ACTA MATHEMATICA VIETNAMICA
Vol. 14 No.2 (1989)

STRONG CONVERGENCE OF TWO – PARAMETER VECTOR – VALUED MARTINGALES AND MARTINGALES IN THE LIMIT

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1. INTRODUCTION

Real—valued martingales indexed by $N^2 = N \times N$ were first introduced and considered by Cairoli [3], Cairoli and Walsh [2] and later developed by Chatterji [4, 5], Brossard [1], Ledoux [8, 9], Millet [9], Millet and Sucheston [10] and others.

The main convergence result of Cairoli [3] asserts that under a so-called condition (F_4), every LlogL—bounded real—valued martingale $(X_t, \mathcal{F}_t)$ converges almost surely, (a. s.). Recently, Talagrand [13] has introduced the class of discrete mils: a class strictly larger than martingales in the limit [Mucci (1976)], pramarts [Egghe (1981)] and amarts [Edgar and Sucheston (1977)] and proved that every L^1 — bounded mil taking values in a Banach space having the Radon—Nikodym property (RNP), converges a. s.

In the present paper, this notion of discrete mils is extended to the multi-parameter case. Our main result (Theorem 2) says that under condition (F_4) every LlogL—bounded two-parameter mil taking values in a Banach space with the (RNP) still converges a.s.

2. NOTATIONS AND DEFINITIONS

Throughout this paper let N be the set of all positive integers. We shall denote by I the set N^2 with the usual order given by $(s_1, s_2) \leqslant (t_1, t_2)$ if $s_1 \leqslant t_1$ and $s_2 \leqslant t_2$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $(\mathcal{F}_t)$ be a filtration indexed by I, i. e. an increasing family of complete subsigma—algebras of $\mathcal{F}$. For every $t = (t_1, t_2)$ set $\mathcal{F}^1_t = \bigvee_u \mathcal{F}_{t_1 u}$ and $\mathcal{F}^2_t = \bigvee_u \mathcal{F}_{u, t_2}$

ıd set $\mathcal{F}_\infty = \bigvee_{t \in I} \mathcal{F}_t$. The family $(\mathcal{F}_t)$ is said to fulfill condition (F_4) of airoli and Walsh [2] if for every bounded $\mathcal{F}$—measurable function $X : \Omega \to E$ ıd for all $t \in I$ we have

$$E(X/\mathcal{F}_t) = E(E(X/F^1_t)/F^2_t)).$$

B—valued (F_t) — adapted Bochner integrable process (X_t) is called a martingale (submartingale) if for all $s, t \in I$, $s \leqslant t$, we have $E(X_t/F_s) = X_s$ ($B = R$ and $E(X_t / F_s) \geqslant X_s$, respectively). (X_t) is LlogL—bounded if $\sup_{t \in I} E(\| X_t \| \log^+ \| X_t \|) < \infty$. In the sequel we assume ıat B is a Banach space with (RNP). We now introduce

DEFINITION. An adapted sequence (X_{ij}) of Banach-space-valued random ıriables is called a mil if for every $\varepsilon > 0$, there sxists $\bar{p} = (p, p)$, $p \in N$ such ıat for every $\bar{n} = (n, n) \geqslant \bar{p}$ we have

$$P\left(\sup_{p \leqslant i, j \leqslant n} \| X_{ij} - E(X_{\bar{n}} / \mathcal{F}_{ij}) \| \geqslant \varepsilon\right) \leqslant \varepsilon.$$

emark. a) If $(Y_n, \mathcal{F}_n)$ is a mil in the sense of Talagrand [13] then the sequence $(X_{ij}, \mathcal{F}_{ij})$ is also a mil, where $X_{ij} = Y_i$, $\mathcal{F}_{ij} = \mathcal{F}_i$, $(i, j) \in N^2$.

b) If M_{ij} is a martingale, it is also a mil. The methods of the theory of set ınction processes developed by Schmidt [11] allow us to strengthen Cairoli's esult in [3], (see also [6], Theorem 1). By using the maximal inequality for a ositive 1-submartingale [10] we prove that under condition (F_4), evry LlogL-ounded mil converges almost surely

4. MAIN RESULTS

Before proving the main theorem we sketch a short proof of the following esult which is a vector-valued version of the corresponding theorem of airoli [3].

HEOREM 1. *Let B be a real Banach space with (RNP) and let (X_{nm}) be a B-valued ıartingale. Suppose that (X_{mn}) satisfies Doob's condition*

$$\sup_{m,n} E \| X_{mn} \| < -\infty. \tag{1}$$

hen for each $m, n \in N$ there exist $X_{m\infty}, X_{\infty n} \in L^1_B(\mathcal{F})$ such that

$$\lim_m X_{mn} = X_{\infty n} \text{ a.s. for every } n \geqslant 1, \tag{2}$$

$$\lim_n X_{mn} = X_{m\infty} \text{ a.s. for every } m \geqslant 1 \tag{3}$$

and

$$\lim_n X_{\infty n} = \lim X_{m\infty} \qquad \text{a.s.} \tag{3}$$

Proof. Let B an (X_{mn}) be as in the theorem. Then by definition, for each m and n, $((X_{mk}, \mathcal{F}_{mk}))_{k\geqslant 1}$ and $((X_{kn}, \mathcal{F}_{kn}))_{k\geqslant 1}$ are one-parameter martingales satisfying Doob's condition. Hence it follows from Chatterji [4] (see also [11, Proposition V — 2 — 10]) that the assertions (2) and (3) are true. The main part of the proof consists in showing that (4) is also satisfied.

To do this, for each $(m, n) \in N^2$ we define $\mu_{mn} : F_{mn} \to B$ by

$$\mu_{mn}(A = E(I_A \cdot X_{mn}),\ A \in F_{mn}. \tag{5}$$

It follows from the integrability of X_{mn} that μ_{mn} is a B-valued measure with $\| \mu_{mn} \| = E \| X_{mn} \|$. It is also easy to check that the set function process $((\mu_{mn}, \mathcal{F}_{mn}))$ is a martingale and by (1)

$$\sup_{m,n} \| \mu_{mn} \| = \sup_{m,n} E \| X_{mn} \| \quad .$$

Next, for any $m,n \in N$ we define

$$\mathcal{F}'_{\infty n} = \bigcup_m F_{mn}\ F,'_{m\infty} = \bigcup_n F_{mn} \quad ,$$

$$F'_{\infty\infty} = \bigcup_{m,n} F_{mn} = \bigcup_m F'_{m\infty} = \bigcup_n F'_{\infty n} \quad .$$

It is clear that $F'_{\infty n}$, $F'_{m\infty}$, $F'_{\infty\infty}$ are algebras. Thus, the limit measures of the martingales $(\mu_{mn})_{m\geqslant 1}$, $(\mu_{mn})_{n\geqslant 1}$ and (μ_{mn}), resp., denoted by $\mu_{\infty n}$, $\mu_{m\infty}$ and $\mu_{\infty\infty}$, resp., are given by

$$\mu_{\infty n}(A) = \lim_m \mu_{mn}(A),\ A \in F_{\infty n}\ ; \tag{6}$$

$$\mu_m(A) = \lim_n \mu_{mn}(A),\ A \in F'_{m\infty}\ . \tag{7}$$

and

$$\mu_{\infty\infty}(A) = \lim_{m,n} \mu_{mn}(A),\ A \in F'_{\infty\infty}\ . \tag{8}$$

It is easily checked that $\mu_{\infty n}$, $\mu_{m\infty}$ and $\mu_{\infty\infty}$ are well-defined and finitely additive measures and the set function processes $((\mu_{m\infty}, F_{m\infty}))$ and $((\mu_{\infty n}) F_{\infty n}))$ are martingales satisfying

$$\lim_m \mu_{m\infty}(A) = \lim_n \mu_{\infty n}(A) = \mu_{\infty\infty}(A) \tag{9}$$

for each $A \in F'_{\infty\infty}$. Moreover, for each $m,n \in N$ the processes $(\mu_{mn})_{n\geqslant 1}$, $(\mu_{mn})_{m\geqslant 1}$, $(\mu_{m\infty})$ and $(\mu_{\infty n})$ are bounded. Hence, it follows from [12, Corollary 3.3.9] that

$$\lim_n D_{mn}\mu_{mn} = D_{m\infty}\mu_{m\infty} \text{ a.s. for each } m, \tag{10}$$

$$\lim_m D_{mn} \mu_{mn} = D_{\infty n} \mu_{\infty n} \quad \text{a.s. for each } n, \tag{11}$$

$$\lim_m D_{m\infty} \mu_{m\infty} = D_{\infty\infty} \mu_{\infty\infty} \quad \text{a.s.}, \tag{12}$$

and

$$\lim_n D_{\infty n} \mu_{\infty n} = D_{\infty\infty} \mu_{\infty\infty} \quad \text{a.s.} \tag{13}$$

'here $D_{mn} \mu_{mn}$ denotes the generalized Radon-Nikodym derivative of μ_{mn} ı.r.t. the probability measure P (see [12]).

Furthermore, it follows from (5) that

$$D_{mn} \mu_{mn} = X_{mn} \quad \text{a.s. for every } (m,n) \in N^2. \tag{14}$$

ionsequently, by (2), (3), (10), (11) and (14) we have

$$X_{m\infty} = D_{m\infty} \mu_{m\infty} \quad \text{a.s. for every } m \geqslant 1 \tag{15}$$

$$X_{\infty n} = D_{\infty n} \mu_{\infty n} \quad \text{a.s. for every } n \geqslant 1. \tag{16}$$

'inally, (4) follows from (12), (13), (15) and (16). The proof of the theorem s thus complete.

For a class of mils we have the following theorem.

THEOREM 2. *Let B have (RNP) and a B-valued mil w.r.t. (F_{ij}) which satis- fies the condition (F_4), Furthermore, assume the sequence $(X_{\overline{n}})$ is $L\log L$-bounded Then (X_{ij}) converges a.s.*

For the proof of this theorem, we need two lemmas.

LEMMA 1. *Let B have (RNP), let (F_{ij}) satisfy condition (F_4) and (X_{ij}) be B-valued $L\log L$-bounded martingale. Then (X_{ij}) converges a. s.*

Proof. For the real—valued case, this lemma was proved by Cairoli (1970). Chatterji (1975) and Millet—Sucheston (1981). It is worth noting that only Chatterji's proof has been extended to the B valued case. Here, for the sake of. completeness we present another proof of the lemma which is based on a result of Millet—Sucheston [10].

Indeed, let (X_t) be as in the lemma. It is easy to see that the martingale (X_t) is uniformly integrable and hence by (RNP) of B and Proposition V—2—10 [11, p. 112] it follows that there exists a B-valued integrable random element X such that (X_t) converges to X in L^1. Hence $X_t = E(X/F_t)$, $t \in N^2$ and $E \| X \| \log^+ \| X \| < \infty$.
To prove that $X_t \to X$ a.s. we first apply Lemma V—2—4 [11] to choose a sequence of simple elements (X^k) in L^1_B such that

$$\| X^k \| \leqslant \| X \|, \text{ a.s. for every } k = 1, 2..., \tag{17}$$

$$X^k \to X \text{ a.s.}$$

Set $X_t^k = E(X^k/F_t)$, $t \in N^2$.

By a result of Cairoli in [3], it follows that

$$X_t^k \xrightarrow{\text{a.s.}} X^k \text{ as } t \to \infty \tag{18}$$

for each $k = 1, 2, \ldots$

Further, applying Theorem 1.5 in [10] to the positive 1-submartingale $(\| X_t - X_t^k \|)_{t \in I}$, we obtain for every $\lambda > 0$ and every $\delta \geqslant 0$

$$p(\sup_t \| X_t - X_t^k \| \geqslant \lambda) \leqslant \frac{1}{\lambda} \frac{e}{e-1} [\delta + | \log\delta | E \| X - X^k \| + E \phi (\| X - X^k \|)] \tag{19}$$

where $\phi(x) = x \log^+ x$, $x > 0$.

Fix $\lambda > 0$, choose sequences (δ_n) and later (k_n) such that

$$\delta_n \downarrow 0,\ k_n \uparrow \infty \text{ as } n \to \infty \text{ with}$$

$$| \log \delta_n | E \| X - X^{k_n} \| \to 0 \text{ as } n \to \infty. \tag{20}$$

Hence, it follows from (19), (20) and $E\phi(\| X - X^{k_n} \|) \to 0$ that for every $\lambda > 0$

$$\lim_n p(\sup_t \| X_t - X_t^{k_n} \| \geqslant \lambda) = 0.$$

This means that the sequence $(\sup_t \| X_t - X_t^{k_n} \|)_{n \geqslant 1}$ converges in probability to zero and hence one can choose an increasing subsequence (P_n) of (k_n) such that

$$\sup_t \| X_t - X_t^{p_n} \| \to 0 \text{ a.s as } n \to \infty \tag{21}$$

Finally, (17), (18), (21) together with the inequality

$$\| X_t - X \| \leqslant \| X_t - X_t^{p_n} \| + \| X_t^{p_n} - X^{p_n} \| + \| X^{p_n} - X \|$$

yield that $X_t \xrightarrow{\text{a.s}} X$. This completes the proof.

LÉMMA 2. *Let T^1 be the set of all bounded $(\mathcal{F}^1_{ij})$-stopping times and let (p_{ij}) be a B-valued (F_{ij})-adapted sequence. If $p_\tau \xrightarrow[\tau \in T^1]{P} 0$, then (p_{ij}) converges* **a.s.**

Proof. Suppose that $p_\tau \xrightarrow[\tau \in T^1]{P} 0$, but (p_{ij}) does not converge a.s. to zero. Then, there exists $\varepsilon > 0$ such that for every $p \in N$, we can find $n \in N$, $n \geqslant p$ such that

$$P \left(\sup_{p \leqslant i,\ j \leqslant n} \| p_{ij} \| \geqslant \varepsilon \right) \geqslant \varepsilon. \tag{22}$$

Take any pair (p, n) with $p \leqslant n$ satisfying (22). We shall construct a stopping time $\tau = \tau_\varepsilon \in T^1$ such that $\bar{p} \leqslant \tau \leqslant \bar{n}$ and

$$P(\| p_\tau \| \geqslant \varepsilon) = P(\sup_{p \leqslant i,\, j \leqslant n} \| p_{ij} \| \geqslant \varepsilon) > \varepsilon. \tag{23}$$

To this end, first define $\tau_1 : \Omega \to \mathrm{N}$ by

$$\tau_1(\omega) = \begin{cases} \inf \{ i \in \{p, p+1, \dots, n\} : \sup\limits_{\substack{p \leqslant k \leqslant i \\ p \leqslant l \leqslant n}} \| p_{kl} \| \geqslant \varepsilon \} \text{ if } \{.\} \neq \phi. \\ n \qquad \text{if } \{.\} = \phi. \end{cases}$$

Next, define $\tau_2 : \Omega \to N$ by

$$\tau_2(\omega) = \begin{cases} \inf \{ j \in \{p, p+1, \dots, n\} : \| P_{\tau_1(\omega),\, j}(\omega) \| \geqslant \varepsilon \text{ if } \{.\} \neq \phi, \\ n \ \text{ if } \{.\} = \varnothing. \end{cases}$$

Finally set $\tau(\omega) = (\tau_1(\omega), \tau_2(\omega))$. It is easy to see that τ is a map from Ω into $\{p, \dots, n\}^2$ such that for every (i, j), $\bar{p} \leqslant (i, j) \leqslant \bar{n}$, $\{\tau = (i, j)\} \in F^1_{i_j}$

Hence $\tau \in T^1$. Furthermore, $\{ \| p_\tau \| \geqslant \varepsilon \} = \{ \sup\limits_{p \leqslant i,\, j \leqslant n} \| p_{i_j} \| \geqslant \varepsilon \}$ a.s. Thus, we have proved (23) which implies that $(p_\tau)_{\tau \in T^1}$ does not converge in probability to zero. This contradiction establishes the lemma.

Proof of Theorem 2. Let (X_{i_j}) be a mil. Then the sequence $X_{\bar{n}}, F_{\bar{n}})$ is also a mil in the sense of Talagrand [13]. This with the hypothesis on $(X_{\bar{n}})$ yields that $(X_{\bar{n}})$ is uniformly integrable. Hence, by [13, Theorem 8, p. 1194], there exists a unique decomposition $X_{\bar{n}} = Y_{\bar{n}} + Z_{\bar{n}}$, where $Y_{\bar{n}}$ is a uniformly integrable martingale and

$$Z_{\bar{n}} \xrightarrow[L_1]{\text{a.s.}} 0 \text{ as } n \to \infty \tag{24}$$

Furthermore, one can check that $(Y_{\bar{n}})$ is Llog$^+$L-bounded. Thus if we put

$$\mu_{i_j} = E(Y_{\bar{n}} / F_{i_j}) \text{ for } (i, j) \leqslant \bar{n},\ n = 1, 2, \dots \text{ and}$$

$$p_{i_j} = X_{i_j} - \mu_{i_j} \text{ for } (i, j) \in \mathrm{N}^2,$$

then (μ_{i_j}) is a LlogL-bounded martingale. Moreover, by Lemma 1, (μ_{i_j}) converges a.s. Hence, to prove that (X_{i_j}) converges a.s. it remains to show that (P_{i_j}) converges a.s. to zero. But, by Lemma 2, it is sufficient to prove that

$$p_\tau \xrightarrow[\tau \in \mathrm{T}^1]{\mathrm{P}} 0. \tag{25}$$

To see this, let $\varepsilon > 0$. It follows, from the definition that there exists $p \in N$ such that for every $m \in N$, $m \geqslant p$

$$P(\sup_{p \leqslant i, j \leqslant m} \| E(X_{\overline{m}}/F_{i_j}) - X_{i_j} \| \geqslant \varepsilon) \leqslant \varepsilon/2. \tag{26}$$

Let $\tau \in T1$, $\tau \geqslant \bar{p}$ be arbitrary but fixed. Then, there exists $n_0 \geqslant p$ such that $\bar{p} \leqslant \tau \leqslant \bar{n}_0$.

It follows from (24) that there exists $n_1 \in N$, $n_1 \geqslant n_0$ such that for every $n \geqslant n_1$, we have

$$E \| X_{\bar{n}} - Y_{\bar{n}} \| \leqslant \varepsilon^2/2\,(n_0 - p)^2. \tag{27}$$

Now, for every $n \geqslant n_1$, by (26) and (27) we get

$$p(\| p_\tau \| \geqslant 2\varepsilon) = p(\| \sum_{p \leqslant i,j \leqslant n_0} 1_{\{\tau = (i,j)\}}(X_{i_j} - \mu_{i_j}) \| \geqslant 2\varepsilon)$$

$$\leqslant p(\| \Sigma\, 1_{\{\tau = (i,j)\}}[X_{i_j} - E(X_{\bar{n}}/F_{ij})] \| \geqslant \varepsilon)$$

$$+ p(\| \Sigma\, 1_{\{\tau = i,\, j)\}}[\mu_{i_j} - E(X_{\bar{n}}/F_{i_j})] \| \geqslant \varepsilon)$$

$$\leqslant p(\sup_{p \leqslant i,j \leqslant n} \| X_{i_j} - E(X_{\bar{n}}/F_{i_j}) \| \geqslant \varepsilon)$$

$$+ p(\sum_{p \leqslant i,j \leqslant n_0} 1_{\{\tau = (i,j)\}} \| E(Y_{\bar{n}} - X_n / \mathcal{G}_{ij}) \| \geqslant \varepsilon)$$

$$\leqslant \varepsilon/2 + \sum_{p \leqslant i,j \leqslant n} E \| X_{\bar{n}} - Y_{\bar{n}} \| / \varepsilon$$

(Tsebyshev's inequality)

$$\leqslant \varepsilon/2 + (n_0 - p)^2 \frac{\varepsilon^2}{2\varepsilon(n_0 - p)^2} = \varepsilon.$$

This shows that $p_\tau \xrightarrow[\tau \in T^1]{P} 0$. Thus, (25) and hence the theorem is proved

Finally, by applying the above method and a result of Chatterji [4] on the convergence of martingales in Banach spaces without (RNP), one can also prove the following

THEOREM 3. *Let B be a Banach space (not necessarily having RNP) and (X_{ij}) a B-valued mil satisfying all the hypotheses of Theorem 2. Then (X_{ij}) converges a.s if and only if the set $\{N_{\bar{n}}(\omega),\ n \in N\}$ is weakly compact a.s.*

Acknowledgment. The author would like to express his deep gratitude to Prof. Nguyen Zuy Tien and Dr. Dinh Quang Luu for great helps,

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Received August 20, 1988

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