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MULTIDIMENSIONAL QUANTIZATION. V THE MECHANICAL SYSTEMS WITH SUPERSYMMETRY

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0. INTRODUCTION

«Graded Lie algebras have recently become a topic of interest in physics in the context of «supersymmetry» relating particles of different statistics», as it is pointed out in the survey [1]. In the physical systems where the Bose-Einstein particles and Fermi-Dirac particles interact together the symmetry must be replaced by a supersymmetry. So Lie superalgebras and Lie super-groups are important mathematical tools of physics and they must be studied seriously. There have been a number of important developments in the last few years such as: the remarkable Kac-Kaplansky's classification of simple Lie superalgebras [9], Berezin-Leites' theory of supermanifolds, Kostant's theory of prequantization [8], Manin's suggestion on the geometry of supergravitation [10], ...

B. Kostant has developed in his work [8] the representation theory of Lie supergroups along the line of the Orbit Method for ordinary Lie groups. His theory is founded on differential geometry and utilizes symplectic structures Hamiltonian formalism, integrality condition, line bundles with connection and prequantization for the supergroup case. A quantization procedure, rather than the prequantization, requires also Hilbert spaces of quantum states and unitary representations of supersymmetry. Perhaps, the main difficulty is the fact that Lie's theorem is not true in the case of solvable Lie superalgebras. In general, they have also irreducible finite-dimensional representations rather than (one-dimensional) characters.

Using the new notion of polarization developed in Parts I and II of this contribution, the author proposes a supersymmetry approach to the quantization problem of the Hamiltonian systems with supersymmetry. The main result is the construction of the multidimensional quantization procedure using Hilbert superbundles with connection and some so called weak Lagrangian invariant tangent superdis tributions (called superpolarizations). In particular, we prove that the derivatives of the representations induced from the superpolarizations are just the Lie superalgebra representations deduced from the multidimensional quantization.

The main stimulus for this work comes from many discussions during the «QUANG BA Mathematical Physics Seminar», May 29 – June 1, 1985. His deepest thanks are addressed to all participants of this seminar.

1. HILBERT SUPERSPACES

1.1. Let us denote by C the ground field of complex numbers, by $Z/2$ the residue field consisting of two elements 0 and 1. Recall that (see [8]) a *vector superspace* V is by definition a $Z/2$—graded vector space

$$V = V_0 \oplus V_1$$

The elements of V_0 are called *even*, $|x| = 0$; those of V_1 *odd*, $|x| = 1$. Throughout what follows, if $|x|$ of x occurs in an expression, then it is assumed that x is homogeneous, and that the expression extends to other elements by linearity.

Suppose that the vector superspace V admits a form $b\colon V \times V \to C$ (a *scalar product*) which is linear relative to the first variable, and:

— *superhermitian*, i.e. for all homogeneous x, y in V

$$b(x, y) = (-1)^{|x| \cdot |y|}\,\overline{b(y, x)},$$

— *consistent*, i.e. $b(x, y) = 0$ if x and y are of different graded degrees, and

— *nondegenerate*, i.e. if $b(x, y) = 0$ for all $y \in V$ then $x = 0$.

1.2. COROLLARY. *Let (V, b) be a vector superspace with scalar product.*

(i) *The restriction of b to V_0 is a scalar product, in other words V_0 is a prehilbertian space. V_1 is a symplectic vector space, i.e. the restriction of b to V_1 is nondegenerate skew symmetric.*

(ii) *The correspondence $z \to b(\cdot, z)$ establishes an antilinear monomorphism $V \subsetneq V^*$.*

Proof. (i) is clear from the definition of b.

(ii) is also an easy consequence of the nondegeneracy property of b.

1.3. In the category of vector superspaces there is a special functor of changing the graduation degrees:

$$(\Pi V)_0 = V_1, \ (\Pi V)_1 = V_0.$$

Thus if V has a scalar product, then by Corollary 1.2 $V \subsetneq V^*$ and we can define Πx for every element x of V. So we have an adjoint superform b^{Π} on ΠV:

$$b^{\Pi}(\Pi x, \Pi y) = (-1)^{|x|} b(x,y).$$

COROLLARY. *V_1 is a symplectic vector space if there exists an even-dimensional prehilbertian space W such that $V_1 = \Pi\ W$.*

Proof. We see that $\Pi^2 = \mathrm{Id}$, thus we define $W = \Pi\, V_1$, so $V_1 = \Pi\ W$. By definition, for all homogeneous w_1, w_2 in W,

$$b^{\Pi}(w_1, w_2) = b^{\Pi}(\Pi\, v_1, \Pi\, v_2) = (-1)^{|v_1|}\, b(v_1, v_2) = -b(v_1, v_2).$$

So we have

$$b^{\Pi}(w_2, w_1) = (-1)^{|v_2|}\, b(v_2, v_1) = (-1)^{|v_1|}\, b(v_1, v_2) = b^{\Pi}(w_1, w_2).$$

Hence b^{Π} is a hermitian form. Other properties are straighforward.

1.4. if $u \in \mathrm{Aut}\,(V, b)_0$, we define

$$u^{\Pi} = \Pi_0\, u_0\, \Pi \in \mathrm{Aut}\,(\Pi\, V, b^{\Pi}).$$

We have a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\ u\ } & V \\ \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \Pi} \\ \Pi\, V & \xrightarrow{\ u^{\Pi}\ } & \Pi\, V \end{array}$$

COROLLARY. *u is symplectic in V_1 if and only if u^{Π} is unitary on $W = \Pi\, V_1$.*

Proof. $u \in \mathrm{Sp}\,(V_1)$ if for all $x, y \in V_1$

$$b(ux, uy) = b(x, y).$$

By definition, for all x, y in W,

$$\begin{aligned} b^{\Pi}(u^{\Pi} x, u^{\Pi} y) &= b^{\Pi}(\Pi\, u\, \Pi\, x, \Pi\, u\, \Pi\, y) \\ (-1)^{|u\Pi x|}\, b(u\Pi x, u\, \Pi y) &= (-1)^{|u|+|\Pi x|}\, b(\Pi x, \Pi y) \\ (-1)^{|\Pi x|}\, b(\Pi x, \Pi y) &= b^{\Pi}(\Pi\, \Pi\, x, \Pi\Pi\, y) \\ &= b^{\Pi}(x, y). \end{aligned}$$

But on W, b^{Π} is a scalar product. Thus the corollary is proved.

1.5. Let us denote by q and q^{Π} the quadratic forms associated to the bilinear forms b and b^{Π} on V and $\Pi\, V$, respectively.

COROLLARY: (1) *q is nondegenerate on V_0, but is identically O on V_1*

(ii) *q^{Π} is identically O on $\Pi\, V_0$, but is nondegenerate on $W = \Pi V_1$.*

Proof. See Corollary 1.3.

1.6. We now define a norm of supervectors on V by

$$\| v \| = \sqrt{q(x) + q^{\Pi}(\Pi\, x)}$$

and a usually associated operator norm

$$\| A \| = \operatorname*{Sup}_{\| x \| \leqslant 1} \| Ax \| .$$

Now by a *Hilbert superspace* we mean a vector superspace with scalar product, which is complete with respect to the preceding norm of vectors. By an *(even) unitary-symplectic operator* $u \in$ USp (V) we mean an even automorphism $u \in$ Aut $(V, b)_0$.

COROLLARY. *The set of all unitary-symplectic operators forms a group USp (V). On this group the strong topology is equivalent to the weak topology.*

Proof. On the one hand, we have by the Cauchy-Bounjakowsky inequality that for every fixed vectors v and w in V_0, or in W

$$\| \langle (A_n - A)\, v,\, w \rangle \| \leqslant \| (A_n - A)\, v \| \cdot \| w \|$$

On the other hand, by taking $w = (A_n - A)\, v$, we have

$$\langle (A_n - A)\, v, (A_n - A) v \rangle = \| (A_n - A)\, v \|^2 .$$

The proof will be completed by the following

1. 7. LEMMA. *The topological group USp (V) is isomorphic to the direct product of unitary groups of Hilbert spaces V_0 and $W = \Pi V_1$.*

Proof. By definition, $u \in$ USp $(V) =$ Aut $(V, b)_0$ iff $u V_i \subset V_i$, $i = 0,1$ and for all (homogeneous) x, y

$$b(ux, uy) = b(x, y).$$

So we have

$$u = \left(\begin{array}{c|c} u|_{V_0} & 0 \\ \hline 0 & u|_{V_1} \end{array} \right)$$

and
$$\begin{aligned} \text{Aut } (V, b)_0 &= \text{Aut } (V_0, b) \times \text{Aut } (V_1, b) \\ &\cong \text{Aut } (V_0, b) \times \text{Aut } (W, b^{\Pi}) \\ &\cong U(V_0) \times U(W), \end{aligned}$$

by Corollary 1.4. The rest of the proof is easy.

1.8. A linear superoperator $A : V \to V$ is said to be *antisymmetric* iff for every homogeneous $x, y \in V$,

$$b(Ax, y) = -(-1)^{|A|,|x|}\, b(x, Ay).$$

LEMMA. *Antisymmetric superoperators form a Lie superalgebra the even part of which is a Lie algebra consisting of all pairs of antisymmetric (unbounded) operators on the direct product $V_0 \times W$ of the Hilbert spaces V_0 and W.*

Proof. We must verify that the supercommutator

$$[A, B] = AB - (-1)^{|A| \cdot |B|} B \cdot A$$

stabilizes the set of all antisymmetric superoperators. But this is clear from the definition: see also [9,5.3.4(b)].

1.9. STONE THEOREM. *Every one (real) parameter continuous group* $\{u(t)\}_{t\in R}$ *of even unitary — symplectic superoperators on a Hilbert superspace V admits a generator iA, which is antisymmetric continuous (perhaps, unbounded) superoperator.*

Proof. We have for all real values of parameter $t\in R$ $u(t)\in \mathrm{USp}(V)\simeq \mathrm{U}(V_0)\times \times U(W)$, then up to this isomorphism, $u(t)=(u_0(t), u_1(t))$ and

$$(u_0(t+s), u_1(t+s)=u(t+s)=u(t).u(s)=(u_0(t)\,u_0(s), u_1(t)\,u_1(s)).$$

Thus we can apply the usual Stone theorem to $\{u_0(t)\}_{t\in R}$ on V_0 and to $\{u_1(t)\}_{t\in R}$ on W.

1.10. *Unitary representation*. Let (G,A) be a Lie supergroup and $\mathcal{G}$ be the Lie superalgebra of (G, A), $U(\mathcal{G})$ the enveloping superalgebra of $\mathcal{G}$ and $A(G)^* = E(G, \mathcal{G}) = R(G)\times U(\mathcal{G})$ the Lie-Hopf coalgebra, see B. Kostant [8].

Let (V, b) be a Hilbert superspace and let End V be the superalgebra of continuous superoperators on V.

By a *smooth unitary representation* of (G, A) on V, we mean a homomorphism of superalgebras

$$r: A(G)^* \to \mathrm{End}\ V$$

such that:

(1) The restriction $r\mid_G$ is a continuous representation of the Lie group G in the group of even unitary-symplectic automorphisms

$$r\mid_G : G\to \mathrm{USp}\ (V).$$

(2) Each vector $v\in V$ is smooth, i.e. the map $G\to V$ $g\to r(g)v$ is of class C^∞.

(3) $r(X)v = \dfrac{d}{dt}(r(\exp\ tX)V)\Big|_{t=0}$

for each $v\in V$ and each $X\in \mathcal{G}_0$.

2. VECTOR SUPERBUNDLES WITH CONNECTION

This section is much similar to the corresponding one in B. Kostant [8, §4], where a theory of line superbundles has been developed. Our interest is essentially in the multidimensional case. So we are trying to modify these results for the multidimensional situation.

2.1. DIFFERENTIAL SUPERFORMS WITH COEFFICIENTS IN A SUPERBUNDLE. Let (X, A) be a supermanifold and let $U\subset X$ be an open set, $\mathrm{Der}A(U)$ be the

Lie superalgebra of all superderivations in $A(U)$, which is also a $A(U)$ — module, $T(U)$ be the tensor algebra of Der $A(U)$ over $A(U)$

$$T(U) = \bigoplus_{b=0}^{\infty} T^b(U) = \bigoplus_{b=0}^{\infty} \underbrace{\text{Der } A(U) \otimes_{A(U)} \cdots \otimes_{A(U)} \text{Der } A(U)}_{b \text{ times}}$$

which is $Z \oplus Z/2$ — (bi) graded.

Now let $J(U)$ be the two side, $Z \oplus Z/2$-graded ideal in $T(U)$ generated by all elements in $T^2(U)$ of the form $\xi \otimes \eta + (-1)^{|\xi||\eta|}\eta \otimes \xi$, where $\xi, \eta \in$ Der $A(U)$ are homogeneous. Also let $J^b(U) = T^b(U) \cap J(U)$.

Denote by E a projective (locally free) A-module sheaf on (X, A). Then for every $b \in$ N, $\text{Hom}_{A(U)}(T^b(U), E(U))$ can be regarded as the set of all b-linear maps on Der $A(U)$ with values in $E(U)$ which satisfy the condition

$$\langle \xi_l, \ldots, f\xi_l, \ldots, \xi_b \mid \beta \rangle = (-1)^{|f| \sum_{i=l}^{l} |\xi_i|} f \langle \xi_l, \ldots, \xi_b \mid s \rangle$$

Now let $\Omega^b(U, E)$ be the set of all $\beta \in \text{Hom}_{A(U)}(T^b(U), E(U))$ which vanish on $J^b(U)$. It is easy to see that the elements β in $\Omega^b(U, E)$ are characterized by the additional condition

$$\langle \xi_1, \ldots, \xi_j, \xi_{j+1}, \ldots, \xi_b \mid \beta \rangle = (-1)^{1+|\xi_j||\xi_{j+1}|} \langle \xi_l, \ldots, \xi_{j+1}, \xi_j, \ldots \xi_b \mid \beta \rangle$$

COROLLARY. $\Omega^b(U, E) = \Omega^b(U, A) \otimes E(U)$.

Proof. It is enough to mention that for the locally free sheaf $E(U)$

$$\text{Hom}_{A(U)}(T^b(U), E(U)) \cong \text{Hom}_{A(U)}(T^b(U), A(U)) \otimes_{A(U)} E(U).$$

Denote by End E the projective (locally free) A-module sheaf of all endomorphisms of the superbundle E. Then it is easy to see that $\Omega^b(U,F)$ and $\text{Hom}_{A(U)}(T^b(U), E(U))$ are End $E(U)$ — modules

$$< \xi_1, \ldots, \xi_b \mid \beta f > = < \xi_1, \ldots, \xi_b \mid \beta > f \text{ for all } f \in \text{End } E(U)$$

and

$$< \xi_1, \ldots, f\xi_l, \ldots, \xi_b \mid \beta > = (-1)^{|f| \sum_{i=l}^{b} |\xi_i|} < \xi_1, \ldots, \xi_b \mid f\beta >, \text{ for all } f \in A(U).$$

Also $\Omega^b(U,E)$ is $Z/2$ — graded:

$$< \xi_1, \ldots, \xi_b \mid \beta > \in E(U)_k.$$

$$k = |\beta| + \sum_{i=l}^{b} |\xi_i|.$$

$$\Omega^0(U,E) \overset{\text{def}}{=} E(U),$$

$$\Omega(U,E) = \bigoplus_{b=o}^{\infty} \Omega^b(U,E).$$

Let $\beta \in \Omega^b(U,E)$, $\alpha \in \Omega^a(U,E')$, $< \dots >$ be a pairing of E and E' with values in F, then one defines $\Omega^{a+b}(U,F)$ in the usual way; for example, $< .,. > = \mathrm{Hom}(E,E')$ is an important case.

Note that if $V \subseteq U$ is an open subset, one has a restriction map $\rho_{U,V}: \Omega(U,E) \to \Omega(V,E)$, such that if $\xi_i \in \mathrm{Der}\, A(U)$, $\beta \in \Omega^b(U,E)$ then $\rho_{U,V}\beta \in \Omega^b(V,E)$ is characterized by

$$< \rho_{U,V}\, \xi_1, \dots, \rho_{U,V}\, \xi_b \mid \rho_{U,V}\, \beta > = \rho_{U,V} < \xi_1, \dots, \xi_b \mid \beta >.$$

It is clear that $U \to \Omega(U,E)$ defines the sheaf of differential superforms with values in a projective (locally free) A-module sheaf E.

Assume $\dim (X,A) = (m,n)$. An open set U is called A- *parallelizable* if there exists $\eta_l \in \mathrm{Der}\, A(U)$, $l = 1, \dots m+n$ such that $\eta_l \in \mathrm{Der}\, A(U)_0$ if $l \leqslant m$ and $\eta_l \in \mathrm{Der}\, A(U)_1$ if $l > m$, and such that every $\xi \in \mathrm{Der}\, A(U)$ can be uniquely written in the form

$$\xi = \sum_{l=1}^{m+n} f_l\, \eta_l$$

where $f_l \in A(U)$.

Note that every A-coordinate neighbourhood (see [8, §2.8]) is A-parallelizable:

Now if the open U is A-parallelizable, one defines $\alpha_l \in \Omega^1(U, A)$ by the condition that

$$\langle \xi, \alpha_l \rangle = f_l, \text{ if } \xi = \sum_{l=1}^{m+n} f_l\, \eta_l \in \mathrm{Der}\, A(U).$$

Thus

$$\langle \eta_k, \alpha_l \rangle = \delta_{kl} \,.\, 1_U \,.$$

Put $\beta_l = \alpha_l$, $l \leqslant m$ and $\gamma_l = \alpha_{l+m}$, $1 \leqslant l \leqslant m$, then $\beta_l \in \Omega^1(U,A)_0$ and $\gamma_l \in \Omega^1(U, A)_1$ and

$$\beta_l\, \alpha_i = -\alpha_i\, \beta_l\,,$$

$$\gamma_i \gamma_j = \gamma_j \gamma_i\,.$$

We introduce the usual multiindex notation (see [8, § 4,2])

$$\beta\mu\, \gamma^\nu = \beta\mu_1 \dots \beta\mu_k\, \gamma_1^{\nu_1} \dots \gamma_n^{\nu_n}$$

We see that in the A-parallelizable open U, $\Omega(U, E)$ is a free $A(U)$-module such that every differential superform ω can be written in the form

$$\omega = \sum_{\mu,\nu} \beta_\mu \ \gamma^\nu f_{\mu\nu}, \ f_{\mu\nu} \in E(U).$$

Now assume that U is an A-splitting coordinate neighbourhood with an coordinate system $\{r_i, s_j\}$, $i = 1,\dots, m$, $j = 1,\dots, n$. Then Der $A(U)$ is a free $A(U)$-module with $\{\partial/\partial r_i, \partial/\partial s_j\}$ as basis. We can choose just $\{\beta_i, \gamma_j\} = \{dr_i, ds_j\}$. So

$$dr_\mu \, ds^\nu \in \Omega^{k(\mu)+|\nu|}(U, A) \subsetneq \Omega^{k(\mu)+|\nu|}(U,E)$$

and every $\beta \in \Omega(U, E)$ can be uniquely written as

$$\beta = \sum_{\mu,\nu} dr_\mu \, ds^\nu \cdot f_\mu,$$

where $f_{\mu\nu} \in E(U)$.

Recall that, as in the line bundle case (see [8]) we can also construct the map σ^*: $\Omega(Y, E') \to \Omega(X, E)$, which is associated to a supperbundle sheaf morphism σ: $E \to E'$:

Finally, by a *superbundle* we mean a projective (locally free) A-module sheaf E such that there exists a covering from opens which are *principal* for E in the sense that:

$E(U)$ is a free $A(U)$-module with a basal system of even generators $t_i \in E(U)_0$, $i = 1, \dots, \mathrm{rank}_{A(U)} E(U)$.

2. 2. VECTOR SUPERBUNDLES WITH CONNECTION. Let E be a vector bundle over the supermanifold (X, A). By a connection ∇ on E we mean a covariant superderivation such that for any open set $U \subseteq X$ and any vector superfield $\xi \in \mathrm{Der}\, A(U)$, one has a linear map

$$\nabla_\xi : E(U) \to E(U),$$

where $|\nabla_\xi| = |\xi|$, which is compatible with the restriction maps to smaller open sets and is such that

(1) $\nabla_\xi(ft) = (\xi f)t + (-1)^{|f| \cdot |\xi|} f \cdot \nabla_\xi(t)$,

for $f \in A(U)$ and $t \in E(U)$, and

(2) The map Der $A(U) \to$ End $E(U)$, given by $\xi \mapsto \nabla_\xi$ is $A(U)$ — linear.

The complexification of Der $A(U)$ may be taken to be the complex Lie superalgebra Der $A_{\mathbb{C}}(U)$ of superfields of vectors. By linearity we may take ξ and f in Der $A_{\mathbb{C}}(U)$ and $A_{\mathbb{C}}(U)$, resp. and $\Omega_{\mathbb{C}}(U, E)$ will denote the complexification of $\Omega(U, E)$.

Now assume that (E, ∇) is a complexified vector superbundle with connection, $U \subseteq X$ is principal for E and t_i, $i = 1,\dots, rk_A E$. Then for any $\xi \in$ Der

$A_C\,(U)$ there exists $g = g\,(\xi)$ such that $\nabla_\xi t = g\,(\xi).\ t$. The correspondence $\xi \to g(\xi)$ defines an $A_C\,(U)$ — linear map Der $A_C\,(U) \to$ End $E\,(U)$. Therefore, there exists a unique element $\alpha\,(t) \in \Omega^1_C\,(U, \text{End } E)$ such that

$$\nabla_\xi t = 2\pi i < \xi \,|\, \alpha(t) > t$$

for all $\xi \in$ Der $A_C(U)$ and $|\alpha(t)| = |\alpha| = 0$, i. e.

$$|< \zeta \,|\, \alpha(t) >| = |\zeta|.$$

Now if $s_i \in E\,(U)_0$ then clearly s_i, $i = 1\ldots, \text{rank}_{A(U)} E(U)$ is a basal system of generators iff $s = tf$ for some $f \in$ Aut $E(U)$. In this case we have

LEMMA. $\alpha\,(s) - f\,\alpha(t)f^{-1} = (1/2\pi i)\ df\,/\,f.$

Proof. Because $s = ft$, we have by definition

$$\nabla_\xi s = 2\pi i < \xi \,|\, \alpha(s) > s = 2\pi i < \xi \,|\, \alpha(ft) > ft$$

$$= \nabla_\xi (ft) = (\xi f)t + (-1)^{|\xi||f|} f\, \nabla_\xi t$$

$$= 2\pi i < \xi \,|\, (1/2\pi i)\text{d}f > t + (-1)^{|\xi||f|}\ f.\ 2\pi i < \xi \,|\, \alpha(t) > t$$

$$= 2\pi i < \xi \,|\, (1/2\pi i)\ \text{d}f f^{-1} > ft + 2\pi i < \xi \,|\, f\alpha(t) > f^{-1}.\, ft$$

from which the lemma follows

2.3. DE RHAM COMPLEX $(\Omega(X.E), d_\nabla)$.

The classical formulas for the differential of a vector valued function hold in supercase. However we must be careful in using the right (or the left) $A(U)$— module structure in $\Omega^1\ (U, E)$; for example, with $f \in E(U)$ arbitrary, one has

$$df = \sum_{i=1}^{m} dr_i\ \partial f/\partial r_i + \sum_{j=1}^{n} ds_j\ \partial f/\partial s_j.$$

So for an arbitrary open set $U \subseteq X$ the map

$$d : \Omega^0 (U, E) \to \Omega^1(U, E)$$

has $Z/2$-graded degree equal to zero.

Observe that $\Omega\ (U, E)$ is $Z + Z/2$-graded and End $\Omega\,(U, E)$ is also $Z + Z/2$-graded.

Thus $u \in$ End $\Omega\ (U, E)$ is of bidegree (c, j) if

$$u(\Omega^b(U, E)_i \subseteq \Omega^{b+c}(U, E)_{i+j}$$

for any $(b, i) \in Z \oplus Z/_2$-graded degree.

If $u \in$ End $\Omega\ (U, E)$ is of bidegree (c, j) we will say that u is a derivation of bidegree (c, j) if for any $\alpha \in \Omega^b(U, E)_i$ and $\beta \in \Omega\ (U, E')_j$, and if E and E' are paired with values in F one has

$$u(\alpha\beta) = u(\alpha)\beta + (-1)^{bc+ij}\,\alpha u(\beta).$$

Note that a derivation must make sense for any superbundle together: so it has a functor sense.

Let us now denote by ∇ the affine connection of our superbundle E, and by $\alpha = \alpha_{\nabla}$ its connection form. Let $\beta \in \Omega^b(U, E)$. Then we define the differential d_{∇}, and the internal product $i(\xi)$ and Lie derivative $\Theta(\xi)$ for any vector superfield $\xi \in \operatorname{Der} A(U, E)$ by the following formulas

$$\langle \xi_1, \dots, \xi_{b+1} | d\beta \rangle = \sum_{i=1}^{b+1} (-1)^{i-1+j_{i-1}|\xi_i|} \nabla_{\xi_i} \langle \xi_1, \dots, \check{\xi}_i, \dots, \xi_{b+1} | \beta \rangle$$

$$+ \sum_{k<l} (-1)^{d_{k,l}} \langle [\xi_k, \xi_l], \xi_1, \dots, \check{\xi}_k, \dots, \check{\xi}_l, \dots, \xi_{b+1} | \beta \rangle,$$

where $j_i = \sum_{k=1}^{i} |\xi_k|$, $d_{k,l} = |\xi_k|\, j_{k-1} + |\xi_l|\, j_{l-1} + |\xi_k||\xi_l| + k + l$,

$$\langle \xi_1, \dots, \xi_{b-1} | i(\xi)\beta \rangle = (-1)^{|\xi| \sum_{i=1}^{b} |\xi_i|} \langle \xi, \xi_1, \dots, \xi_{b-1} | \beta \rangle.$$

Finally we set by definition

$$\theta_{\nabla}(\xi) = d_{\nabla} \cdot i(\xi) + i(\xi)\, d_{\nabla}.$$

It is easy to see that d_{∇}, $i(\xi)$, and $\theta_{\nabla}(\xi)$ are the derivations of bidegrees $(1, 0)$, $(-1, |\xi|)$, $(0, |\xi|)$, respectively.

We have the superbracket relations:

(1) $i(\xi)\, i(\eta) + (-1)^{|\xi|.|\eta|}\, i(\eta)\, i(\zeta) = [i(\xi), i(\eta).] = 0,$

(2) $\theta(\xi)\, i(\eta) - (-1)^{|\xi|\cdot|\eta|}\, i(\eta)\, \theta(\xi) = [\theta(\xi), i(\eta)] = i([\xi, \eta]),$

(3) $\theta(\xi)\, \theta(\eta) - (-1)^{|\xi||\eta|}\, \theta(\eta)\theta(\xi) = [\theta(\xi,), \theta(\eta)] = \theta([\xi, \eta])$

and the following relationship between contraction and Lie derivation

$$\xi < \xi_1 \dots, \xi_b | \beta > = \sum_{i=1}^{b} (-1)^{|\xi| \sum_{k=1}^{i-1} |\xi_k|} < \xi_1, \dots, [\xi, \xi_i], \dots, \xi_b | \beta > +$$

$$+ (-1)^{|\xi| \sum_{k=1}^{b} |\xi_k|} < \xi_1, \dots, \xi] \, \theta(\xi_b)\, \beta >.$$

So we have a de Rham complex of global sections of E

$$\Omega^*(X, E): \dots \to \Omega^b(X, E) \xrightarrow{d_{\nabla}} \Omega^{b+1}(X, E) \to \dots$$

iff the connection ∇ is flat, i.e. for all $\xi, \eta \in \operatorname{Der} A(U)$

$$\operatorname{Curv}(\nabla)(\xi, \eta) = [\nabla_\xi, \Delta_\eta] - \nabla_{[\xi, \eta]} = 0.$$

Let $p \in U \subseteq X$, $T_p(X, A)$, the tangent superspace at p, $\Omega^b_E(p)$ be the linear space of all $E(U)$-valued b-linear forms z on $T_p(X, A) = T_p(X) \oplus T_p(X, A)$, such that

$$\langle v_1, \ldots, v_j, v_{j+1}, \ldots, v_b \mid z \rangle = (-1)^{1+|v_j||v_{j+1}|} \langle v_1, \ldots, v_{j+1}, v_j, \ldots \mid z \rangle$$

Note that $\Omega^b_{E(p)}$ is $Z/2$-graded such that if z is homogeneous then $\langle v_1, \ldots, v_b \mid z \rangle$ vanishes unless

$$|z| = \sum_{i=1}^{b} |v_i| .$$

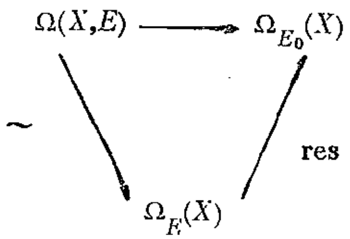
So $z \mid_{T_p(X)}$ is an $E(U)_0$ – valued form on $T_p(X)$ and $z \mid_{T_p(X, A)_1}$ is a symplectic b-linear form on $T_p(X, A)_1$. We define $\Omega^0_E(p) = E(U)_0$

$$\Omega_E(p) = \bigoplus_{b=0}^{\infty} \Omega^b_E(p) .$$

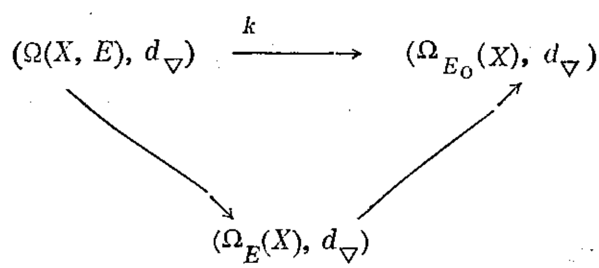
We observe that the map $A(U) \to C^\infty(U)$, $f \to \widetilde{f}$ extends to a homomorphism $\Omega(U, E) \to \Omega_E(U)$, $\beta \to \widetilde{\beta}$,

$$(\langle \xi_1, \ldots, \xi_b \mid \beta \rangle)^{\sim} = \langle \widetilde{\xi}_1, \ldots, \widetilde{\xi}_b \mid \widetilde{\beta} \rangle$$

Now let $\Omega_E(X) \to \Omega_{E_0}(X) = \mathrm{Hom}\,(T_p(X), E(U)_0)$ be the restriction map from the complex $\Omega_E(X)$ to the ordinary $E(U)_0$– valued de Rham complex. Then we have a commutative diagram



Note that $k : \Omega(X, E) \to \Omega_{E_0}(X)$ commutes with d_∇ and suppose that the connection ∇ is flat. Then we have actually a complex homomorphism commutative diagram



2.4. POINCARÉ LEMMA. *Suppose $f \in E(U) = \Omega^0(U, E)$. Then $d_\nabla f = 0$ in the connected open U if and only if $f = \lambda \cdot 1_U$ where λ is a constant function with* *single value in End $E(pt)$. If, in addition, U is a contractible A-coordinate neighbourhood and $\beta \in \Omega^b(U, E)$, $d_\nabla \beta = 0$ then there exists $\omega \in \Omega^{b-1}(U, E)$ such that $\beta = d_\nabla \omega$.*

Proof. Our connection is flat by assumption, so locally we can take the trivial form of connection. Then $d_\nabla = d$

$$df = \sum_{i=1}^{m} dr_i\, \partial f/\partial r_i + \sum_{j=1}^{n} ds_j \,.\, \partial f/\partial s_j$$

and because $f = \sum_\mu f_\mu s_\mu$ for every scalar superfunction the first assertion is trivial.

Locally the map $(\Omega(U, E), d_\nabla) \xrightarrow{k} (\Omega_{E_0}(X), d_\nabla)$ is a complex isomorphism.

So for a contractible open U, the acyclicity of the usual de Rham complex implies the acyclicity of our graded de Rham complex, proving the lemma.

2. 5. DE RHAM THEOREM. *One has a commutative diagram of algebra isomorphisms*

$$\begin{array}{ccc} \mathrm{Coh}\,(\Omega(X,E), d_\nabla)) & \xrightarrow{\quad \mathbf{k} \quad} & \mathrm{Coh}\,(\Omega_{E_0}(X)), d_\nabla) \\ & \cong \searrow \qquad \nearrow \cong & \\ & H(X\ ;\ \mathrm{End}\ E(X)_0) & \end{array}$$

Proof. For the complex $\Omega(X, A)$, B. Kostant [8, § 4.7] has constructed a flasque resolution of the constant sheaf. Our complex is, by Corollary 2.1, its tensor product with End $E(pt)$. Thus we have a flasque resolution by our complex for the constant sheaf.

2. 6. CURVATURE. Let (E, ∇) be a vector superbundle with connection form $\alpha = \alpha_\nabla$ on a supermanifold (X,B). Then there exists a unique differential 2-superform $\omega \in \Omega^2(X, \mathrm{End}\ E)$ such that $\omega = d_\nabla \alpha$, i.e.

$\langle \xi,\eta \mid \omega \rangle = \langle \eta \mid \xi\, \alpha \rangle - (-1)^{|\xi| \cdot |\eta|} \langle \xi \mid \eta\alpha \rangle - \langle [\xi,\eta] \mid \alpha \rangle + [\langle \xi \mid \alpha \rangle, \langle \eta \mid \alpha \rangle]$.
This 2-superform is called the *curvature form* of the connection ∇.

PROPOSITION.

$\langle \xi.\eta \mid \mathrm{Curv}\,(E,\nabla) \rangle = [\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]} = 2\pi i \langle \xi, \eta \mid \omega \rangle$
The proof is the same as in the classical case.

2. 7. *The action of* $H^1(X;\ \mathrm{Aut}\ E(X)_0)$ *on* $\mathcal{L}_\omega(X,\ A)$. Let (E, ∇) be a vector superbundle with connection over (X, A) and $\{(U_i\ , t_{ij})\}\ i \in I\ ,\ j = 1, \ldots,\ \mathrm{rank}_A E$, be a *local system* for E. We denote $t_i = (t_{i1}\ , \ldots, t_{i\ \mathrm{rank}_A\ E})$ and $c_{ij} \in \mathrm{End}\ E(U)$, the transition functions defined by $t_i\ c_{ij} = t_j$, and we will then refer to the set $(c_{ij},\ \alpha_i)$,

$$\alpha_j - c_{ij}\,\alpha_i\, c_{ij}^{-1} = 1/(2\pi i)\, dc_{ij}\, .\, c_{ij}^{-1}$$

as *local data* for $(E,\ \nabla)$.

If $(c_{ij}\ , \alpha'_i)$ is an another local data of some vector superbundle $(E',\ \nabla')$, then (E, ∇) is equivalent to $(E',\ \nabla')$ if and only if there exists $\lambda_i \in \mathrm{ISO}\ (E,\ E')_0$ such that

$$\lambda_i\ c_{ij}\, \lambda_j^{-1} = c'_{ij} \quad \text{and}$$

$$\alpha'_i - \lambda_i\ \alpha_i\, \lambda_i^{-1} = \frac{1}{2\pi i}\ d\ \ \lambda_i \cdot \lambda_i^{-1}$$

Since every vector superbundle with connection admits at least one local data with respect to a contractive covering, it follows that the notion of curvature is an equivalentce invariant and hence $\mathrm{Curv}\ [(E,\ \nabla)] := [\ \mathrm{Curv}\ \ (E,\nabla)]$ is well defined. Note that the set $\mathcal{L}_c\ (X,\ A)$ of all equivalent classes of vector superbundles with conection has the structure of an abelian group:

$$[(E,\ \nabla)] = [(E',\ \nabla')] + [(E'',\ \nabla'')],$$

$$c_{ij} = c'_{ij} \cdot c''_{ij}$$

$$\alpha_i = \alpha'_i + \alpha''_i$$

$$\mathrm{Curv}\ [(E,\ \nabla)] = \mathrm{Curv}\ [(E',\ \nabla')] + \mathrm{Curv}\ [(E'',\ \nabla'')].$$

Now for any closed 2-superform $\omega \in \Omega^P_{\mathbb{C}}(X,\ \mathrm{End}\ E(U)_0)$ let $\mathcal{L}_\omega(X,\ A)$ be the set of all classes $[(E,\ \nabla)] \in \mathcal{L}_c(X,\ \ A)$ such that $\omega = \mathrm{Curv}\ [(E,\ \nabla)]$,

$$\mathcal{L}_c(X,\ A) = \bigcup_{\omega} \mathcal{L}_\omega\ (X,\ A)$$

is a disjoint union over the set of all closed 2-superform $\omega \in \Omega^2\,(X,\ \mathrm{End}\ \ E(U)_0)_0$.

Now given a closed 2-superform $\omega \in \Omega^2\,(X,\ \mathrm{End}\ E\,(U))$ the question is to decide whether $\mathcal{L}_\omega\ (X,\ A)$ is empty or not. We observe that one has the same answer as in the ungraded case.

If Aut $E(pt)$ is the automorphism group of fiber transformations. Then the cohomology group $H^1\,(X;\ \mathrm{Aut}\ E(X)_0)$ operates on $\mathcal{L}_c\,(X,\ A)$.

Let $\{U_i\}_{i \in I}$ be a contractible covering of X and assume that (E, ∇) is a vector superbundle with connection over (X,A). Let (c_{ij}, α_i) be the corresponding local data for (E,∇). Let z_{ij} be a cocycle for the constant sheaf Aut $E(pt)_0$.

LEMMA. *H^1 (X, Aut $E(X)_0$) operates as a group endomorphisms of $\mathcal{L}_c$ (X, A) in such a fashion that one has*

$$[z_{ij}] \cdot [(E, \nabla)] = [(E', \nabla')]$$

where (E', ∇') has the local data $(c_{ij} z_{ij}, \alpha'_i)$ with respect to the covering $\{U_i\}_{i \in I}$, $\alpha'_i = z_{ij} \alpha_i z_{ij}^{-1}$.

Proof. It is easy to see that $(c_{ij} z_{ij}, z_{ij} \alpha_i z_{ij}^{-1})$ is a local data of some superbundle with connection (E', ∇'). We must prove that Curv $(E', \nabla') = \omega$. Since $(z_{ij}, 0)$ is also a local data, there exists a flat superbundle with connection (E_z, ∇_0) such that $(z_{ij}, 0)$ is its local data. By the abelian group structure on $\mathcal{L}_c(X, A)$, we have

$$\begin{aligned} \text{Curv } (E', \nabla') &= \text{Curv } (E, \nabla) + \text{Curv } (E_z, \nabla_0) \\ &= \omega + 0 = \omega . \end{aligned}$$

2. 9. COROLLARY. $\mathcal{L}_\omega$ $(X, A) = H^1$ $(X;$ *Aut* E $(X)_0)$

2.10. Let us denote by exp the exponential map

$$f \in \text{End } E\ (U)_0 \longmapsto \exp\ (2\pi i f) \in \text{Aut } E(U).$$

$$\exp\ (2\pi i f) = \sum_{n=0}^{\infty} (2\pi i f)^n / n!$$

The exponential series converges absolutely on the operator norm topology, as usual.

Note that the elements of the form $I +$ End $E^1(U)_0$, the unipotents, have unique logarithms in End $E^1(U)_0$, hence there is an isomorphism

$$\text{End } E^1 (U)_0 \cong I_U + \text{End } E^1 (U)_0 .$$

Denoting by Γ the kernet of the exponential map, one has an exact sequence of sheafs of groups

$$0 \to \Gamma \longrightarrow (\text{End } E)_0 \xrightarrow{\exp} \text{Aut } E_0 \longrightarrow 1.$$

So by the long exact cohomology sequence we have

2. 11 COROLLARY. *Let (X, A) be a supermanifold and let $\mathcal{L}_c$ (X, A) be the group of equivalence classes of superbundles over (X,A). Then one has an isomorphism of group*

$$\mathcal{L}_c(X, A) = H^2 (X, \Gamma)$$

Remark. By the Kuiper type theorems, Aut $E(U)$ can be homotopically trivial in the infinite–dimensional case. The interesting case is the classification of all infinite dimensional Hilbert superbundles associated to a principal superbundle with a Lie-supergroup as the structural group. Nothing in this case is trivial and it is the main subject of the multidimensional quantization.

2.12. LEMMA. *Let (X, A) be a supermanifold and let $\omega \in \Omega^2_C\ (X,\ End\ E)_0$ be a closed 2-form. Then $\mathcal{L}_\omega(X, A)$ is non empty if and only if the cohomology class $[\omega]$ belongs to the cohomology group $H^2(X, \Gamma) \hookrightarrow H^2(X, Aut\ E(X)_0)$.*

Proof. Assume that the class $[\omega]$ is Γ—valued. Let $\{U_i\}$, $i \in I$, be a contractible covering of X. By the Poincaré Lemma there exists $\alpha_i \in \Omega^1(U_i, E)$ such that $d\alpha_i = \omega|U_i$. Hence in the intersection $U_i \cap U_j$, $d(\alpha_i - \alpha_j) = 0$. Thus there exists $f_{ij} \in$ Aut $E(U_i \cap U_j)$ such that $\alpha_j - \alpha_i = df_{ij}$. But then in the intersection $U_i \cap U_j \cap U_k$ $d(f_{ij} + f_{jk} - f_{ik}) = 0$. So one has some $z_{ijk} \in$ Aut $E(X)_0$ such that $f_{ij} + f_{jk} - f_{ik} = z_{ijk}\, Id_{U_i \cap U_j \cap U_k}$. But now the class of ω is Γ—valued, and z_{ijk} can be chosen in Γ. So $c_{ij} = \exp\ (f_{ij}) \in$ Aut $E(U)_0$. It is easy to verify that (c_{ij}, α_i) is a local data of some vector superbundle with connection (E, ∇). Clearly $\omega =$ Curv (E, ∇) proving that $\mathcal{L}_\omega(X, A)$ is nonempty.

The converse assertion is proved in the same way as the classical one:

3. QUANTIZATION OPERATOR

3.1. Let (M, A, ω) be a symplectic supermanifold of dimension (m, n). Since $(X, k\omega)$ is a symplectic manifold the even dimension m must be even $m = 2m_0$ and there exists at every point a Darboux coordinate system (p_i, q_i) in a coordinate neighbourhood U. On the other hand, because the restriction of ω to the odd part of the tangent superspaces is a nondegenerate symmetric form, there exists (see [8, § 5. 3]) a Morse canonical system of coordinates s_j, $j = 1, \ldots, n$. So altogether there exists at every point a so called *A-Darboux* coordinate system $((p_i, q_i)_{i=1, \ldots, m_0}, s_j, j = 1, \ldots, n)$, where $p_i, q_i \in A(U)_0$, and $s_j \in A(U)_1$, and in which

$$\omega = \sum_{i=1}^{m_0} dp_i \wedge dq_i + \sum_{i=1}^{n} \frac{\varepsilon_j}{2} (ds_j)^2$$

where ε_j is either $+1$ or -1.

It is easy to see that in this A-Darboux coordinate system the hamiltonian vector superfield ξ_f corresponding to a superfunction f, $i(\xi_f)\,\omega = df$ (see [8, §5. 2]) now becomes

$$\xi_f = \sum_{i=1}^{m_0} \left(\frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} \right) + \sum_{j=1}^{n} (-1)^{|f|} \varepsilon_j \frac{\partial f}{\partial s_j} \frac{\partial}{\partial s_j}$$

In particular

$$\xi p_k = - \frac{\partial}{\partial q_k} \ , \ \xi q_k = \frac{\partial}{\partial p_k},$$

$$\xi s_i = \varepsilon_j \frac{\partial}{\partial s_j}$$

In such a A-Darboux coordinate system the Poisson superbrackets has the form

$$f,g \in A(U) \to \{f,g\} := \xi_f(g) = -(-1)^{|f|\,|\xi|}\, \xi_g(f) = \langle \xi_f, \xi_g \mid \omega \rangle$$
$$= \sum_{i=1}^{m_0} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) + \sum_{j=1}^{n} (-1)^{|f|} \varepsilon_j \frac{\partial f}{\partial s_j} \frac{\partial g}{\partial s_j}$$

In particular we have for the Poisson brackets the canonical supercommutator relations

$$\{p_i, p_k\} = \{q_i, q_l\} = \{s_j, p_k\} = \{s_j, q_l\} = 0,$$
$$\{q_k, p_l\} = \delta_{kl} 1_U,$$
$$\{s_i, s_j\} = \varepsilon_j \delta_{ij} 1_U .$$

Now we consider the quantization problem. By definition, a *quantization procedure* is a correspondence associating to each superfunction f a supersymmetric superoperator f in some fixed Hilbert superspace which is antiautoadjoint superoperator if f is a real superfunction and

$$\widehat{\{f_1, f_2\}} = \frac{i}{\hbar} [\widehat{f_1}, \widehat{f_2}],$$

$$\widehat{1} = Id$$

where $\hbar = h/2\pi$ and h is the Planck's constant in physics.

3.2. Let us denote by (E, ∇) a vector superbundle with connection and with Hilbert superspace fibers such that the connection preserves the Hilbert structure. Let us denote also by $(\hbar/i).\alpha(.)$ the connection superform of ∇. Then the values of the superform α are the antisymmetric superoperators for complex vector superfields and are anti-autoadjoint for real vector superfields..

Now we define for each superfunction $f \in A(M)$ the corresponding quantized superoperator $\widehat{f}$

$$f = \widehat{f} + \hbar/i \cdot \nabla_{\xi_f} = f + \mathrm{h}/2\pi i \cdot \nabla_{\xi_f}$$

3. 3. THEOREM I. *The three following conditions are equivalent:*
(1) *The superoperator-valued differential 2-superform satisfies the following (non-linear in multidimensional case) equation*

$$\langle \xi, \eta \mid d_\nabla \alpha \rangle = \langle \eta \mid \xi \alpha \rangle - (-1)^{|\xi||\alpha|} \langle \xi \mid \eta \alpha \rangle + \frac{i}{\hbar} [\langle \xi \mid \alpha \rangle, \langle \eta \mid \alpha \rangle] = \langle \xi, \eta \mid \omega \rangle \mathrm{Id}$$

(2) *The curvature of* ∇ *is symplectic. More precisely,*

$$\langle \xi, \eta \mid Curv\,(E, \nabla) \rangle = [\nabla_\xi, \nabla_\eta] - \nabla_{[\xi, \eta]} = -\frac{i}{\hbar} \langle \xi, \eta \mid \omega \rangle$$

(3) *The correspondence* $f \to \widehat{f}$ *is a prequantization procedure.*

3. 4. Remark 1. Here we talk about a prequantization procedure because as usual the covariant derivatives ∇_ξ, $\xi \in$ Der $A(U)$ operates as superoperators in End $E(U)$ for every open set U.

Only with regards to the superpolarization can we construct the corresponding Hilbert superspace of the quantum states of the physical system under consideration, and have then a quantization procedure.

Remark 2. In condition (1) of the above theorem, we have a nonlinear superoperator-valued differential equation

$$\langle \xi, \eta \mid d_\nabla \alpha \rangle = \langle \xi, \eta \mid \omega \ \mathrm{Id} \rangle$$

In condition (2) this nonlinear equation takes on the equivalent form of some curvature equation

$$\mathrm{Curv}\,(E, \nabla) = \frac{-i}{\hbar} \langle .,. \mid \omega \rangle \,\mathrm{Id}.$$

Equations of this kind related to the curvature of some connections are encountered in modern physics. An interesting example is the Yang — Mills equation

$$* \,\mathrm{Curv}\,(E, \nabla) = \pm\, \mathrm{Curv}\,(E, \nabla).$$

where the star means the Hodge star. By Radon-Penrose transform this equation becomes the Cauchy-Riemann conditions for the corresponding bundle in the space of light rays of the complexified compactified Minkowski space.

CONJECTURE. Our quantization equation could be also converted by the Radon-Penrose transform into some algebraic condition of the corresponding superbundle.

3.5. *Proof of Theorem 1.* (1) $\Leftrightarrow$ (2):
We have

$$[\nabla_\xi, \nabla_\eta] - \nabla_{[\xi, \eta]} = [\theta(\xi) + \frac{i}{\hbar} \langle \xi \mid \alpha \rangle, \ \theta(\eta) + \frac{i}{\hbar} \langle \eta \mid \alpha \rangle] -$$
$$- \theta\,([\xi, \eta]) - \frac{i}{\hbar} \langle [\xi, \eta] \mid \alpha \rangle.$$

$$= [\theta(\xi), \theta(\eta)] + \frac{i}{\hbar}\ [\theta(\xi), \langle \eta \mid \alpha \rangle] - \frac{i}{\hbar}(-1)^{|\eta||\xi|}[\theta(\eta), \langle \eta \mid \alpha \mid \rangle] +$$

$$+ \left(\frac{i}{\hbar}\right)^2 [\langle \xi \mid \alpha \rangle, \langle \eta \mid \alpha \rangle] - \theta([\xi, \eta]) - \frac{i}{\hbar}\langle [\xi, \eta] \mid \alpha \rangle =$$

$$= [\theta(\xi), \theta(\eta)] - \theta([\xi, \eta]) +$$

$$+ \frac{i}{\hbar}\Big\{[\theta(\xi), \langle \eta \mid \alpha \rangle] - (-1)^{|\xi||\eta|}[\theta(\eta), \langle \xi \mid \alpha \rangle] -$$

$$- \langle [\xi, \eta] \mid \alpha \rangle + \frac{i}{\hbar}[\langle \xi \mid \alpha \rangle, \langle \eta \mid \alpha \rangle]\Big\}$$

In view of superbracket relations 2,3 (3), $[\theta(\xi), \theta(\eta)] = \theta([\xi, \eta])$ the rest of the proof is immediate

LEMMA $[\theta(\zeta), \langle \eta \mid \alpha \rangle] = \langle \eta \mid \zeta \alpha \rangle$

Proof. For all supersections $f \in E(U)$ we have

$$[\theta(\xi), \langle \eta \mid \alpha \rangle] f = \theta(\xi)(\langle \eta \mid \alpha \rangle f) + (-1)^{|\xi||\eta|}\langle \eta \mid \alpha \rangle \theta(\xi) f -$$

$$- (-1)^{|\xi||\eta|}\langle \eta \mid \alpha \rangle \theta(\xi) f =$$

$$= \theta(\xi) \langle \eta \mid \alpha \rangle f$$

3.6. (2) $\Leftrightarrow$ (3):

By the definition of the quantization operators we have

$$\frac{i}{\hbar}[\widehat{f_1}, \widehat{f_2}] = \frac{i}{\hbar}[f_1 + \frac{\hbar}{i}\nabla_{\xi_{f_1}}, f_2 + \frac{\hbar}{i}\ \nabla_{f_2}] =$$

$$= \frac{i}{\hbar}\Big\{\frac{\hbar}{i}[f_1, \nabla_{\xi_{f_2}}] + \left(\frac{i}{\hbar}\right)^{-1}[\nabla_{\xi_{f_1}}, f_2] + \left(\frac{\hbar}{i}\right)^2[\nabla_{\xi_{f_1}}, \nabla_{\xi_{f_2}}] =$$

$$= [f_1, \nabla_{\xi_{f_2}}] + \frac{\hbar}{i}\ \nabla_{\xi_{\{f_1, f_2\}}} - \frac{\hbar}{i}\nabla_{\xi_{\{f_1, f_2\}}} +$$

$$+ [\nabla_{\xi_{f_1}}, f_2] + \frac{\hbar}{i}[\nabla_{\xi_{f_1}}, \Delta_{\xi_{f_2}}]$$

It is easy to see that $\xi_{\{f_1, f_2\}} = [\xi_{f_1}, \xi_{f_2}]$. So the proof of the theorem will be completed by proving the following

LEMMA. $[f_1, \nabla_{\xi_{f_2}}] = [\nabla_{\xi_{f_1}}, f_2] = \theta(\xi_{f_1}) f_2 = -(-1)^{|\xi_{f_1}||\xi_{f_2}|}\theta(\xi_{f_2})(f_1)$

$$= \langle \xi_{f_1}, \xi_{f_2}\omega \rangle = \{f_1, f_2\}$$

Proof. The multiplication by a superfunction is supercommuting with the multiplication by any superoperator-valued superfunction. So

$$[f_1, \nabla_{\xi_{f_2}}] = [f_1, \theta(\xi_{f_2})].$$

For every section $s \in E(U)$, one has

$$[f_1, \theta(\xi f_2)]\, s = f_1(\theta(\xi f_2)s) - (-1)^{|f_2||f_1|}\theta(\xi f_2)(f_1 s)$$

$$= f_1(\theta(\xi f_2)s) - (-1)^{|\xi f_2||f_1|}(\theta(\xi f_2) f_1)s - (-1)^{2|\xi f_2||f_1|} \cdot f_1(\theta(\xi f_2)s)$$

$= -(-1)^{|\xi_{f_2}||f_1|}(\theta(\xi_{f_2})f_1)s$

$= \langle \xi_{f_1}, \xi_{f_2} \mid \omega \rangle s.$

Similarly,

$$[\nabla_{\xi_{f_1}}, f_2] = \theta(\xi_{f_1})(f_2) = \langle \xi_{f_1}, \xi_{f_2} \mid \omega \rangle.$$

The lemma is proved.

4. APPLICATION TO SUPERGROUP REPRESENTATION

4. 1. COCOMMUTATIVE HOPF SUPERALGEBRAS [8, § 3. 3]. We recall that if G is any group and K is the fixed ground field (R or C) the group algebra $K(G)$ is a cocommutative Hopf algebra with antipode over K,

$$\triangle : K(G) \longrightarrow K(G) \otimes K(G)$$

so that for $g \in G$, $\triangle(g) = g \otimes g$, $s(g) = g^{-1}$, $1_E(g) = 1$.

Now assume also that $\mathcal{G}$ is a Lie superalgebra and one has a representation $\Pi : G \longrightarrow \mathrm{Aut}\, \mathcal{G}$. Then Π extends uniquely to a representation of G by automorphisms of the universal enveloping superalgebra $U(\mathcal{G})$. Now the *smash product*

$$K(G) \# U(\mathcal{G})$$

with respect to Π (or simply smash product if Π is selfevident), is by definition a cocommutative Hopf superalgebra with antipode such that

(1) as a vector superspace it is the graded tensor product $K(G) \# U(\mathcal{G})$.

(2) as an algebra $K(G)$ and $U(\mathcal{G})$ are subalgebras but $gug^{-1} = \Pi(g)\, u$ for g in G and u in $U(\mathcal{G})$,

(3) with respect to the diagonal map $\triangle$ the elements of G are group-like and the elements of $\mathcal{G}$ are primitive and.

(4) one has $s(g) = g^{-1}$, $s(X) = -X$, and $1_E(g) = 1$, $1_E(X) = 0$ for $g \in G$, $X \in \mathcal{G}$.

Conversely, let E be any commutative Hopf superalgebra with antipode over an algebraically closed field K of characteristic zero, G be the group of all group-like elements in E and $\mathcal{G}$ be the Lie superalgebra of all primitive elements in E. Then there is a representation $\Pi : G \to \mathrm{Aut}\, \mathcal{G}$ such that for any $g \in G$, $X \in U(\mathcal{G})$, $gXg^{-1} = \Pi(g)\, X$ and E is isomorphic to the smash product with respeet to Π

$$E \cong K(G) \# U(\mathcal{G}).$$

4. 2. *Lie-Hopf superalgebras.* Now assume that G is a group, $\mathcal{G}$ is a Lie superalgebra over R and E is the Hopf superalgebra with respect to some re

presentation $\Pi : G \to \operatorname{Aut} \mathcal{G}$. Following B. Kostant [8, § 3. 4], we will say that E has the structure of a Lie-Hopf superalgebra if:

(1) G has the structure of a (not necessarily connected) Lie group,

(2) $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_1$ is a finite dimensional Lie superalgebra where $\mathcal{G}_0 \cong \operatorname{Lie} G$ is the Lie algebra of G,

(3) $\operatorname{Ad}_{\mathcal{G}}$ is defined on the identity component G_0 of G and $\Pi \mid G_0 \simeq \operatorname{Ad}_{\mathcal{G}}$.

If $E(G, \mathcal{G})$ is a Lie-Hopf superalgebra, let $E(G, \mathcal{G}_0)$ be the Lie-Hopf algebra obtained by replacing $\mathcal{G}$ by its even part. As a Hopf algebra it satisfies

$$E(G, \mathcal{G}_0) = C^{\infty}(G)^*$$

where $C^{\infty}(G)^*$ is the set of distribution with finite support.

The collection of Lie-Hopf algebras form a category in which a morphism

$$E(G, \mathcal{G}) \to E(H, \mathcal{H})$$

is a morphism of Hopf superalgebras such that the restriction to Hopf even parts is a Hopf algebra morphism

$$E(G, \mathcal{G}_0) \to E(H, \mathcal{H}_0)$$

induced by a morphism $G \to H$ of Lie groups.

Of course $E(G,\mathcal{G})$ and $E(H, \mathcal{H})$ are isomorphic if there exists an isomorphism

$$E(G, \mathcal{G}_0) \overset{\simeq}{\to} E(H, \mathcal{H}_0)$$

4.3. *Superalgebra* $B(Y)^*$. Let (Y, B) be a supermanifold $B(Y)$ the superalgebra of superfunctions on (Y, B). Consider the full dual $B(Y)'$ of $B(Y)$. One has certainly an injection

$$0 \to B(Y)' \otimes B(Y)' \to (B(Y) \otimes B(Y))'$$

and also the diagonal map

$$\Delta : B(Y)' \to (B(Y) \otimes B(Y))'$$

defined by the relation

$$v(f \otimes g) = v(fg)$$

for $v \in B(Y)'$, $f, g \in B(Y)$.

Consider the subspace $B(Y)^*$ defined as the supersubspace of $B(Y)'$ of all $v \in B(Y)'$ which vanish on some ideal of finite codimension of $B(Y)$. One knows that if $v \in B(Y)'$ $\Delta v \in B(Y)' \otimes B(Y)'$ iff $v \in B(Y)^*$. So one has a morphism

$$\Delta : B(Y)^* \to B(Y)^* \otimes B(Y)^*$$

which induces on $B(Y)^*$ the structure of a cocommutative superalgebra.

Recall that if $\Delta: C \to C \otimes C$ is a superalgebra an element $\sigma \in C$ is called *group-like* if it is a non zero even element and $\Delta\sigma = \sigma \otimes \sigma$. An element v is called *primitive* with respect to a group-like element σ if

$$\Delta v = \sigma \otimes v + v \otimes \sigma.$$

Note that $B(Y)^*$ is just the set of all distributions of finite support on (Y,B).

Recall that a morphism

$$\tau : B(Y)^* \to C(Z)^*$$

of supercoalgebras is said to be smooth iff it is an induced morphism $\tau = \sigma^*$ for some morphism of supermanifolds

$$\sigma : (Y, B) \to (Z, C)$$

Now let (G, A) be a supermanifold of dimension (m,n), let

$$\Delta : A(G)^* \to A(G)^* \otimes A(G)^*$$

be the diagonal map with respect to which $A(G)^*$ is a cocommutative superalgebra. The counit is given by the identity element $1_G \in A(G)$, $1_G(v) = v(1_G)$ for $v \in A(G)^*$.

Recall that (G, A) has the structure of a *Lie supergroup* if $A(G)^*$ has also the structure of an algebra such that

(1) $(A(G)^*, 1_G, \Delta)$ is a Hopf superalgebra with antipode s,

(2) the map

$$A(G)^* \otimes A(G)^* \to A(G)^*$$

given by the multiplication and the map

$$s : A(G)^* \to A(G)^*$$

given by the antipode *are smooth.*

It is well known (see B. Kostant [8, §3.5]) that if (G, A) is a Lie supergroup, then G with respect to its manifold and group structure is a Lie group with Lie algebra Lie $G \simeq \mathcal{G}_0$, the even part of the Lie superalgebra $\mathcal{G} = \{$primitive elements of $A(G)^*\}$ and

$$A(G)^* = E(G, \mathcal{G})$$

with respect to the representation $\Pi : G \to \text{Aut}\, \mathcal{G}$ $\Pi(g)x \overset{def}{=} gxg^{-1}$ for $x \in \mathcal{G}$, $g \in G$ and $\Pi \mid G_e = \text{Ad}_{\mathcal{G}}$ the restriction to the identity component.

4.4. *Homogeneous superspaces and isotropy supergroups.*

Let (G, A) be a Lie supergroup and let (Y, B) be a supermanifold. We shall say that (G, A) operates on (Y, B) or (Y, B) is a (G, A)—space iff the following map is smooth

$$\Delta : A(G)^* \times B(Y)^* \to G(Y)^*$$

$$\Delta(vw) = \sum_{i=j} (-1)^{u''_i w'_j} u'_i \cdot w'_j \otimes u''_i w''_j,$$

$$\text{if} \quad \Delta(u) = \sum_i u'_i \otimes u''_i$$

$$\text{and} \quad \Delta(w) = \sum_j w'_j \otimes w''_j,$$

for all $u \in A(G)^*$, $w \in B(Y)^*$.

So in this case, $B(Y)^*$ becomes an $A(G)^*$-module. By duality the commutative superalgebra $B(Y)$ becomes also some $A(G)^*$-module:

$$\langle w, u.f\rangle = (-1)^{|u|\cdot|w|}\langle s(u).w, f\rangle$$

Observe that if $f, g \in B(Y)$ then

$$u.fg = \Sigma\,(-1)^{|f|\,|u_i''|}(u_i'.f)(u_i''.g)$$

Recall that a Lie subsupergroup (H, B) in (G, A) will be called closed if H is a closed Lie subgroup of G. Let

$$\rho : G \longrightarrow H\diagdown G,\ g \longrightarrow Hg$$

be the coset projection map. Put $V = \rho^{-1}(U)$ if U is some open set in $H\diagdown G$. Then $(V, A(V))$ is some (H, B)-superspace and the restriction map $\rho_{G,V} : A(G)\to$ $\to A(V)$ is some (H, B)-module map. Now put

$$B\diagdown A(U) = \{f \in A(V)\,;\, R^V_W f = (-1)^{|w|\,|f|} f s(w) = \langle w, 1_H\rangle f\}$$

It is easy to see that $B\diagdown A(U)$ is a commutative superalgebra contained in $A(V)$ and the correspondence $U \mapsto B\diagdown A\,(U)$ is a sheaf of commutative superalgebras. As it is pointed out in [8. §3. 9], the sheaf $B\diagdown A$ on $H\diagdown G$ together with the homomorphism $B\diagdown A\,(U) \to C^\infty\,(U)$ define the structure of a supermanifold $(H\diagdown G, B\diagdown A)$ of dimension $(m\text{-}m', n\text{-}n')$ if $\dim\,(G, A) = (m, n)$ and $\dim\,(H, B) =$ $= (m', n')$. Furthermore, we have also the local triviality of the projection map, i.e. for sufficiently small open sets U, one has an isomorphism

$$\Theta : (U\times H,\ B\diagdown A\times B) \to (V, A)$$

So we obtain some principal superbundle associated to each Lie subsupergroup.

It is not hard to show that if (H, B) is a closed Lie subsupergroup of a Lie supergroup (G,A) then with respect to the action of (G,A) on $(H\diagdown G, B\diagdown A)$, $(H\diagdown G, B\diagdown A)$ is a homogeneous superspace (see [8, 3. 10. 3]). Conversely, if (X', A') is a homogeneous superspace for (G, A) then $(X', A') \simeq (H\diagdown G, B\diagdown A)$ where (H, B) is the stabilizer of a point $p \in X'$.

4. 5. *Integral Poisson representation.* Suppose that our Lie supergroup (G,A) acts on a symplectic supermanifold $(M, B\,;\,\omega)$ by a representation $\Pi\,(\,.\,)$: $A\,(G)^* \longrightarrow \operatorname{End} B(M)$ such that its restriction to the Lie superalgebra $\mathcal{G}$ is a representation of the Lie superalgebra $\mathcal{G}$ by the canonical transformations $X \in \mathcal{G} \longrightarrow \xi_X \in \operatorname{Ham} B(M) \subseteq \operatorname{Der} B(M)$.

Denote by L_X the Lie derivative along the vector superfield ξ_X, i.e. $L_X = H(\xi_X)$.

We have the natural exact sequence of super Lie algebras

$$0 \longrightarrow R\,1_M \longrightarrow B\,(M) \longrightarrow \operatorname{Ham} B\,(M) \longrightarrow 0$$

Hence for each $X \in \mathcal{G}$ there exists a superfunction, the generating function, $f_X \in B\,(M)$.

By the calculus on supermanifolds (see § 2.3), we have

$$[L_X , L_Y] = L_{[X,Y]}$$

and

$$L_X f = \{f_X , f\}$$

Now suppose that f_X depends linearly on X we then have a 2-cocycle of Lie superalgebra

$$c(X, Y) = \{f_X , f_Y\} - f_{[X,Y]}$$

The quantization procedure yields

$$[Л(X), Л(Y)] = Л([X, Y]) + c(X, Y)$$

by setting $Л(X) = \frac{i}{\hbar} \cdot \widehat{f}_X = \frac{i}{h} f_X + \nabla_{\xi_X}$.

DEFINITION. We shall say that the action of (G, A) by canonical transformations on the symplectic supermanifold (M, B) is *flat* if the 2-cocycle c (...) is zero.

Remark. If the (G, A)-action on (M, B) is flat, the Lie superalgebra homomorphism $\mathcal{G} \to \mathrm{Ham}_{loc}(M,B)$, $X \to \xi_X$ can be lifted to the superalgebra homomorphism $\mathcal{G} \to B(Y)$. So we recover Kirillov's notion of strictly hamiltonian action in Lie group situation.

Obviously, the Lie superalgebra representation $\mathcal{G} \to B(Y)$, $X \to \{f_X, .\}$ is an intergrable Poisson representation in Kostant's sense [8, §5. 4].

In the flat action case we have a representation of our Lie superalgebra $\mathcal{G}$ by the superfunction, the classical (physical) quantities, and a representation $Л$ of our Lie superalgebra by the quantum quantities, the antisymmetric superoperators. If E. Nelson's conditions on integrality are satisfied, we could obtain a Lie supergroup representation $\exp\left(\frac{i}{\hbar} \widehat{f}_X\right)$ of the universal covering $\widetilde{G}$ of (G, A).

DEFINITION. By a *mechanical system with supersymmetry* we mean a symplectic supermanifold together with a flat homogeneous action of some Lie supergroup (of supersymmetry).

Thus starting from a mechanical system with flat action of Lie supergroup (G, A) we can obtain some representations, i. e. the corresponding quantum systems, by using the quantization procedure.

5. INTEGRALITY CONDITION

We see that to construct the quantum systems one needs a prequantization procedure and a Hilbert superbundle of quantum states (with internal symmetry) with connection $(E, \nabla) \in \mathcal{L}_\omega(M, B)$. This section is devoted to the question of whether the set $\mathcal{L}_\omega(M, B)$ is non empty. In general this question is similar to the classical question of line superbundles with connection (see B. Kostant [8, §6. 4] and the preceding section 2.12).

5.1. *Integral functional.* Let $\mathcal{G}$ be any finite dimensional real Lie superalgebra and let (G, A) be the corresponding simply connected Lie supergroup. Let (M, B) be an arbitrary homogeneous (G, A)-superspace with flat action of (G, A). Let $m \in M$ and let (G_m, A_m) be the isotropy subsupergroup at the point m. Let $\mathcal{G}_m \subseteq \mathcal{G}$ be the Lie superalgebra of (G_m, A_m). The point m can be considered as a superfunctional in $\mathcal{G}^*$ in the following sense. Actually every element of G is just some group-like element of the Lie-Hopf superalgebra. So $m \leftrightarrow \delta_m \in A(G)^*$.

$$\langle m, X\rangle \overset{\text{def}}{=} \chi_m\left(\frac{i}{\hbar} f_X\right) = \frac{i}{\hbar} f_X(m)$$

In the flat action case we have some character χ_m

$$\chi_m(\text{Exp } X) = \exp\left(\frac{i}{\hbar} f_X(m)\right).$$

So we shall say that the point m is *integral* iff this character admits a continuation to some smooth unitary representation of the whole stabilizer subsupergroup (G_m, A_m).

5.2. *Existence of quantum superbundles at an integral point*

Suppose that the character χ_m can be extended to some unitary representation σ of the stabilizer (G_m, A_m) in some Hilbert superspace H. Taking a fixed connection on the principal superbundle

$$(G_m, A_m) \rightarrowtail (G,A) \twoheadrightarrow (M,B)$$

that is formed by some (G, A)-invariant tangent superdistribution and the differential of our extended representation, we obtain some affine connection on the associated superbundle. It is easy to see that this vector superbundle is just some quantum superbundle, i.e. its curvature is the symplectic form of our mechanical system. In fact the character χ_m gives a connection on some line superbundle. Its continuations to representations are enumerated by the cohomology group $H^1(M, \text{Aut}_0(H))$. Thus we have the following results:

PROPOSITION. *If (M, B) is some homogeneous flat (G,A) — superspace then the following statements are equivalent:*

(1) Some point $m \in M$ is integral.

(2) Every point of M is integral

(3) The set $\mathcal{L}_\omega(M, B)$ is nonempty.

PROPOSITION. *If* (M, B) *is non empty, its elements are parametrized by the elements of the cohomology group* $H^1(M, \mathrm{Aut}\, E(pt))$ *for some* $(E, \nabla) \in \mathcal{L}_\omega(M, B)$.

Proof. The essence of the proof of these two propositions has been pointed out above. The rest is trivial.

6. SUPERPOLARIZATIONS AND INDUCED REPRESENTATIONS

In this section we shall derive for the supersymmetry some results analogous to those in [3, §II.3]. The main point is the supervariant of the Frobenius theorem.

6. 1. *Tangent superdistributions.* Let (M,B,ω) be a symplectic flat homogenous (G,A)-manifold, i.e. a mechanical system with supersymmetry, $O(M) = M/G$ the (G,A)-orbit space, Ω a fixed (G,A)-orbit, $x \in \Omega$ a fixed point, (G_x, A_x) the stable supersubgroup, $\mathcal{G}_x$ the Lie superalgebra of (G_x, A_x), $((G_x)_0, A_x^0)$ the connected component of the stabilizer, and finally $\tilde{\sigma}$ a unitary representation of (G_x, A_x) the kernel of which contains $((G_x)_0, A_x)$.

Let $T(\Omega,C)$ be the complexified tangent superbundle of the orbit (Ω,C). Note that the sheaf C is just the quotient sheaf $A_x \backslash A$, for some point x in Ω. Assume that L is an invariant, integrable (i. e. such that the Frobenius conditions are satisfied) smooth tangent distribution such that $L + \bar{L}$ is also integrable, where the bar indicates the complex conjugation.

LEMMA *(Frobenius integrability condition). The sets of all global sections of the distributions* L, $L + \bar{L}$ *form Lie subsuperalgebras iff the corresponding distributions are integrable.*

LEMMA. *(Invariant condition), The Lie superalgebra of invariant global sections to integrable distributions* L, $L + \bar{L}$ *are isomorphic to the quotient tangent superspaces* L_x, $L_x + \bar{L}_x$ *iff the distributions are invariant.*

LEMMA $L_x \simeq \mathcal{P}/(\mathcal{G}_x)_c$, $L_x + \bar{L}_x \approx (\mathcal{P} + \bar{\mathcal{P}})/(\mathcal{G}_x)_c$ *for some complex Lie subsuperalgebra* $\mathcal{P} \subset \mathcal{G}_c$.

The proof of these lemmas is trivial if we remark that the invariant smooth sections of L, $L + \bar{L}$ respectively form the invariant vector superfields. Then

$$\mathcal{L},\ \mathcal{L} + \bar{\mathcal{L}} \hookrightarrow \mathrm{Der}_{(G,A)}(\Omega,C)_c$$

$$\text{and so } \mathcal{L} \simeq L_x,\ \mathcal{L} + \bar{\mathcal{L}} \simeq L_x + L_x$$

As vector superspaces $\mathcal{P} \simeq L_x + (\mathcal{G}_x)_c$. The invariance garantees that $[\mathcal{G}_x, L_x] \subseteq L_x$. So $\mathcal{P}$ is a complex superalgebra.

6. 2. DEFINITION. We shall say that the *distribution L is closed* if the connected subsupergroups (H_0, F), (M_0, I) corresponding to the Lie superalgebras $\mathcal{H} = \mathcal{P} \cap \mathcal{G}$ $\mathcal{M} = (\mathcal{P} + \overline{\mathcal{P}}) \cap \mathcal{G}$, and the subsupergroups

$(H, F) = (G_x, A_x) \times (H_0, F)$, $(M, I) = (G_x, A_x) \ltimes (M_0, I)$ are closed.

6.3. DEFINITION. We say that (L, ρ, σ_0) is a (χ_x, σ) — *polarization and L is weakly Lagrangian* if:

(a) σ_0 is an irreducible unitary representation of the supergroup H_0, F) in some Hilbert superspace V such that:

(1) The restriction $\sigma_0|_{(G_x, A_x) \cap (H_0, F)} = \text{mult}\, \chi_x . \tilde{\sigma}$,

(2) The point σ_0 in the dual $(H_0, F)^\frown$ is fixed under the action of the supergroup (G_x, A_x),

(3) ρ is a representation of the complex Lie superalgebra $\mathcal{P}$ in the Hilbert superspace V such that its restriction to the real part $\mathcal{H}$ is the Lie derivative (or the differential as called usually) of the representation σ_0 and all the E. Nelson conditions are satisfied.

6.4. $(\tilde{\sigma}, x)$ — *polarizations.* we modify the motion of $(\tilde{\sigma}, x)$ — (see [3]). We say that $(\mathcal{P}, \rho, \sigma_0)$ is a (σ, x)—polarization if

(a) $\mathcal{P}$ is some complex subsuperabgebra of $\mathcal{G}_e$, containing $\mathcal{G}_x$.

(b) The subsuperalgebra $\mathcal{P}$ is $\text{Ad}_{\mathcal{G}c}(G_x)$—invariant.

(c) The vector superspace $\mathcal{P} + \overline{\mathcal{P}}$ is the complexification of some real superalgebra, i. e. $\mathcal{M} = (\mathcal{P} + \overline{\mathcal{P}}) \cap \mathcal{G}$.

(d) The subsupergroups $(M_0, \tilde{I})$, $(H_0, \tilde{F})$, (M, I), (H, F) are closed, where $(M_0, \tilde{I})$ (resp., $(H_0, \tilde{F})$) is the connected subsupergroup of (G, A) with the Lie superalgebra $\mathcal{M}$ (resp., $\mathcal{H} = \mathcal{P} \cap \mathcal{G}$) and

$$(M, I) = (G_x, A_x) \ltimes (M_0, \tilde{I}),$$

$$(H, F) = (G_x, A_x) \ltimes (H_0, \tilde{F}).$$

(e) σ_0 is an irreducible unitary representation of the supergroup (H_0, F) such that the restriction $\sigma_0 \mid_{(G_x, A_x) \cap (H_0, F)}$ is a multiple of the representation

$\chi_x . \tilde{\sigma} \mid_{(G_x, A_x) \cap (H_0, F)}$, where by definition

$$\chi_x\ (\exp\ X) = \exp\left(\frac{2\pi i}{h}\langle x,\ X\rangle\right)$$

and (2) the point σ_o is fixed under (G_x, A_x) — action on the dual $(H_o, F)^\wedge$.

(f) ρ is some representation of the complex Lie superalgebra $\mathcal{P}$ in the Hilbert superspace V such that its restriction to the real part $\mathcal{H}$ is equivalent to the derivative of the representation σ_o.

6.5. THEOREM 2. (L, ρ, σ_o) is a $(\chi_x, \tilde{\sigma})$-polarization iff $(\mathcal{P}, H, \rho, \sigma_o)$ is a $(\tilde{\sigma}, x)$ - polarization.

Proof. If (L, ρ, σ_o) is a $(\chi_x, \tilde{\sigma})$ - polarization, L is a weakly Lagrangian, invariant, integrable tangent superdistribution of the tangent superbundle $T(\Omega, A_x\setminus A)$. In § 6.1 we have reconstructed the Lie subsuperalgebra $\mathcal{P}$ by L, $\mathcal{P} = (\mathcal{G}_x)_c \oplus L_x$. It is easy to verify that we have in this case a $(\tilde{\sigma}, x)$ — polarization. Conversely, it is easy to reconstruct a $(\chi_x, \tilde{\sigma})$-polarization (L, ρ, σ_o) starting from some $(\tilde{\sigma}, x)$-polarization $(\mathcal{P}, H, \rho, \sigma_o)$.

6.6. COROLLARY. *Suppose Ω to be an integral orbit of a mechanical system with supersymmetry $(M, B;\ G, A, \omega)$, (L, ρ, σ_o) a $(\chi_x, \tilde{\sigma})$-polarization, where $\tilde{\sigma}$ is a representation of (G_x, A_x) the kernel of which contains the connected component $((G_x)_o, A_x)$. Then:*

(1) The homogeneous space Ω admits the structure of a mixed manifold of type (k, l, m) in the sense of [3,2] (2) There exists a unique irreducible unitary representation σ of the supergroup (H, F) such that its restriction to the stable supersubgroup (G_x, A_x) is a multiple of the representation $\chi_x \cdot \tilde{\sigma}$ and its derivative is the restriction of the representation ρ to the real part $\mathcal{H} = \mathcal{P} \cap \mathcal{G}$

Proof. The proof is the same as in the ordinary Lie group situation (see [3, II.3.8]).

6.7. INDUCED REPRESENTATIONS (see B. Kostant [8,§6,1] for the line superbundle case). Assume now that $(L, \rho, \tilde{\sigma}_0)$ is a $(\chi_x, \tilde{\sigma})$ — polarization of our orbit Ω. So by the preceding consideration we obtain some unitary representation σ of the polarization (closed) subsupergroup (H, F) in some Hilbert superspace V. Let us denote by τ_G the natural projection $(G, A) \longrightarrow (H\setminus G,\ F\setminus A)$. Let $U \subset H\setminus G$ be an open set and let $V \equiv \tau_G^{-1}(U) \subseteq G$. One thus has $A(V)^* B(H)^* \subseteq A(V)^*$.

Denote by $A(V,\sigma)$ the superspace of all V-valued superfunction $f \in E_V(U)$ such that

$$\langle wv,f\rangle = \sigma(w)\langle v,f\rangle,\ v \in A(V)^*,\ w \in B(H)^*$$

It is easy to see that $(F\setminus A)(U)$ can be embedded into $A(V)$ as a subsuperspace of superfunctions such that

$$\langle wv, g\rangle = \langle w, 1_H\rangle\langle v, g\rangle,\ g \in (F\setminus A)(U),\ gf \in E_V(U)$$

for all $v \in A(V)^*$, $w \in B(H)^*$. So if $f \in E_V(U)$, $g \in (F\setminus A)(U)$, then $gf \in E_V(U)$. Furthermore, $U \subset H\setminus G \longmapsto E_V(U) = E_V(\tau_G^{-1}(U), \sigma)$ is a sheaf on $H\setminus G$. Let $\sigma^\frown$ be the element of $E_V(\tau_G^{-1}(U)$ such that for any $w \in B(H)^*$,

$$\langle w, \sigma^\frown\rangle = \sigma(w)t_0, \text{ and}$$

for a fixed t_0 in $E_V(\tau_G^{-1}(U)$. Because the representation is irreducible we see that the open set U is principal for the sheaf E_V. Thus it is some superbundle sheaf.

It is also easy to see that $E_V(G, A, \sigma)$ is a closed subsuperspace of V-valued superfunctions on (G, A) and it is stable under the action of our supergroup (G, A) on the right. In particular, the elements of the Lie superalgebra $\mathcal{G}$ acts via the Hamiltonian superfields on the right. Hence for each open set U in $H\setminus G$ the subsuperspace $E_V(\tau_G^{-1}(U), \sigma, \rho)$ consisting of the sections which have the covariant derivatives zero along the vector superfields from our complex polarization $\mathcal{P}$ form a subsheaf which gives us also an invariant closed subsuperspace of global sections of our quantum superbundle E_{V,ρ,σ_0}. We refer to this invariant subsuperspace of global sections of our quantum superbundle as the induced representation Ind $(G, A; \mathcal{P}, H, \rho, \sigma_0)$.

6. 8. COROLLARY. *The natural representation called partially invariant holomorphically induced representation and denoted by Ind* $(G, A; L, x, \rho, \sigma_0)$ *of the Lie supergroup* (G, A) *in the superspace of so called partially invariant partially holomorphic sections of the induced superbundle* E_{V,ρ,σ_0} *is equivalent to the natural right regular representation by right translations on the superspace* $A(G, A; L, x, \rho, \sigma_0)$ *of smooth superfunctions on* G *with values in* V *such that*

$$\langle wv, f\rangle = \sigma(w)\langle v, f\rangle,$$
$$\nabla_{\xi_X} f = 0 \text{ for all } X \in \overline{\mathcal{P}},$$

where $\nabla_{\xi_X} = \theta(\xi_X) + \langle X, \rho\rangle$.

The aim of this final section is to compute the Lie derivative of our representation as a (G, A)-homomorphism of superbundles. We will show that this Lie derivative is just the Lie superalgebra representation obtained from the corresponding multidimensional quantization procedure.

7.1. STATEMENT OF THEOREM 3. The Lie derivative of the partially invariant holomorphically induced representation. Ind $(G, A; \mathcal{P}, (H,F), \rho, \sigma_0)$ is equivalent to the Lie superalgebra representation

$$X \longmapsto \frac{i}{\hbar} \widehat{f}_X$$

of the superalgebra $\mathcal{G}$ of our supersymmetry group (G, A) via the multidimensional quantization procedure.

The proof of this theorem is lengthy and we divide it into several steps.

7.2. Suppose that E_{V,ρ,σ_0} is an induced superbundle. Then we have a homomorphism

$$A(G)^* \to \operatorname{End} E_{V,\rho,\sigma_0}(\Omega).$$

such that its restriction to the Lie superalgebra part is equal to the Lie derivation of the action of the corresponding one parameter subgroup action

$$X \,.\, u = \frac{d}{dt} (\exp(tX) u \exp(-tX)\Big|_{t=0}$$

It is easy to see that $X \,.\, u$ is then some differential superoperator of degree 1

$$X.u = X_E . u - (-1)^{|u||x|} u . X_E$$

7.3. CONNECTION. We consider the similar case for section $u \in E_{V,\rho,\sigma_0}(U)$. Then we have the Lie derivation X. u as the covariant derivation, so that

$$\nabla_{\xi X} = \Theta(\xi_X) + \frac{i}{\hbar} \langle \xi_X \mid \alpha_1 \rangle$$

By identifying $E_{V,\rho,\sigma_0}(U)$ with the corresponding superspace of V-valued superfunctions on U which are σ-invariant $\mathcal{P}$—parallelizable, the covariant derivation is of the form

$$\Theta(\xi_X) + \langle X \mid \rho \rangle$$

for all X in our polarization subsuperalgebra $\mathcal{P}$.

7.4. *Differential superform* β. Note that each point y of our symplectic supermanifold (M, B) can be considered as some linear functional on the Lie superalgebra $\mathcal{G}$:

$$\langle X, y \rangle = f_X(y) \,, \; X \in \mathcal{G} \,,$$

where f_X is the potential of the corresponding Hamiltonian field vector superfield ξ_X.

Define a 1-form $\beta \in \Omega^1(\Omega)$ on Ω by

$$\langle \xi \mid \beta \rangle (y) \text{ def } \langle \xi(e) \mid y \rangle.$$

Then so we have

$$f_X(y) = \langle X, y \rangle = \langle \xi_X \mid \beta \rangle (y).$$

7.5. We see now that the Lie derivative of our induced representation is

$$\nabla_{\xi_X} = \Theta(\xi_X) + \frac{i}{\hbar} \alpha_1(\xi_X)$$

$$= \Theta(\xi_X) + \frac{i}{\hbar} f_X + \frac{i}{\hbar} (\langle \xi_X \mid \alpha_1 \rangle - \langle \xi_X \mid \beta \rangle)$$

Denoting by α the differential 1-superform $\alpha_1 - \beta$, we have

$$\nabla_{\xi_X} = \Theta(\xi_X) + \frac{i}{\hbar} \langle \xi_X \mid \alpha \rangle + \frac{i}{\hbar} f_X$$

$$= \frac{i}{\hbar} \left(\frac{\hbar}{i} \Theta(\xi_X) + f_X + \langle \xi_X \mid \alpha \rangle \right) = \frac{i}{\hbar} \widehat{X}.$$

This completes the proof of the theorem.

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ERRATA

Page	Line	Error	Should be corrected as
75	9	$\mu_t = (\mu_t, \ldots, \mu_t)$	$\mu_t = (\mu_t^1, \ldots, \mu_t^m)$
76	14	$m_t = \gamma_t - \int_0^t \pi(h_s)\, ds$	$m_t^i = \gamma_t^i - \int_0^t \pi(h_s^i)\, ds$ $(i = 1 \ldots, n)$
78	1	O-mean	0-mean