{}^1 L\{\varphi_\alpha, \Omega\} \supset \overset{\circ}{W}{}^2 L\{\varphi_\alpha, \Omega\} \supset ...$$

$$\| u \|_{(1)} \leq \| u \|_{(2)} \leq \cdots$$

and each space $\overset{\circ}{W}{}^N L\{\varphi_\alpha,\Omega\}$ is Banach space. Further, by Lemma 1. 3

$$\| u \|_{(\infty)} = \lim_{N \to \infty} \| u \|_{(N)} ,$$

i.e. $L\overset{o\infty}{W} \{\varphi_\alpha, \Omega\}$ is the limit of the decreasing sequence of the spaces $\overset{oN}{W}L\{\varphi_\alpha, \Omega\}$ in the sense of [7,2]. By Theorem 4. 1. 1 [2] $L\overset{0\infty}{W}\{\varphi_\alpha, \Omega\}$ is Banach space with the norm $\| \cdot \|_{(\infty)}$. The theorem is proved.

DEFINITION 1. 5. *We say that the space* $L\overset{0\infty}{W}\{\varphi_\alpha , \Omega\}$ *is nontrivial if there exists a function* $u(x) \in \overset{\infty}{C_0}(\Omega)$ *such that* $u(x) \not\equiv 0$ *and* $\| u \|_{(\infty)} < + \infty$.

As in [3], we introduce the sequence of numbers M_α

$$M_\alpha = \begin{cases} \varphi_\alpha^{-1} (1/\mathrm{mes}\, \Omega), & \varphi_\alpha \not\equiv 0 , \\ + \infty , & \varphi_\alpha \equiv 0 , \end{cases}$$

where $\varphi_\alpha^{-1}(t)$, is the inverse function of the N-function $\varphi_\alpha(t)$, $| \alpha | = 0,1,...$

THEOREM 1. 6. *The space* $L\overset{0\infty}{W}\{\varphi_\alpha, \Omega\}$ *is nontrivial if and only if the sequence* M_α. $| \alpha | = 0,1,...$ *defines a nonquasianalytic class of functions of n real variables.*

Proof. Sufficiency is immediate from Theorem 1. 1. [3] and Lemma 1. 2.

Necessity. Let M_α generate a quasianalytic class $C[M_\alpha]$ of functions of n real variables, and let $u \in \overset{\circ\infty}{LW}\{\varphi_\alpha, \Omega\}$. We prove that $u \equiv 0$.

The conclusion $u \in \overset{\circ\infty}{LW}\{\varphi_\alpha. \Omega\}$ shows, in particular, that there exists a number $K_0 > 0$, such that $K_0^{-1} u \in \overset{\circ\infty}{\mathcal{L}W}\{\varphi_\alpha, \Omega\}$. By Theorem 1.1 [3] we have $k_0^{-1} u \equiv 0$. Hence $u \equiv 0$, that is, $\overset{\circ\infty}{LW}\{\varphi_\alpha, \Omega\}$ is trivial. The theorem is thus proved.

Using the Lelong criterion [8] it is not difficult to formulate Theorem 1.6 in terms of quasianalytic classes of functions of a single real variable. Further, from Theorem 1.6 and Corollaries 1.1, 1.2 and 1.3 in [3], we obtain the following algebraic necessary and sufficient conditions for nontriviality of $L\overset{\circ}{W}{}^\infty\{\varphi_\alpha, \Omega\}$.

COROLLARY 1.7. *Each of the following conditions is necessary and sufficient for the space $L\overset{\circ}{W}{}^\infty\{\varphi_\alpha, \Omega\}$ to be nontrivial:*

a) If $\beta_n = \inf_{N \geqslant n} (\min_{|\alpha|=N} M_\alpha)^{1/N}$, *then* $\sum_{n=1}^{\infty} \beta_n^{-1} < +\infty$

b) If $T(r) = \operatorname{Sup}\left(\frac{r^N}{M_N}\right)$, *then* $\int_1^\infty \frac{\ln(T(r))}{r^2}\, dr < +\infty$

c) $\lim_{N\to\infty} M_N^{1/N} = \infty$, $\sum_{N=0}^{\infty} \frac{M_N^c}{M_{N+1}^c} < +\infty$

Let $\chi(\Omega)$ be the characteristic function of the domain and let $(\varphi, \overline{\varphi})$ be a pair of complementary N—functions.

COROLLARY 1. 8. *The space* $\overset{\circ\infty}{LW}\{\varphi_\alpha, \Omega\}$ *is nontrivial if and only if the sequence* $\| \chi(\Omega) \|_{\varphi_\alpha}$, $|\alpha| = 0, 1, \ldots$ *defines a nonquasianalytic class of functions of n real variables**.

COROLLARY 1.9. *The space* $LW^\infty\{\varphi_\alpha, \Omega\}$ *is nontrivial if and only if the sequence* $\| \chi(\Omega) \|_{\varphi_\alpha}^{-1}$, $|\alpha| = 1, 2, \ldots$, *defines a nonquasianalytic class of functions of n real variables**.

1. 2. The space $LW^\infty\{\varphi_\alpha, T^n\}$. Let us denote by T^n the n-dimensional torus. Consider the space

* $\| \cdot \|_\varphi$ denotes the Orlicz norm in the Orlicz space $L_\varphi(\Omega)$.

$$LW^{\infty}\{\varphi_{\alpha}, T^n\} \equiv \{u(x) \in C^{\infty}(T^n), \| u \|_{(\infty)} < +\infty,$$

$$\| u \|_{(\infty), T^n} = \inf_k \{k : \sum_{|\alpha|=0}^{\infty} \int_{T^n} \varphi_{\alpha}\left(\frac{D^{\alpha} u}{k}\right) dx \leqslant 1\}, \tag{1.6}$$

where $u(x)$ is a periodic function of period 2π. As usual, the question of the nontriviality of the space $LW^{\infty}\{\varphi_{\alpha}, T^n\}$ arises. We are interested only in the space $LW^{\infty}\{\varphi_{\alpha}, T^n\}$ which has infinite dimension, i. e. which contains an infinite set of linearly independent periodic functions. Such spaces are called nontrivial.

Using Theorem 4. 1. 1 [2] and considering $LW^{\infty}\{\varphi_{\alpha}, T^n\}$ as the limit of the decreasing sequence of Banach spaces $W^N L\{\varphi_{\alpha}, T^n\}$ one can prove the following property.

THEOREM 1. 10. *The space* $LW^{\infty}\{\varphi_{\alpha}, T^n\}$ *is a Banach space with the norm (1.6). Moreover* $LW^{\infty}\{\varphi_{\alpha}, T^n\}$ *contains class* $\mathcal{L}W^{\infty}\{\varphi_{\alpha}, T^n\}$ *as a subset.*

THEOREM 1. 11. *The space* $LW^{\infty}\{\varphi_{\alpha}, T^n\}$ *is nontrivial if and only if there exists a sequence of distinct multi-indices* $q_{\nu} = (q_{1_{\nu}}, \ldots, q_{n_{\nu}})$ $\nu = 0,1,\ldots,$ *such that*

$$\sum_{|\alpha|=0}^{\infty} \varphi_{\alpha}(q_{\nu}^{\alpha}) < +\infty \tag{1.7}$$

where $q_{\nu}^{\alpha} = q_{1_{\nu}}^{\alpha_1} \ldots q_{n_{\nu}}^{\alpha_{\nu}}$

Proof. Sufficiency is immediate from Theorem 2. 8 [4] and Theorem 1. 10.

Necessity. Let us assume the contrary that the series (1.7) is convergent for not more than a finite set of multi-indices $q = (q_1, \ldots, q_n)$, namely, for $|q_1| \leqslant N_1, \ldots, |q_n| \leqslant N_n$, where N_j are integers. Further, for any $u \in LW^{\infty}\{\varphi_{\alpha}, T^n\}$ there exists $k_0 > 0$, such that $k^{-1}u \in \mathcal{L}W^{\infty}\{\varphi, T^n\}$. From the proof of the necessity of Theorem 2.8 [4] it follows that the function $k_0^{-1}u$, and consequently, the function u belongs to $L(\exp(iq,x))$, where $L(\exp(iq,x))$ is the linear hull of the functions $\exp(iq,x)$ with $|q_j| \leqslant N_j$, $j=1,\ldots,n$, i.e. $LW^{\infty}\{\varphi_{\alpha} T^n\} \subset L(\exp(iq,x))$. The latter contradicts the condition that $LW^{\infty}\{\varphi_{\alpha}, T^n\}$ is of infinite dimension. The theorem is thus proved.

1. 3. The space $LW^{\infty}\{\varphi_{\alpha}, R^n\}$. We shall consider the following space of function $u(x) : R^n \to C^1$

$LW^{\infty}\{\varphi_\alpha, \mathbf{R}^n\} = \{u(x) : D^\alpha u \in L_{\varphi_\alpha}(\mathbf{R}^n), \|u\|_{(\infty), \mathbf{R}^n} \angle +\infty,$

$\|u\|_{(\infty), \mathbf{R}^n} = \inf_k \{k>0 : \sum_{|\alpha|=0}^{\infty} \int_{\mathbf{R}^n} \varphi_\alpha\left(\frac{D^\alpha u}{k}\right) dx \leq 1\}$, (1. 8) where D^α denotes the generalized derivative of order α.

By a method similar to that of Sections 1. 1, 1. 2 and by Theorem 4.1.1 [2], Theorem 2 [5] we can prove the following results.

THEOREM 1. 12. *The space $LW^{\infty}\{\varphi_\alpha, \mathbf{R}^n\}$ is a Banach space with the norm (1.8). Moreover, $LW^{\infty}\{\varphi\alpha, \mathbf{R}^n\}$ contains the class $\mathcal{L}W^{\infty}\{\varphi_\alpha R^n\}$ as a subset*

THEOREM 1. 13. *The space $LW^{\infty}\{\varphi_\alpha, R^n\}$ is nontrivial if and only if there exists a number $q > 0$, such that*

$$\sum_{|\alpha|=0}^{\infty} \varphi_\alpha(q^{|\alpha|}) \angle +\infty.$$

REMARK. It is obvious that the problem of nontriviality of the space $LW^{\infty}\{\varphi_\alpha, \mathbf{R}^n\}$ becomes meaningful if $\varphi_0 \not\equiv 0$, since otherwise the function $u(x) \equiv$ const belongs to $LW^{\infty}\{\varphi_\alpha, \mathbf{R}^n\}$. By this reason we can always assume that $\varphi_0 \not\equiv 0$.

§2. The space $EW^{-\infty}\{\bar{\varphi}_\alpha, \Omega\}$. We consider the case of functions defined in a bounded domain $\Omega \subset \mathbf{R}^n$.

As well known, the space $W^{-N}E\{\bar{\varphi}_\alpha, \Omega\}$ is dual to $\overset{0}{W}{}^N L\{\varphi_\alpha, \Omega\}$ [6,9]. We consider the increasing sequence of Banach spaces

$$W^{-1}E\{\bar{\varphi}_\alpha, \Omega\} \subset W^{-2}E\{\bar{\varphi}_\alpha, \Omega\} \ldots \subset W^{-N}E\{\bar{\varphi}_\alpha, \Omega\} \subset \ldots$$
$$\|x\|_{-1} \geq \|x\|_{-2} \geq \ldots \geq \|x\|_{-N} \geq \ldots$$

We denote by $EW^{-\infty}\{\bar{\varphi}_\alpha, \Omega\}$ the completion of

$$\bigcup_{m=0}^{\infty} W^{-m}E\{\bar{\varphi}_\alpha, \Omega\},$$

in the norm

$$\|x\|_{-\infty} = \lim_{m\to\infty} \|x\|_{-m}.$$

By Theorem 4. 1. 2 in [2, p. 113] we have

THEOREM 2.1. *Let the space $\overset{0}{LW}{}^{\infty}\{\varphi_\alpha, \Omega\}$ be nontrivial. Then the space $EW^{-\infty}\{\bar{\varphi}_\alpha, \Omega\}$ is nontrivial too.*

THEOREM 2.2. *The following representation is valid*

$$EW^{-\infty}\{\bar{\varphi}_\alpha, \Omega\} \equiv \{h(x) : h(x) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha h_\alpha(x),$$

$$h_\alpha(x) \in E\{\bar{\varphi}_\alpha, (\Omega)\}, \sum_{|\alpha|=0}^{\infty} \|h_\alpha\|_{(\bar{\varphi}_\alpha)} < +\infty\}.$$

Proof. Let us show that one and only one series

$$\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha h^\alpha(x),$$

appearing in the theorem, corresponds to an element of the space $E\overline{W}^{\infty}\{\bar{\varphi}_\alpha, \Omega\}$

Indeed, let $h_s(x) \in \bigcup_{m=1}^{\infty} W^{-m}E\{\bar{\varphi}_\alpha, \Omega\}$, $s = 1, 2, \ldots$ be a fundamental sequence in the norm $\|\cdot\|_{(-\infty)}$. Let $m_0 = m_0(s)$ be the smallest among all the numbers m for which $h_s(x) \in W^{-m}E\{\varphi_\alpha, \Omega\}$. Then, as it is known from the theory of Sobolev—Orlicz spaces, the functions $h_s(x)$ can be represented as

$$h_s(x) = \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^\alpha h_{s\alpha}(x),$$

where $h_{s\alpha} \in E\varphi_\alpha(\Omega)$ (see [9]).

Now we consider the family of Dirichlet problems of infinite order

$$L(u_s) \equiv \sum_{|\alpha|=0} (-1)^{|\alpha|} D^\alpha(\mu_\alpha(D^\alpha u_s(x))) = h_s(x), \qquad (2.1)_s$$

$$D^\omega u_s \Big|_{\partial\Omega} = 0, \quad = 0,1, \ldots \qquad (2.2)_s$$

where μ_α is the derivative of φ_α. According to our results of § 3 below, the problem $(2.1)_s - (2.2)_s$ has an unique solution $u_s(x) \in L\mathring{W}^{\infty}\{\varphi_\alpha, \Omega\}$, and, for any $m \geqslant m_0$, the inequality

$$\|u_s\|_{(m)} \leq K < +\infty, \qquad (2.3)$$

is valid. The estimate (2.3) implies that there exists a function $u(x) \in L\mathring{W}^{\infty}\{\varphi_\alpha, \Omega\}$ such that $u_s(x) \to u(x)$ uniformly with respect to all its derivatives.

We shall show that $u(x)$ is the unique limit point of the sequence $u_s(x)$. Indeed, let $u_k(x)$ be another subsequence of the sequence $u_s(x)$ such that

$$u_k(x) \to w(x) \text{ in } C_0^\infty(\Omega),$$

where $w(x)$ is a function from $L\mathring{W}^{\infty}\{\varphi_\alpha, \Omega\}$. We must show $w(x) \equiv u(x)$.

It is obvious that

$$\langle L(u_r) - L(u_k), v\rangle \leqslant \langle h_r - h_k, v\rangle_m$$

where $v \in L\overset{0\sim}{W}\{\varphi_\alpha, \Omega\}$, $m=\max[m_0(r), m_0(k)]$ (see the definition of number $m_0(r)$ at the beginning of our proof). Hence

$$\langle L(u_r) - L(u_k), v\rangle \leqslant \| h_r - h_k \|_{W^{-m}E\{\overline{\varphi}_\alpha, \Omega\}} \cdot \| v \|_{\overset{0}{W}{}^m L\{\varphi_\alpha, \Omega\}} .$$

Passing to the limit when $m \to \infty$, we get the inequality

$$\langle L(u_r) - L(u_k), v\rangle \leqq \| h_r - h_k \|_{(-\infty)} \| v \|_{(\infty)}$$

Since the original sequence $h_k(x)$ is fundamental in the norm $\|\cdot\|_{(\infty)}$, we see by letting $k \to \infty$, $r \to \infty$ that

$$\langle L(u) - L(w), v\rangle \leqq 0,$$

where $v(x) \in L\overset{0\infty}{W}\{\varphi_\alpha, \Omega\}$ is an arbitrary function.

In view of uniqueness of the solution of the Dirichlet problem of infinite order (Theorem 3. 2 and 3. 3 below) $u(x) \equiv w(x)$. Thus for every fundamental sequence $h_s(x)$, $s = 1, 2, \dots$, there exists one and only one function

$u(x) \in L\overset{0\infty}{W}\{\varphi_\alpha, \Omega\}$ such that

$$u_s(x) = L^{-1}(h_s) \to u(x),$$

uniformly with respect to all its derivatives. It is easy to see that if the fundamental sequences $h_s(x)$ and $h'_s(x)$ are equivalent the corresponding functions $u(x)$ and $u'(x)$ are equal, i. e. $u(x) \equiv u'(x)$. It remains to remark that for any function $u(x) \in L\overset{0\infty}{W}\{\varphi_\alpha, \Omega\}$ the series $L(u)$ defines an element of the completion $\bigcup_{m=1}^{\infty} W^{-m}E\{\overline{\varphi}_\alpha, \Omega\}$ in the norm $\|\cdot\|_{(-\infty)}$. Therefore, the desired map

$$L(u) \leftrightarrow h_s(x), s = 1,2,\dots$$

is defined. The theorem is thus proved.

§3. BOUNDARY VALUE PROBLEMS OF INFINITE ORDER

In this paragraph we study Dirichlet problems for the nonlinear differential equations of infinite order in a bounded domain. However, all the results obtained below, are still valid in the cases of torus T^n, the full Euclidean space $\mathbf{R}^n$, etc.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 1$, with boundary $\partial\Omega$. We consider the problem

$$L(x, D)u = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, u, \ldots, D^{\alpha}u) = h(x), \qquad (3.1)$$

$$D^{\omega}u\big|_{\partial\Omega} = 0, \quad |\omega| = 0,1,\ldots, \qquad (3.2)$$

Here $A_{\alpha}(x, \xi)$ are, generally speaking, nonlinear functions of $\xi = (\xi_0, \ldots, \xi_{\alpha})$, ($\xi_{\alpha} \in \mathbb{R}^1$). We assume that for each α the function $A_{\alpha}(x, \xi)$ satisfies the Caratheodory condition and the following conditions:

I. There exist N-function $\varphi_{\alpha}(t)$, a function $a_{\alpha}(x) \in L\{\varphi_{\alpha}, \Omega\}$, a continuous bounded function $c^1_{\alpha}(|t|)$ $(1 \leqslant c^1_{\alpha}(|t|) \leqslant \text{const})$ and a constant $b > 0$ such that

$$|A_{\alpha}(x, \xi)| \leqslant a_{\alpha}(x) + b\,\overline{\varphi}_{\alpha}^{-1}\varphi_{\alpha}(c^1_{\alpha}(|\xi_{\alpha}|)\xi_{\alpha}),$$

where

$$\sum_{|\alpha|=0}^{\infty} \| a_{\alpha} \|_{(\overline{\varphi}_{\alpha})} < +\infty.$$

II. There exist functions $b_m(x) \in L_1(\Omega)$, $g_{\alpha}(x) \in E\{\overline{\varphi}_{\alpha}, \Omega\}$ a continuous bounded function $c^2_{\alpha}(|t|) \geqslant c^1_{\alpha}(|t|)$ and a constant $d > 0$ such that

$$\sum_{|\alpha|=m} (A_{\alpha}(x, \xi) - g_{\alpha}(x))\xi_{\alpha} \geqslant$$

$$\geqslant d \sum_{|\alpha|=m} \varphi_{\alpha}(c^2_{\alpha}(|\xi_{\alpha}|)\xi_{\alpha}) - b_m(x),$$

where

$$\sum_{|\alpha|=0}^{\infty} \|g_{\alpha}\|_{(\overline{\varphi}_{\alpha})} < +\infty, \sum_{m=0}^{\infty} \int_{\Omega} |b_m(x)|\,dx < +\infty.$$

III. The N-functions $\varphi_{\alpha}(t)$ are such that the space $L\overset{0}{W}{}^{\infty}\{\varphi_{\alpha}, \Omega\}$ is nontrivial (see §1).

IV. The strict monotonicity condition. For arbitrary $\xi = (\xi_0, \ldots, \xi_{\alpha})$, $\xi' = (\xi'_0, \ldots, \xi'_{\alpha})$ and $x \in \Omega$ we have the inequality

$$\sum_{|\alpha|=m} (A_{\alpha}(x, \xi) - A_{\alpha}(x, \xi'))(\xi_{\alpha} - \xi'_{\alpha}) \geqslant 0$$

and the equality is valid if and only if when $\xi = \xi'$, $m = 0, 1, 2, \ldots$

We assume that the right hand side of (3.1) belongs to $EW^{-\infty}\{\bar{\varphi}_\alpha, \Omega\}$ (see §2). The duality of the spaces $L\overset{o}{W}^{\infty}\{\varphi_\alpha, \Omega\}$ and $E\bar{W}^{\infty}\{\bar{\varphi}_\alpha \Omega\}$ is determined by the expression

$$\langle h, v\rangle \equiv \sum_{|\alpha|=0} \int_\Omega h_\alpha(x) D^\alpha v(x)\, dx\,,$$

which is obviously correct.

DEFINITION 3. 1. *A function $u(x) \in L\overset{o}{W}^{\infty}\{\varphi_\alpha, \Omega\}$ is said to be a solution of the problem (3. 1) — (3-2) if for an arbitrary function $v(x) \in L\overset{o}{W}^{\infty}\{\varphi_\alpha, \Omega\}$ we have*

$$\langle L(x, D)u, v\rangle = \langle h, v\rangle$$

Remark. Definition 3. 1 is more exact than Definition 2. 1 in [3] because the test function space $L\overset{o}{W}^{\infty}\{\varphi_\alpha, \Omega\}$ is larger than $\overset{o}{W}^{\infty}L\{\varphi_\alpha, \Omega\}$ i.e. $\overset{o}{W}^{\infty}L\{\varphi_\alpha, \Omega\} \subset L\overset{o}{W}^{\infty}\{\varphi_\alpha, \Omega\}$.

THEOREM 3. 2. *Let the conditions I—III be satisfied. Then for an arbitrary right-hand side $h(x) \in EW^{-\infty}\{\overline{\varphi}_\alpha, \Omega\}$ there exists at least one solution $u(x)$ of the problem (3. 1)—(3. 2) in the space $L\overset{o}{W}^{\infty}\{\varphi_\alpha, \Omega\}$.*

THEOREM 3. 3. *Let the condition IV be satisfied. Then the solution of the problem (3.1)—(3.2) is unique.*

The proof of these theorems is essentially similar to that of Theorem 2.1.1 and 2.1.2 in [4] since $\overset{o}{W}^{\infty}\mathcal{L}\{\varphi_\alpha, \Omega\} \subset L\overset{o}{W}^{\infty}\{\varphi_\alpha, \Omega\}$ and the spaces $L\overset{o}{W}^{\infty}\{\varphi_\alpha, \Omega\}$ and $EW^{-\infty}\{\overline{\varphi}_\alpha, \Omega\}$ are linear.

Examples. We consider the following problem:

$$L(D)\, u \equiv \sum_{|\alpha|=0}^{\infty} (-1)^\alpha D^\alpha(\mu_\alpha(D^\alpha u(x))) = h(x),\ x \in \Omega, \tag{3. 3}$$

$$D^\omega u|_{\partial\Omega} = 0,\ |\omega| = 0, 1, \dots \tag{3. 4}$$

For each α the function $\mu_\alpha : \mathbf{R}^1 \to \mathbf{R}^1$ is continuous, odd, and nondecreasing, and $\mu_\alpha(+\infty) = +\infty$; We put

$$\varphi_\alpha(t) = \int_0^t \mu_\alpha(\tau)\, d\tau,\ \alpha = 0, 1 \dots$$

The functions $\varphi_\alpha(t)$ are N-functions. Obviously, as in [3], the conditions I,II are satisfied for the equation (3.3). In addition. let us assume that the functions $\varphi_\alpha(t)$ are such that $\overset{0\infty}{LW}\{\varphi_\alpha, \Omega\}$ is nontrivial. Then by virtue of Theorem 3.2 the problem (3.3)—(3.4) has a nontrivial solution in $\overset{0\infty}{LW}\{\varphi_\alpha, \Omega\}$ for $h(x) \in EW^{-\infty}\{\overline{\varphi}_\alpha, \Omega\}$. If the functions $\mu_\alpha(t)$ are strictly increasing, it follows from Theorem 3.3 that the solution is unique. We examine some concrete examples.

a) Case of coefficients with rapid growth. We consider the problem

$$\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha (a_\alpha D^\alpha u \exp [a_\alpha (D^\alpha u)^2] = h(x), \tag{3.5}$$

$$D^\alpha u|_{\partial\Omega} = 0, \ |\alpha| = 0,1,\ldots \tag{3.6}$$

For $a_\alpha = (|\alpha|!)^{-2\nu}$, $\nu > 1$ the space $\overset{0\infty}{LW}\{\exp [a_\alpha t^2] - 1, \Omega\}$ is nontrivial (see § 1). Consequently, the problem (3.5) — (3.6) has a unique solution $u(x) \in \overset{0\infty}{LW}\{\exp [a_\alpha t^2] - 1, \Omega\}$ for arbitrary function $h(x)$ from corresponding space $EW^{-\infty}\{\overline{\varphi_\alpha}, \Omega\}$. For $a_\alpha = (\alpha!)^{-2\nu}$, $\nu \leqslant 1$, the problem (3.5)—(3.6) reduces to the triviality $0 \equiv 0$.

b) Case of coefficients with power-law growth. We consider the problem.

$$\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha (a_\alpha |D^\alpha u|^{p_\alpha - 2} D^\alpha u) = h(x), \tag{3.7}$$

$$D^\alpha u|_{\partial\Omega} = 0, \ |\alpha| = 0,1,\ldots \tag{3.8}$$

For $a_\alpha = (|\alpha|)^{-\nu|\alpha| p_\alpha}$, $\nu > 1$, $p^\alpha \geqslant 1$, in accordance with Theorems 3.2, 3.3, the problem (3.7) — (3.8) has a unique solution in the corresponding space $\overset{0\infty}{LW}\{\varphi_\alpha, \Omega\}$.

c) Case of coefficients with slow growth. Let

$$\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha (a_\alpha \operatorname{sign} (D^\alpha u) \ln (1 + a_\alpha |D^\alpha u|)) = h(x), \ x \in \Omega, \tag{3.9}$$

$$D^\alpha u|_{\partial\Omega} = 0. \ |\alpha| = 0,1,\ldots \tag{3.10}$$

If $a_\alpha = (|\alpha|!)^{-\nu}$, $\nu > 1$, then the problem (3.9), (3.10) has a unique solution in the space $\overset{0\infty}{LW}\{1+a_\alpha|t|)\ln(1+a_\alpha|t|)-a_\alpha|t|, \Omega\}$ for $h(x) \in EW^{-\infty}\{\exp [a_\alpha^{-1}|t|] - 1, \Omega\}$.

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