A NOTE ON THE FIXED-POINT SET FOR MULTIVALUED MAPPINGS

NGUYEN HUY VIET

The aim of this note is to extend the main results of [1] on characterization of the fixed-point set in terms of the maximal total-variant subsets to the case of multi-valued mappings. Also, the invalidity of Theorem 4 of [1] is shown.

1. NOTATION AND DEFINITIONS. Let X be a set $2^X = \{B/B \subset X\}$, $Y \subset X$ a subset of X and $F : X \to 2^X$ a multivalued mapping. Let $F_{ix}F = \{x \in X/x \in F(x)\}$ be the set of fixed points of F, $F(Y) = \cup \{F(y)/y \in Y\}$,

$$F^-(Y) = \{x \in X/F(x) \cap Y \neq \phi\},$$

$$F^+(Y) = \{x \in X/F(x) \subset Y\} \qquad \text{and} \qquad \overline{Y} = X \setminus Y.$$

A subset Y of X is called total F — variant if $Y \cap F(Y) = \phi$. If $x \in F(x)$ for all $x \in X$, it is evident that there exist no total F — variant subsets. But this case is trivial and so up to Theorem 3 it will be assumed that there is some F — variant point. Our assumption assures that

$$\mathcal{A} = \{Y \subset X/Y \cap F(Y) = \phi\} \neq \phi.$$

$F : X \to 2^X$ is called injective if $F(x) \cap F(y) = \phi$ for all $x \neq y$.

LEMMA. *If $F : X \to 2^X$ is a multi-valued mapping then $\mathcal{A}$ has maximal elements (with respect to set inclusion).*

Proof. If $\mathfrak{B} \subset \mathcal{A}$ is a linear ordered subset of $\mathcal{A}$, i. e., for all B_1 and $B_2 \in \mathfrak{B}$ we have $B_1 \supset B_2$ or $B_2 \supset B_1$. Then, it is evident that the set $\cup \{B, B \in \mathfrak{B}\}$ belongs to $\mathcal{A}$ and contains every $B \in \mathfrak{B}$. Further, the conclusion of the lemma follows from the principle of maximal elements.

2. MAIN RESULTS.

THEOREM 1. *If $F : X \to 2^X$ is a multivalued mapping and $Y \subset X$ a maximal total F-variant subset of X. Then*

$$\overline{Y} \cap \overline{F(Y)} \cap \overline{F^-(Y)} \subseteq F_{ix}F. \tag{1}$$

Proof. Let $x \in \overline{Y} \cap \overline{F(Y)} \cap \overline{F^-(Y)}$. Because $x \in \overline{Y}$, the maximality of Y implies that $F(Y \cup \{x\}) \cap (Y \cup \{x\}) \neq \phi$ and then $((F(Y) \cap Y) \cup (Y \cap F(x)) \cup (F(Y) \cap \{x\}) \cup (\{x\} \cap F(x)) \neq \phi$. But $F(Y) \cap Y = \phi$, $x \in \overline{F(Y)}$, $x \in \overline{F^-(Y)}$. So $\{x\} \cap F(x) \neq \phi$, i. e., $x \in F_{ix}F$.

Using Theorem 1 and de Morgan's laws we obtain the following factorization theorem.

THEOREM 2. *Let* $F : X \to 2^X$ *be a mapping and* $Y \subset X$ *a total F-variant maximal subset of X. Then*

$$X = F_{ix}F \cup Y \cup F(Y) \cup F^{-}(Y).$$

THEOREM 3. *Let* $F ; X \to 2^X$ *be an injective multi-valued mapping and* $Y \subset X$ *a maximal total F-variant subset of X. Then*

$$\overline{Y} \cap \overline{F(Y)} \cap \overline{F^{-}(Y)} \subseteq F_{ix}F \subseteq \overline{Y} \cap \overline{F(Y)} \cap \overline{F^{+}(Y)} \qquad (2)$$

Proof. In view of (1) it suffices to prove the last inclusion. Let $x \in F_{ix}F$, i. e. $x \in F(x)$. It is clear that $x \in \overline{Y}$. If $x \in F(Y)$ then $x \in F(y)$ for some $y \in \overline{Y}$, and we have $F(y) \cap F(x) \neq \phi$. By injectivity of F, $x = y \in Y$, which contradicts the fact that $x \in \overline{Y}$ Thus, $x \in \overline{F(Y)}$. If $x \in F^{+}(Y)$, then $F(x) \subset Y$ and $x \in Y$, a contradiction. Thus $x \in \overline{F^{+}(Y)}$ as required.

THEOREM 4. *Let X be a Hauskorff topological space,* $F : X \to 2^X$ *an upper semi-continuous mapping with non-empty compact values, Y a compact maximal total F-variant subset. Then Y is open.*

Proof. Let $y \in Y$. It is clear that $F(y) \cap Y = \phi$. Because $F(y)$ and Y are both compact, we can construct an open neighbourhood V_1 of y and an open G such that $G \cap V_1 = \phi$, $G \supset F(y)$ and $G \cap Y = \phi$. By the upper semicontinuity of F, there exists an open neighbourhood V_2 of y such that $F(V_2) \subset G$. Because Y is compact and F is upper semi-continuous with compact values, it is well known that $F(Y)$ is compact [2]. Then there exists a neighbourhood V_3 of y such that $V_3 \cap F(Y) = \phi$.

Setting $V = V_1 \cap V_2 \cap V_3$, we have $F(V) \subset G, F(V) \cap Y = \phi, F(V) \cap V = \phi$ and $V \cap F(Y) = \phi$. Therefore $Y \cup V$ is a total F — variant subset. The maximality of Y implies that $V \subset Y$. Thus Y is open.

THEOREM 5. *Let X be a connected compact Hausdorff topological space,* $F : X \to 2^X$ *an upper semi-continuous mapping with non-empty closed values. Let Y be a closed total maximal F-variant subset of X. Then* $F_{ix}F = X$.

Proof, By Theorem 4, Y is open. By the connectedness of X, the only simultaneously open and closed subsets are ϕ and X. By the definition of total variant subsets, $Y \neq X$.

Thus, $Y = \phi$, i.e. $x \in F(x)$ for every $x \in X$.

Remark. Theorem 5 shows that there is no mapping (except for the trivial case $f = I_X$) which satisfies Theorem 4 of [1].

REFERENCES

[1] M. Deaconescu, *The fixed-point set for injective mappings*, Studia Univ. Babes-Bolyai Mathematica XXIX, (1984), 13 – 15.

[2] C. Berge, *Espaces topologiques*. Dunod Paris, 1966.

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INSTITUTE OF MATHEMATICS, P. O. BOX 631 BO HO 10000 HANOI, VIETNAM