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ON THREE CONCEPTS OF QUASICONVEXITY IN VECTOR OPTIMIZATION*

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0. INTRODUCTION

In nonconvex optimization, quasiconvexity is one of the most important generalizations of convexity, for it possesses various properties which are very close to that of convexity. Up to now, quasiconvex functions with scalar values have been thoroughly studied and there are a large number of papers dealing with their continuity, differentiability and other aspects (see [1 — 3] and references therein). Recent, specialists of vector optimization have turned their attention to this class of functions, taking into account the presence of partial orders in the underlying space (see [4—5] and [7]). They have generalized the concept of quasiconvexity for vector functions and investigated vector optimization problems with objectives quasiconvex in the corresponding sense. The aim of the present paper is to give an overlook at three main generalizations of quasiconvexity. By giving several characterizations of these generalizations we show how they are linked with the usual quasiconvexity of the scalar case.

The paper is structured as follows. §1 is concerned with notations and definitions. §2 deals with characterizations of quasiconvex vector functions in terms of level sets. §3 is devoted to describing the relationship between quasiconvex vector functions and scalar functions. In doing this we point out how these generalizations are different from each other.

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1. NOTATIONS AND DEFINITIONS

Let R^n be the n-dimensional Euclidean space, which is partially ordered by a convex closed pointed cone C with a nonempty interior int$C$. We recall that for two points u and v from R^n, $u > v$ if $u - v \in \text{int}C$, $u \geqq v$ if $u - v \in C$ and $u \geqslant v$ if $u - v \in C \setminus \{0\}$. Further, given a nonempty set $A \subseteq R^n$, a point $a_0 \in A$ is said to be an efficient or minimal point of A (with respect to C) if there is no $a \in A$ satisfying the relation $a_0 \geqslant a$. We shall denote the set of efficient points of A by $E(A \mid C)$ or simply by $E(A)$ if it is clear which cone is concerned.

Now let X be a nonempty convex subset of a topological vector space and let f be a function from X into R^n.

DEFINITION 1.1 (see [5] *f is said to be Q1-quasiconvex on X if for every x and y from X and for every a from R^n with $a \geqq f(x)$ and $a \geqq f(y)$ we have*

$$a \geqq f(\lambda x + (1 - \lambda)y), \text{ for every } \lambda \in [0,1].$$

DEFINITION 1.2 (see [4]) *Given a point $x \in X$, f is said to be Q2-quasiconvex at x if the condition $f(x) \geqq f(y)$, $y \in X$, implies $f(x) \geqq f(\lambda x + (1-\lambda)y)$ for every $\lambda \in [0,1]$; and f is said to be Q3-quasiconvex at x if the condition $f(x) \geqq f(y)$, $y \in X \setminus \{x\}$, implies the existence of a point $z \in X \setminus \{x\}$ such that $f(x) \geqq f(\lambda x + (1 - \lambda)z)$, for every $\lambda \in [0,1]$. When f is Q2-quasiconvex* (resp., Q3-quasiconvex) at *every point of X, we say that it is Q2-quasiconvex* (resp., Q3-quasiconvex) *on X.*

It is easy to see that every Qi-quasiconvex function is $Q(i + 1)$-quasiconvex but not vice versa. However, if $n = 1$, i. e. if f is scalar-valued, then $Q1$-and $Q2$-quasiconvexities coincide and they are the same as quasiconvexity or quasiconcavity in the usual sense, according to whether the cone C is the set of nonnegative numbers or nonpositive ones. Further, in Definition 1.1, it is sufficient to check the required relation for every point a from the set $E((f(x) + C) \cap (f(y) + C))$, which is reduced to a single point whenever the cone C is a polyhedral cone generated by n independent vectors. The latter fact is none than a sufficient and necessary criterium for a space ordered by a cone to be a lattice.

2. LEVEL SETS

In mathematical analysis if a function possesses certain property then this property is often attached to the epigraph while any quasi-property is derived from the corresponding property of the level sets. For instance, a convex function is a function with a convex epigraph, while a function with convex level sets is merely called quasiconvex. A similar terminology is used for closed functions. In this section we will see how this general rule works for the quasiconvexities defined in the previous section.

We recall that the level set of a function f from X into R^n at a point $x \in X$ is the set $L(x) = \{y \in X : f(x) \geqq f(y)\}$ and a set $A \subseteq R^n$ is said to be starshaped with a center at $a \in A$ if it contains all closed intervals linking its points with the point a.

PROPOSITION 2.1. *The following assertions hold:*

i) *f is Q1-quasiconvex if and only if its level set at any point is convex;*

ii) *f is Q2-quasiconvex at a point $x \in X$ if and only if $L(x)$ is starshaped with a center at x.*

iii) *f is Q3-quasiconvex at a point $x \in X$ if and only if $L(x)$ is a single point or it contains an interval with one end at x.*

Proof. This is immediate from the definitions given in the previous section. The last two assertions were formulated in [4].

It is clear that among three generalizations only $Q1$-quasiconvexity follows the rule stated above. In fact, it turns out that this generalization is the closest one in the sense, as it will be established later, that whenever C is the nonnegative orthant of the space, every component function f_i of $f = (f_1, \ldots, f_n)$ is quasiconvex in the usual sense provided f is $Q1$-quasiconvex. This is why the efficient point sets of vector problems with $Q1$-quasiconvex objectives have very nice properties (see [5]) and therefore we will pay more attention to this concept in the next section. The concept of $Q3$-quasiconvexity was introduced in connection with a local and global property of optimal solutions. It is known that any local solution of a quasiconvex optimization problem is also a global solution. This fact still holds for vector optimization problems with $Q3$-quasiconvex objectives. Moreover, the following result was proved in [4]. Let x_0 be a local minimal solution of the vector problem, denoted by (P),

$$\min f(x),$$
$$\text{s.t. } x \in X,$$

i. e. there is a neighborhood U of x_0 in X such that $f(x_0) \in E(f(U))$. Then x_0 is a global minimal solution of (P), i. e. $f(x_0) \in E(f(X))$ if and only if f is $Q3$-quasiconvex at x_0.

In [4] it was constructed a $Q3$ – quasiconvex function which is not $Q2$ — quasiconvex. By using the result of Proposition 2. 1 one can construct without any difficulty $Q2$—quasiconvex functions which are not $Q1$ — quasiconvex. The only thing to remark here is that for these functions the dimension of X and of R^n must be more than one.

3. CONNECTION WITH SCALAR QUASICONVEXITY

In this section we give some characterizations of $Q1$—and $Q2$—quasiconvexities in terms of scalar quasiconvex functions.

Let e be a fixed vector from int C. For every $a \in R^n$ we define a function $g(., a)$ from R^n into R by the relation:

$$g(x, a) = \min \{t : x \leqq a + te\}, \text{ for every } x \in R^n.$$

It is clear that the function g is well defined, i. e. $g(x, a)$ is finite for every $x \in R^n$, and it is continuous in both variables x and a.

PROPOSITION 3. 1. *A function f from X in to R^n is $Q2$—quasiconvex at $x_o \in X$ if and only if the composition function $g(f(.), f(x_o))$ is $Q2$—quasiconvex at that point.*

Proof. By the definition of the function g, it is easy to see that for $x \in X$, $f(x)_o) \geqq f(x)$ if and only if $g(f(x_o), f(x_o)) \geqq g(f(x), f(x_o))$. Now if $f(x_o) \geqq$ $\geqq f(\lambda x_o + (1-\lambda)x)$ then $g(f(x_o), f(x_o)) \geqq g(f(\lambda x_o + (1-\lambda)x), f(x_o))$, $\lambda \in [0,1]$ and vice versa. The proposition is proved.

PROPOSITION 3. 2. *f is $Q1$-quasiconvex on X if and only if $g(f(.), a)$ is quasiconvex (in the usual sense) for every $a \in R^n$.*

Proof. Suppose that f is not $Q1$-quasiconvex, i. e. there exist some $a \in X$, $\lambda \in (0,1)$ and $a \in R^n$ such that

$$a \geqq f(x) \text{ and } a \geqq f(y), \text{ but } f(\lambda x + (1-\lambda)y) \notin a - C. \tag{3. 1}$$

Consider the function $g(f(.), a)$. It is obvious that $g(a, a) = 0$, $0 \geqq g(f(x), a)$ and $0 \geqq g(f(y), a)$. Despite this, by (3. 1), $g(f(\lambda x + (1-\lambda)y, a) > 0$. Hence $g(f(.), a)$ is not quasiconvex.

Conversely, suppose that $g(f(.), a)$ is not quasiconvex for some $a \in R^n$, i.e. there exist $x, y \in X$ and $\lambda \in (0, 1)$ such that

$$g(f(\lambda x + (1-\lambda)y, a) > \max \{g(f(x), a); g(f(y), a)\}. \tag{3. 2}$$

Assume that $t = g(f(x), a) \geqq g(f(y), a)$. Then by (3. 2), we have

$$f(\lambda x + (1-\lambda)y) \notin a + te - C, \tag{3. 3}$$

while $f(x)$ and $f(y)$ belong to $a + te - C$. The latter fact shows that the vector $a + te$ is greater than or equal to $f(x)$ and to $f(y)$. Because of this and (3. 3), the function f cannot be $Q1$-quasiconvex on X. The proof is complete.

Before proceeding to further results we recall that the positive polar cone of C is the cone $C^* = \{p \in R^n : \langle p, c\rangle \geqq 0 \text{ for every } c \in C\}$. We will write $pf(.)$ instead of $\langle p, f(.)\rangle$ for $p \in C^*$ and pc instead of $\langle p, c\rangle$ if no confusion is likely to occur.

PROPOSITION 3. 3. *If $pf(.)$ is quasiconvex for every extremal vector p of C^*, then f is Q1-quasiconvex.*

Proof. We recall that p is an extremal vector of C^* if there are no two linearly independent vectors ξ and ζ of C^* with $p = \xi + \zeta$. Suppose that f is not Q1-quasiconvex, then there exist some $x, y \in X$, $a \in R^n$ and $\lambda \in (0,1)$ such that

$$a \geqq f(x),\ a \geqq f(g) \text{ and } f(\lambda x + (1-\lambda)y) \notin a - C. \tag{3.4}$$

Since C is convex closed, there is an extremal vector $p \in C^*$ such that

$$pf(\lambda x + (1-\lambda)y) > pa. \tag{3.5}$$

Indeed, if that is not the case, then by virtue of Theorem 18.5 of [6] we have $pa \geqq pf(\lambda x + (1-\lambda)y)$ for all $p \in C^*$, consequently $f(\lambda x + (1-\lambda)y) \in a - C$, contradicting relation (3.4). Further, it follows from (3.4) that $pa \geq \max\{pf(x), pf(y)\}$. This fact and (3.5) show that pf is not quasiconvex, completing the proof.

Remark that the inverse assertion of Proposition 3.3 is not always true, i.e. it is not necessary for the function pf to be quasiconvex whenever p is an extremal vector of C^* and f is Q1-quasiconvex.

PROPOSITION 3. 4. *Assume that C is a polyhedral cone generated by n independent vectors. Then f is Q1-quasiconvex if and only if pf is quasiconvex for every extremal vector p of C^*.*

Proof. By Proposition 3.3, it suffices to prove that the Q1-quasiconvexity of f implies the quasiconvexity of pf if p is an extremal vector of C^*. First we note that if $a_1,\dots a_n$ generate C, then the nonzero vectors $b_1,\dots, b_n$ defined by the relation

$$b_i a_j = 0,\ i \neq j,\ b_i a_i = 1,\ i = 1,\dots, n, \tag{3.6}$$

generate C^*.

Now suppose that bf is not quasiconvex for some $b \in \{b_1,\dots, b_n\}$. say $b = b_1$. Then there exist some $x, y \in X, \lambda \in (0, 1)$ such that $bf(\lambda x + (1-\lambda)y) > \max\{bf(x), bf(y)\}$. This means that b strictly separates $\{f(\lambda x + (1-\lambda)y)\}$ and $\{f(x), f(y)\}$. Assume that $bf(x) \geqq bf(y)$. Consider the hyperplane H generated by b and passing through $f(x)$. By (3.6), we have that $H = f(x) + \text{lin}\,(a_2,\dots,a_n)$ where lin(...) denotes the linear subspace stretched on $a_2,\dots, a_n$. Consequently,

$$\{f(y) + C\} \cap H = \{f(y) + \text{lin}\,(a_1)\} \cap H + \text{cone}\,(a_2,\dots, a_n), \tag{3.7}$$

where cone(...) denotes the cone generated by the vectors $a_2,\dots, a_n$. In fact, let $a \in \{f(y) + C\} \cap H$. Then $a = f(y) + \sum_{j=1}^{n} \alpha_j a_j$, $\alpha_j \geqslant 0$. Hence $a = f(y) + \alpha_1 a_1 + \sum_{j=2}^{n} \alpha_j a_j$. But $a \in H$, and so $f(y) + \alpha_1 a_1 \in f(x) + \text{lin}\,(a_2,\dots, a_n) - \sum_{j=2}^{n} \alpha_j a_j = f(x) + \text{lin}(a_2,\dots, a_n)$. In this way, $a \in \{f(y) + \text{lin}\ a_1\} \cap H + \text{cone}\,\{a_2,\dots, a_n\}$.

Conversely, $a \in \{f(y) + lin\, a_1\} \cap H + \text{cone}\,\{a_2, \ldots, a_n\}$ implies that $a = f(y) + \alpha_1 a_1 + C$, where $\alpha_1 \in R^1$, $C \in \text{cone}\,\{a_2, \ldots, a_n\}$, and $f(y) + \alpha_1 a_1 \in H$. It is clear that $a \in H$, To prove $a \in f(y) + C$, it suffices to show that $\alpha_1 \geqslant 0$. But this is trivial because if not, $bf(y) > bf(x)$. Let us consider the $(n-1)$-space $H - f(x)$ with the ordering cone $C_0 = \text{cone}\,(a_2, \ldots, a_n)$. Then the set $E(C_0 \cap (\{f(y) + \text{lin}\,(a_1)\} \cap H - f(x) + C_0) \mid C_0)$ consists of a single point, say c_0. We prove the following properties of the point $c := c_0 + f(x)$

i) $c \in (f(x) + C) \cap (f(y) + C)$

ii) $c \in H$.

In fact, it is clear that $c \in f(x) + C$ because $C_0 \subseteq C$. Moreover, it follows from (3.7) and the definition of C_0 that $c \in (f(y) + C) \cap H$. This shows both i) and ii). With i) and ii) one sees that $c \geqslant f(x)$ and $c \geqslant f(y)$, while $f(\lambda x + (1-\lambda)\, y) \notin H - C$. In this way, f is not $Q1$ — quasiconvex. The proof is complete.

COROLLARY 3.1. *Assume that C is the nonnegative orthant of the space. Then $f = (f_1, \ldots, f_n)$ is $Q1$ — quasiconvex if and only if $f_1, \ldots, f_n$ are quasiconvex as scalar functions.*

Proof. This follows from Proposition 3.4 and the fact that the polar cone of the nonnegative orthant is itself.

Conclusion. We have considered three generalizations of quasiconvexity and their connection with scalar quasiconvex functions. As in the scalar case, similar results concerning continuity and differentiability can be obtained for $Q1$ — and $Q2$ — quasiconvex vector functions (the reader who is interested in this can verify himself by using the technique of papers [2] and [3]). The concept of $Q3$ — quasiconvexity has no other special characterizations except for that presented in [4] about the local and global property of optimal solutions.

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