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ON THE CLOSEDNESS OF THE MAPPING DEFINED BY THE GENERALIZED GRADIENT OF THE SUPPORT FUNCTION OF A LIPSCHITZ SET-VALUED MAP

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0. INTRODUCTION AND DEFINITIONS

In recent years, set-valued maps have become a subject that attracts much attention from researchers dealing with optimization theories. In our previous papers [2—5] we considered a class of locally Lipschitz set-valued maps and, established some results for optimization problems involving set-valued maps, including necessary and sufficient optimality conditions, stability properties,... Furthermore, some local surjectivity theorems, implicit function theorems for set-valued maps were also established. Most of the results were proved under certain assumption on the closedness of the gradient mapping of the support functions of the set-valued map. More precisely, recall that a set-valued map F from a Banach space X into another Banach space Y is said to be locally Lipschitz at a point x_0 if, for somepositive number k and neighborhood U of x_0 the following relation holds

$$F(x_1) \subset F(x_2) + k \| x_1 - x_2 \| B, \text{ for all } x_1, x_2 \in U,$$

where $\| . \|$ stands for the norm and B denotes the unit ball in Banach spaces. F is said to be locally Lipschitz (on X) if it is locally Lipschitz at every point of X. The support function for F is defined as follows

$$c_F(y^*, x) = \sup \{\langle y^*, v\rangle / v \in F(x)\}, \ x \in X, \ y^* \in Y^*,$$

where Y^* is the continuous dual of Y and $\langle . ; . \rangle$ denotes the canonical pairing between Y^* and Y. It is well known that if F is locally Lipschitz then the set

$$Y^*_F(x) = \{y^* \in Y^* \,/\, c_F(y^*, x) < +\infty\}$$

is a nonempty convex cone which does not depend on x (and will be denoted by Y^*_F). Furthermore, $c_F(y^*,.)$ is locally Lipschitz (in x) uniformly for $y^* \in$ $\in Y^*_F \cap B^*$ (B^* is the unit ball of Y^*) The generalized gradient of $c_F(y^*,.)$ at x (in the sense of [1]) is denoted by $\partial_x c_F(y^*, x)$. Most of the results in [2—5] were established under the assumption that F satisfies certain *C1—property* which means that the set—valued map $(y^*, x) \rightarrow \partial_x c_F(y^*, x)$ is closed.

In the present paper we consider this assumption in more detail. First, we give some sufficient conditions which ensure the satisfaction of the assumption. Further on, we construct an example showing the existence of locally Lipschitz set-valued maps with no *Cl-property*. In the rest of the paper we provide a method to circumvent the hindrance from the *Cl-property* assumption.

Although most of the results in this paper can be established for infinite-dimensional spaces, for simplicity of exposition we shall restrict ourselves to finite-dimensional Euclidean spaces, and we shall identify X^*, Y^* with X, Y.

1. SOME CLASSES OF LOCALLY LIPSCHITZ SET-VALUED MAPS WITH CL-PROPERTY

Recall that a locally Lipschitz set-valued map F from X into Y has the Cl-property if the mapping $(y^*, x) \to \partial_x c_F(Y^*, x)$ is closed, or, which amounts to the same, if, for every triplet of sequences $y_n^* \in Y_F^*$, $x_n \in X$, $x_n^* \in \partial_x c_F(y_n^*, x_n)$, from $y_n^* \to y_0^* \in Y_F^*$, $x_n \to x_0 \in X$, $x_n^* \to x_0^*$ it follows that $x_0^* \in \partial_x c_F(y_0^*, x_0)$.

PROPOSITION 1. *If $F(x) = g(x) + K$, where $g\colon X \to Y$ is a locally Lipschitz function and K is a convex subset of Y, then F has the Cl-property.*

Proof. It is clear that

$$Y_F^* = K^\infty := \{ y^* / \sup_{v \in K} \langle y^*, v\rangle < +\infty \},$$

and, for $y^* \in K^\infty$,

$$c_F(y^*, x) = \langle y^*, g(x)\rangle + \sup \{ \langle y^*, v\rangle / v \in K \}.$$

Hence,

$$\partial_x c_F(y^*, x) = \partial_x \langle y^*, g(x) \rangle.$$

It is a simple matter to verify that the mapping $(y^*, x) \to \partial_x \langle y^*, g(x) \rangle$ is closed and the proposition follows.

PROPOSITION 2. *If F is convex, locally Lipschitz and, for every x, $c_F(., x)$ is continuous, then F has the Cl-property.*

Proof. As F is convex, $c_F(y^*,.)$ is concave. Hence, $x^* \in \partial_x c_F(y^*, \bar{x})$ if and only if

$$c_F(y^*, x) - c_F(y^*, \bar{x}) \leqslant \langle x^*, x - \bar{x}\rangle, \text{ for all } x \in X. \tag{1}$$

Now let $y_n^* \in Y_F^*, x_n \in X$, $x_n^* \in \partial_x c_F(y_n^*, x_n)$ be such that

$$y_n^* \to y_0^* \in Y_F^*, x_n \to x_0 \in X, x_n^* \to x_0^*.$$

By (1) we have, for all $x \in X$,

$$c_F(y_n^*, x) - c_F(y_n^*, x_0) + c_F(y_n^*, x_0) - c_F(y_n^*, x_n) \leqslant \langle x_n^*, x - x_n\rangle . \tag{2}$$

Clearly

$$\left| c_F(y_n^*, x_0) - c_F(y_n^*, x_n)\right| \leqslant k \| y_n^* \| \| x_n - x_0 \| ,$$

where k is a Lipschitz constant for F at x_0. On the other hand $c_F(. , x)$ is continuous, then from (2), by letting $n \longrightarrow \infty$, we get

$$c_F(y_o^*, x) - c_F(y_o^*, x_o) \leqslant \langle x_o^*, x - x_o \rangle .$$

That means

$$x_o^* \in \partial_x c_F (y_o^*, x_o) ,$$

and the proposition is proved.

COROLLARY 1. *If F is locally Lipschitz, convex and, for every x, $F(x)$ is bounded, then F has the Cl-property.*

Indeed, in this case $Y_F^* = Y$ and the function $c_F(\cdot, x)$, being concave, is well defined on the whole space Y. Hence, $c_F(\cdot, x)$ is continuous and the corollary follows from Proposition 2.2.

Remark 1. The same argument shows that Proposition 2 and Corollary 1 remain valid for the case where F is concave.

Remark 2. It can be shown that the continuity of $c_F(\cdot, x)$ implies the closedness of Y_F^*. The converse holds if the cone Y_F^* is polyhedral.

PROPOSITION 3. *Let U be a metric compact space, and f be a (single-valued) function from $X \times U$ into Y which* is continuous on $X \times U$ *together with $f'_x(\cdot, \cdot)$ Then the set-valued map*

$$F(x) = \{f(x, u) \mid u \in U\}$$

is locally Lipschitz and has the Cl-property.

Proof. It is easy to prove that F is locally Lipschitz. Further, $F(x)$ is bounded, hence $Y_F^* = Y$. Note that

$$c_F(y^*, x) = \max \{\langle y^*, f(x, u)\rangle \mid u \in U\} .$$

According to a result of [2] (Lemma 2.8.2) we have

$$\partial_x c_F(y^*, x) = \mathrm{co}\, \{ y^* f'_x(x, u) \mid u \in I(y^*, x)\} ,$$

where $I(y^*, x) = \{u \in U \mid \langle y^*, f(x, u)\rangle = c_F(y^*, x)\}$.

Observe that the mapping $(y^*, x) \longrightarrow I(y^*, x)$ is closed, and $I(y^* x)$ is a compact set. Hence, I is upper semicontinuous. By assumption, $f'_x(\cdot, \cdot)$ is continuous, so the set-valued map

$$(y^*, x) \longrightarrow G(y^*; x) = \{y^* \cdot f'_x(x, u) \mid u \in I(y^*, x)\}$$

is upper semicontinuous. This implies the upper semicontinuity of the map $\partial_x c_F(y^*, x) = \mathrm{co}\, G(y^*, x)$, which is equivalent to its closedness. The proof is thus complete.

Remark 3. It should be noted that the assumption of the above proposition can be weakened.

2. EXAMPLE

In this part we construct an example of set-valued maps having no Cl-property.

In the 3-dimensional Euclidean space R^3 we take a convex cone

$$K = \{a = (x, y, z) \in R^3 \ / \ z \geqslant \sqrt{x^2 + y^2}\}.$$

The dual of K is

$$K* = \{a' \in R^3 \ / < a', a > \leqslant 0, \text{ for all } a \in K\}$$

$$= \{a' = (u, v, w) \in R^3 \ / \ w \leqslant -\sqrt{u^2 + v^2}\}.$$

Let $\alpha_n \in (0,1)$ satisfy $\lim\limits_{u \to \infty} \alpha_n = 0$ and $\cos \alpha_n > 1/2$ for all n.

Denote $a_n = \dfrac{1}{1-\cos\alpha_n}(1, \operatorname{tg}\alpha_n, 1) \in R^3$,

$$A = \{a_n \ / \ n = 1, 2, 3, \ldots\},$$

$$M = \overline{\text{co}}\{A \cup K\},$$

and observe that

$$M \subset K + 2B, \tag{3}$$

since $k_n = \dfrac{1}{1-\cos\alpha_n}\left(1, \operatorname{tg}\alpha_n, \dfrac{1}{\cos\alpha_n}\right) \in K$ for all n, and

$$d(a_n ; K) \leqslant \| \ a_n - k_n \| = \frac{1}{\cos\alpha_u} < 2 \text{ for all } n.$$

Define the set-valued map F from R^1 to R^3 as follows

$$F(t) = \begin{cases} K + (0, 0, -3t), \text{ if } t \leqslant 0 \\ t.M \qquad , \text{ if otherwise.} \end{cases}$$

It is obvious that $Y_F^*(t) = K*$ (independent of t) and, for $a' \in K*$,

$$S(a') = \sup \{ < a', m > / \ m \in M\} \leqslant 2 \| a' \|,$$

because of (3). Direct computation shows that, for $a' = (u, v, w) \in K*$,

$$c_F(a', t) = \begin{cases} -3t, & \text{if } t \leqslant 0 \\ t.\, S(a'), & \text{if otherwise.} \end{cases}$$

Clearly, $c_F(a', t)$ is locally Lipschitz in t uniformly for $a' \in K* \cap B$ (with common Lipschitz constant 3). From a result of [2] (see Property 1.5, p.113) it follows that F is locally Lipschitz. Further, we shall show that F has no Cl-property. Take $a'_n = (\cos\alpha_n, \sin\alpha_n, -1) \in K*$. It is obvious that

$a'_n \to a'_0 = (1, 0, -1) \in K*$, and

$$c_F(a'_n, t) = \begin{cases} -3t, \text{ if } t \leqslant 0 \\ t.\, S(a'_n), \text{ otherwise,} \end{cases}$$

$$c_F(a'_0, t) = \begin{cases} -3t, & \text{if } t \leqslant 0 \\ 0, & \text{otherwise} \end{cases}.$$

Hence, $\partial_t\, c_F\, (a'_n, 1) = \{S\ (a'_n)\}$ and $\partial_t\, c_F\,(a'_0, 1) = \{0\}$. Note that

$$S(a'_n) \geqslant \langle a'_n, a_n \rangle = \frac{1}{\cos \alpha_n} > \frac{1}{2} \text{ for all } n.$$

From this we deduce that the mapping $\partial_t\, c_F\, (.,.)$ is not closed.

Remark 4. It is clear that in the given example, the support function $c_F\,(.,\ t)$ is discontinuos if $t = 1$.

3. CONCLUDING REMARK

The previous example shows that there exist locally Lipschitz set-valued maps with no Cl-property. Even though these maps are rarely encountered in practice, for wider applicability of the previously established results, we should like to point out a method to overcome the hindrance from the Cl-property assumption. To this end, let us consider a function $f : T \times X \to R$, where T is a topological space. Suppose that for some point $x \in X$, each function $f(t,.)$ is Lipschitz near x. Following Clarke [1] we define the, relaxed, partial generalized gradient of f (with respect to variable x) as

$$\partial_x^{[T]} f(t, x) = \overline{\mathrm{co}} \left\{ \xi = \lim_{i=\infty} \xi_i \,/\, \xi_i \in \partial x f\, (t_i, x_i),\ t_i \in T,\ t_i \to t\ x_i \to x \right\},$$

where $\partial_x\, f(t,\ x)$ denotes the generalized gradient of $f(t,.)$ at x. Let F be a locally Lipschitz set-valued map from X into Y, and $B\,(F) = Y^*_F \cap B$. We put

$$\bar{\partial}_x\; c_F\; (y^*, x) := \partial_x^{[B(F)]}\; c_F\; (y^*, x)\quad .$$

It is not difficult to prove that

(i) $\bar{\partial}_x C_F\ (y^*, x)$ is nonempty, convex and compact for every (y^*, x),

(ii) the mapping $(y^*, x) \to \bar{\partial}_x c_F\ (y^* x)$ is closed,

(iii) F has the Cl-property if and only if $\partial_x\ c_F\ (y^*, x) = \bar{\partial}_x\, c_F\ (y^*,\ x)$.

Without the Cl-property assumption we can easily verify that all the results previously established in [2—5] remain valid with $\bar{\partial}_x\, c_F\ (y^*,\ x)$ playing the role of $\partial_x\ c_F(y^*, x)$.

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