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DIRECT LIMITS WHICH ARE HILBERT SPACES

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1. INTRODUCTION

For a metric space (X, ρ) let 2^X denote the hyperspace of all nonempty compact subsets of X equipped with the Hausdorff metric

$$d(A, B) = \max \{ \max_{a \in A} \min_{b \in B} \rho (a, b), \max_{b \in B} \min_{a \in A} \rho (a, b)\}.$$

We denote $\exp (X) = 2^X$ and $\exp^n (X) = \exp (\exp^{n-1} (X))$,

$$X' = \lim_{n \to \infty} \exp^n (X),$$

$$X^* = \text{the completion of } X'.$$

In [8] Torunczyk and West proved that if X is a Peano continuum then

$(X^*, X') \cong (l_2, l_2^\sigma)$ where l_2 denotes the Hilbert space of all square summable sequences of real numbers and

$$l_2^\sigma = \left\{ x = (x_n) \in l_2 : \sum_{n=1}^{\infty} (n x_n)^2 < \infty \right\}.$$

Here we write $(X^*, X') \cong (l_2, l_2^\sigma)$ iff there exists a homeomorphism f from X^* onto l_2 such that $f(X') = l_2^\sigma$.

In this note we establish similar results for compact convex sets in normed spaces.

2. MAIN RESULT

By $cc(X)$ we denote the hyperspace of all nonempty compact convex subsets of a compact convex set X lying in a normed space.

A family $F \subset cc(X)$ is convex iff for $A, B \in E$ and $\lambda \in [0, 1]$ we have $\lambda A + (1 - \lambda) B \in F$.

Let $cc^2(X)$ denote the hyperspace of all compact convex families in $cc(X)$. Inductively we put

$$cc^n(X) = cc(cc^{n-1}(X)) \text{ and}$$

$$cc'(X) = \lim_{n\to\infty} cc^n(X),$$

$$cc^*(X) = \text{completion of } cc'(X).$$

Our main result is the following

THEOREM. *Let X be a compact convex set in a normed space.*
(i) If $\dim X = 1$ then $(cc^*(X), cc'(X)) \cong (l_2, l_2^f)$;

(ii) If $\dim X \geq 2$ then $(cc^*(X), cc'(X)) \cong (l_2, l_2^\sigma)$.

Here $l_2^f = \{x = (x_n) \in l_2 : x_n = 0$ for almost all $n\}$.

3. PROOF

The proof of Theorem 1 is based on the following facts

LEMMA 1. (i) *$cc^n(X)$ is an AR-space for every $n \in N$;*
(ii) *$cc'(X)$ and $cc^*(X)$ are AR-spaces.*

Proof. We shall prove Lemma 1 for $cc'(X)$. The proofs for the other cases are the same.

We shall prove that for every simplicial complex K and for every map f_0 from the set K_0 of all vertices of K into $cc'(X)$ there is an extension $f : K \to cc'(X)$ such that $\operatorname{diam} f(\sigma) = \operatorname{diam} f_0(\sigma^0)$ for every $\sigma \in K$. Whence according to [4], $cc'(X) \in AR$.

Let $\sigma = \langle V_1, \dots, V_k \rangle \in K$. Then for every $x \in \sigma$ we have

$$x = \sum_{i=1}^{k} \alpha_i V_i, \; V_i \in \sigma^0, \; i = 1, \dots, k, \; \alpha_i \geq 0 \text{ and}$$

$\sum_{i=1}^{k} \alpha_i = 1$. We define $f(x)$ by

$$f(x) = \sum_{i=1}^{k} \alpha_i f_0(V_i).$$

It is easy to see that f satisfies the required conditions.
Lemma 1 is proved.

LEMMA 2. *If* $\dim X \geqslant 2$ *then* $cc^n(X)$ *is homeomorphic to the Hilbert cube* Q *for every* $n \in N$.

Proof. By Lemma 1 $cc^n(X) \in AR$. Therefore according to [6] it suffices to show that given $\varepsilon > 0$ and maps

$$f_1, f_2 : I^k \to cc^n(X),\ k = 1, 2, \ldots$$

there exist maps $g_1, g_2 : I^k \to cc^n(X)$ such that $d(f_i(x), g_i(x)) < \varepsilon$ for every $x \in I^k$, $i = 1, 2$ and $g_1(I^k) \cap g_2(I^k) = \phi$.

Let $f_1, f_2 : I^k \to cc^n(X)$ and $\varepsilon > 0$ be given. Take a triangulation K of Ik such that $\operatorname{diam} f_i(\sigma) < \frac{1}{2}\varepsilon$ for every $i = 1, 2$ and $\sigma \in k$. Let $\{a_1, \ldots, a_m\}$ denote the set of all vertices of K. Select families $\{p^1_1, \ldots, p^1_m\}$ and $\{p^2_1, \ldots, p^2_m\}$ of convex polyhedra in $cc^n(X)$ such that $d(p^i_k, f_i(a_k)) < \frac{\varepsilon}{2}$ for $i = 1, 2$, and $k = 1, \ldots, m$ and $\left(\max\left\{V(p^1_k) : k = 1, \ldots, m\right\}\right)^m < \min\left\{V(p^2_k) : k = 1, \ldots, m\right\}$, where $V(p)$ denotes the number of all vertices of a polyhedron p.

Define $g_1, g_2 : I^k \to cc^n(X)$ by the formula

$$g_i(x) = \sum_{k=1}^{q} \alpha_k p^i_k \text{ for } x = \sum_{k=1}^{q} \alpha_k a_k \in \sigma \in k, i = 1, 2.$$

It is easy to see that g_1, g_2 are the desired maps. Therefore the lemma is proved.

LEMMA 3 [1]. *Let* A *be a proper closed subset of a metric space* (X, d). *Then there exists an indexed family* $\{U_i, c_i\}$ $i \in I$ *called a Dugundji system for* $X \setminus A$, *such that*

(i) $U_i \subseteq X \setminus A$ *and* $c_i \in A$ *for each* $i \in J$;

(ii) $\mathcal{U} = \{U_i\}_{i \in I}$ *is a locally finite open cover of* $X \setminus A$;

(iii) *If* $x \in U_i$ *then* $d(x, c_i) \leqslant 2d(x, A)$ *for each* $i \in J$.

Let $\{\alpha_i\}_{i \in J}$ *be a locally finite partition inscribed into* $\{U_i\}_{i \in J}$, *We also say that* $\{\alpha_i, c_i\}_{i \in J}$ *is a Dugundji system for* $X \setminus A$.

Denote $G_n = \{(A, t) \in cc^n(X) \times [0, \infty) : A \in t\, cc^n(X)\}$;

$G = \{(A, t) \in cc'(X) \times [0, \infty) : A \in t\, cc'(X)\}$;

$G^* =$ *the completion of* G.

LEMMA 4. *For every compact set* $K \subset G^*$, *for every* $n \in N$ *and for every* $\varepsilon > 0$ *there is a map* $f : K \to G_m$ *for some* $m > n$ *such that* $f|_{K \cap G_n} = Id$ *and* $d(x, f(x)) < \varepsilon$ *for every* $x \in K$.

Proof. Take $m > n$ such that G_m is an $\frac{\varepsilon}{2}$-net for K. Let $\{\alpha_i, c_i\}$ $i \in I$ be a Dugundji system for $K \setminus G_m$. Define $f: K \to G_m$ by the formula

$$f(x) = \begin{cases} x & \text{if } x \in K \cap G_m \\ \sum_{i \in I} \alpha_i(x) c_i & \text{if } x \in K \setminus G_m \end{cases}.$$

An easy computation shows that

$$d(x, f(x)) \leqslant 2d(x, G_m) \text{ for every } x \in K.$$

Therefore f is continuous. Since G_m is an $\frac{\varepsilon}{2}$-net for K we infer that $d(x, f(x)) < \varepsilon$ for every $x \in K$. Obviously $f|_{K \cap G_n} = \mathrm{id}$. Thus the lemma is proved.

LEMMA 5. *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every compact set $K \subset G_n$, there exists an ε-homotopy $h_t: K \to G_m$, for some $m > n$, $0 \leqslant t \leqslant 1$, such that $h_0 = Id_k$ and $d(K, h_1(K)) \geqslant \delta$.*

Proof. Without loss of generality we may assume that $0 \in X$ and there exists $a \in X$ such that $\| a \| = 1$. Denote $a_1 = [0, a]$, $a_2 = [0, a_1], \ldots, a_k = [0, a_{k-1}], \ldots$ Obviously $a_k \in cc^k(X)$ for every $k \in N$. Therefore $(ta_k, t) \in G_k$ for every $k \in N$ and $t \in [0, \infty)$.

Given $\varepsilon > 0$ and $n \in N$, we take $m > n$ and $b = \left(\frac{1}{4}\varepsilon a_m, \frac{1}{4}\varepsilon\right) \in G_m$. Define $h_t: K \to G_m$ by the formula

$$h_t(x) = x + tb \text{ for every } x \in K.$$

It is easily seen that h_t is an ε-homotopy, $h_0 = Id_k$ and $d(K, h_1(k)) \geqslant \frac{1}{4}\varepsilon$. This proves the lemma.

DEFINITION 1 [7]. A metric space X is said to have strong discrete approximation property iff for each map $f: \bigoplus_{n=1}^{\infty} I^n \to X$ and for each map

$\alpha: X \to (0, \infty)$ there is a map

$g: \bigoplus_{n=1}^{\infty} I^n \to X$ such that $d(f(x), g(x)) < \alpha f(x)$

for every $x \in \bigoplus_{n=1}^{\infty} I^n$ and $\{g(I^n)\}_{n=1}^{\infty}$ is a discrete family in X.

Here we say that a family $\{A_i\}_{i \in I}$ is a discrete family in a metric space X iff each point of X has a neighbourhood which interests with at most one member of $\{A_i\}_{i \in I}$.

THEOREM 2 [7]. *Let X be a separable complete AR-space. Then X is homeomorphic to l_2 if and only if X has the strong discrete approximation property.*

LEMMA 6 [5], (see also [2]. *Let X be a locally path connected metric space with a tower $X_1 \subset X_2 \subset \dots \subset X$ satisfying the following conditions*

(i) *For every compact set $K \subset X$, for every $n \in N$ and for every $\varepsilon > 0$ there is a map $f: K \to X_m$ for some $m > n$ such that $f|_{K \cap X_n} = id$ and $d(x, f(x)) < \varepsilon$ for every $x \in K$,*

(ii) *For every $\varepsilon > 0$ there exists a $\delta > 0$, such that for every compact set $K \subset X_n$ there is an ε-homotopy $h_t: K \to X_m$ for some $m > n$, $0 \leqslant t < 1$, such that $h_0 = id_k$ and $d(K, h_1(k)) \geqslant \delta$.*

Then X has the strong discrete approximation property.

From Lemmas 4, 5, 6 and from Theorem 2 we get

COROLLARY 1. $G^* \cong l_2$.

From Corollary 1 we obtain

COROLLARY 2. $cc^*(X) \cong l_2$.

Proof. By Corollary 1 $G^* \cong l_2$. Therefore $G^* \setminus \{(\theta,0)\} \cong l_2$. Obviously $G^* \setminus \{(\theta, 0)\} \cong cc^*(X) \times (0, \infty)$. Therefore by a result of Mogilski [3] $cc^*(X) \cong l_2$, the corollary is proved.

THEOREM 3 [1]. *Let X be a metric space homeomorphic to l_2 and let $\{X_n\}$ be an increasing sequence of compact subsets of X (respectively, of finite dimensional compact subsets of X), and $X' = \bigcup_{n=1}^{\infty} X_n$. Then $(X, X') \cong (l_2, l_2^{\sigma})$ (respectively, $(X, X') \cong (l_2, l_2^{f})$) if and only if the following condition holds:* (SK) *For each compact set (respectively, for each finite dimensional compact set) $K \subset X$, for each $\varepsilon > 0$ and for each $n \in N$ there is an embedding $f: K \to X_m$ for some $m > n$ such that $f|_{X_n \cap K} = id$ and $d(x, f(x)) < \varepsilon$ for every $x \in K$.*

LEMMA 7. *For every finite dimensional compact set $K \subset cc^*(X)$, for every $\varepsilon > 0$ and for every $n \in N$ there is an embedding $f: K \to cc^m(X)$ for some $m > n$ such that $f|_{A \cap cc^n(X)} = id$ and $d(x, f(x)) < \varepsilon$ for every $x \in K$.*

Proof. By Lemma 4 there is a map $g: K \to cc^k(X)$ for some $k > n$ such that $d(x, g(x)) < \frac{1}{2}\varepsilon$ for every $x \in K$ and $g|_{K \cap cc^n(X)} = \mathrm{id}$. Since K is finite dimensional there is an embedding $\varphi : K \to I^q$ for some $q \in N$.

Let φ_i, $i = 1, 2, \dots, q$ denote the i's coordinate function of φ. We may assume $0 \in X$. Let $a \in X$ such that $a \neq 0$. Denote $a_1 = [0, a]$, $a_2 = [0, a_1], \dots, a_n = [0, a_{m-1}], \dots$ Let $m = k + q + 1$ and define $f : K \to cc^m(X)$ by the formula

$$f(x) = (1 - \delta d(x, B))g(x) + \frac{\delta d(x, B)}{q+1}(\varphi_1(x)a_{k+1} + \dots + \varphi_k(x)a_{k+q} + a_{k+q+1}),$$

where $B = K \cap cc^n(X)$ and δ is chosen so small that $d(g(x), f(x)) < \frac{1}{2}\varepsilon$ for every $x \in K$. Obviously f is one-to-one. Since K is compact, f is an embedding. The lemma is proved.

LEMMA 8. *$cc^n(X)$ is a Z-set in $cc^m(X)$ for every $m > n$.*

Here we say that a closed subset A of metric space X is a Z-set iff given a map $f : Q \to X$ and $\varepsilon > 0$ there is a map $g : Q \to X \setminus A$ such that $d(f(x), g(x) < \varepsilon$ for every $x \in Q$.

Proof of Lemma 8. Given $f : Q \to cc^m(X)$ and $\varepsilon > 0$. Take $y_0 \in cc^m(X) \setminus cc^n(X)$ and define $g : Q \to cc^m(X)$ by the formula

$$g(x) = (1 - \delta)f(x) + \delta y_0 \text{ for every } x \in Q,$$

where $\delta > 0$ is chosen so small that $d(f(x), g(x)) < \varepsilon$ for every $x \in Q$. Obviously $g(Q) \cap cc^n(X) = \phi$ and hence the lemma is proved.

LEMMA 9. *For each compact set $K \subset cc^*(X)$, for each $\varepsilon > 0$ and for each $n \in N$ there is an embedding $f : K \to cc^m(X)$ for some $m > n$ such that $f|_{K \cap cc^n(X)} = id_k$ and $d(x, f(x)) < \varepsilon$ for each $x \in K$.*

Proof. By Lemma 4 there is a map $g: K \to cc^p(X)$ for some $p > n$ such that $d(x, g(x)) < \frac{1}{2}\varepsilon$ for every $x \in K$. By Lemma 2 $cc^k(X) \cong Q$ for every $k \in N$ By Lemma 8 $cc^k(X)$ is a Z-set in $cc^l(X)$ for every $k < l$. Therefore by [1] there is an embedding $f : K \to cc^m(X)$ for some $m > p$ such that $f|_{K \cap cc^n(X)} = g|_{K \cap cc^n(X)} = id_k$ and $d(f(x), g(x)) < \frac{1}{2}\varepsilon$ for every $x \in K$.

Now we are already in a position to prove our main result. By Corollary 2, $cc^*(X) \cong l_2$. It is easy to see that if $\dim X = 1$ then $\dim cc^n(X) < \infty$ for every $n \in N$. Therefore (i) follows from Theorem 3 and Lemma 7. Finally from Theorem 3 and Lemma 9 we get (ii) and thereby Theorem 1 is proved.

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