ACTA MATHEMATICA VIETNAMICA
Vol. 14 No 1 (1989)

A NOTE ON MIXED HYPERGEOMETRIC SERIES

M. A. KHAN AND A.H. KHAN

1. INTRODUCTION

In the present paper a new type of hypergeometric series, called « mixed hypergeometric series » will be introduced involving parameters of which some are ordinary and others are on the base q. It will also be shown how naturally such series arise while attempting to find q-analogues of certain results of ordinary hypergeometric series.

The basic number $[\alpha]$ defined by

$$[\alpha] = \frac{1-q^{\alpha}}{1-q}, \; |q| < 1$$

tends to α as $q \to 1$. Thus ordinary hypergeometric series is a limiting case of the q-hypergeometric series. In 1967, Agrawal and Verma [1] introduced generalized basic hypergeometric series with unconnected bases. If one of the two bases tends to 1 the resulting series contains both types of parameters i.e. ordinary as well as those which are on the base q.

In this paper such a mixed hypergeometric series has been introduced and an attempt has been made to show how naturally they occur while establishing q-analogues of certain results involving ordinary hypergeometric series.

2. DEFINITIONS AND NOTATIONS

Let

$$(a)_n = a(a+1)(a+2)\ldots(a+n-1), \; (a)_0 = 1 \tag{1}$$

$$(q^a)_n = (1-q^a)(1-q^{a+1})\ldots(1-q^{a+n-1}), \; (q^a)_0 = 1 \tag{2}$$

The generalized « bibasic » hypergeometric series is then defined as

$$ {}_{A+B}\Phi_{C+D}\left[\begin{matrix} q^{(a)};\, q_1^{(b)};\, x \\ q^{(c)};\, q_1^{(d)};\, q^{\lambda},\, q_1^{\lambda_1} \end{matrix}\right] $$

$$= \sum_{n=0}^{\infty} \frac{[q^{(a)}]_n \, [q_1^{(b)}]_n}{[q^{(c)}]_n \, [q_1^{(d)}]_n} \, x^n \, q^{\frac{1}{2}\lambda n(n+1)} \, q_1^{\frac{1}{2}\lambda_1 n(n+1)} \tag{3}$$

where $\lambda, \lambda_1 > 0$, $|q| < 1$, $|q_1| < 1$ and for $\lambda = 0 = \lambda_1$, $|x| < 1$.

In the numerator and the denominator the terms before the colon are on the base q and those after it are on the base q_1. As usual (a_N) stands for the sequence of N parameters $a_1, a_2, \ldots, a_N$ when $N = A$ it will be dropped out.

Also, the q-fractional derivative $D^{\alpha}_{q,x}$ is defined as (cf. Agarwal [2]).

$$D^{\alpha}_{q,x}\, x^{\mu-1} = (1-q)^{-\alpha}\, \pi_q \left[{\mu-\alpha \atop \mu}\right] x^{\mu-\alpha-1}, \quad (\alpha \neq \mu) \tag{4}$$

where $\pi_q\left[{\alpha \atop \beta}\right]$ stands for $\frac{[\alpha]_\infty}{[\beta]_\infty}$ and $[\alpha]_\infty$ for $\prod_{r=0}^{\infty} [1-\alpha q^r]$.

We shall also need the following results:

$$D^{\alpha-\beta}_{q,x}\, x^{n+\alpha-1} = \frac{(q^\beta)_\infty (q^\alpha)_n}{(q^\alpha)_\infty (q^\beta)_n} (1-q)^{\beta-\alpha}\, x^{n+\beta-1}, \tag{5}$$

$$[x]_n = {}_1\Phi_0\, [q^{-n};\ xq^n]$$

and

$$[a-x]_n = (a-x)(a-xq)\ldots(a-xq^{n-1}) = a^n [x/a]_n \tag{6}$$

(cf. Jackson [7]).

Lastly, we introduce mixed hypergeometric series defined as follows:

$$A+B^{Y}C+D\left[\begin{matrix}(a)\,;\ q^{(b)};\ x \\ (c)\,;\ q^{(d)};\ q^{\lambda}\end{matrix}\right] = \sum_{n=0}^{\infty} \frac{[(a)]_n\, [q^{(b)}]_n}{[(c)]_n\, [q^{(d)}]_n}\, x^n\, q^{\frac{1}{2}\lambda n(n+1)} \tag{7}$$

where $\lambda > 0$, $|q| < 1$ and for $\lambda = 0$, $|x| < 1$. Also, $[(a)]_n$ stands for $(a_1)_n$, $(a_2)_n \ldots (a_A)_n$ and $[q^{(b)}]_n$ stands for $(q^{b_1})_n\, (q^{b_2})_n \ldots (q^{b_B})_n$, where $(a)_n$ is given by (2. 1) and $(q^a)_n$ is given by (2.2).

3. MAIN RESULTS

In this section, we shall consider our main results. Consider the identity

$$[(1-y)-(1-x)]_n = [x-y]_n$$

or
$$\sum_{r=0}^{n} \frac{(q^{-n})}{(q)_r} (1-x)^r (1-y)^{n-r} q^{nr} = \sum_{r=0}^{n} \frac{(q^{-n})}{(q)_r}{}_r\, x^{n-r} y^r q^{nr}$$

or
$$\sum_{r=0}^{n} \frac{(q^{-n})_r\, q^{nr}}{(q)_r} \sum_{k=0}^{r} \frac{(-r)_k}{k!} x^{k+a-1} \sum_{i=0}^{n-r} \frac{(-n+r)_i}{i!} y^{i+x-1}$$

$$= \sum_{r=0}^{n} \frac{(q^{-n})_r\, q^{nr}}{(q)_r} x^{n-r+a-1}\, y^{r+\alpha-1}.$$

Applying $D_{q,x}^{a-b}$ and $D_{q,y}^{\alpha-\beta}$, we get

$$\sum_{r=0}^{n} \frac{(q^{-n})_r\, q^{nr}}{(q)_r}\; {}_{1+1}Y_{1+1}\left[\begin{matrix} -r: q^{a}; x \\ l: q^{b} \end{matrix}\right] l+l\,\gamma\, l+l\left[\begin{matrix} -n+r: q^{\alpha}; y \\ l: q^{\beta} \end{matrix}\right]$$

$$= \frac{(q^{a})_n\, x^{n}}{(q^{b})_n}\; {}_3\Phi_2\left[\begin{matrix} q^{-n}, q^{1-b-n}, q^{\alpha}; \dfrac{yq^{n+b-a}}{x} \\ q^{1-a-n}, q^{\beta} \end{matrix}\right]. \qquad (1)$$

If we continue this process of performing q-derivative operators u times with respect to x and l times with respect to y and then suppress some parameters, we arrived at

$$\sum_{r=0}^{n} \frac{(q^{-n})_r\; q^{nr}}{(q)_r}\; {}_{1+u}Y_{1+v}\left[\begin{matrix} -r: q^{a_1}, q^{a_2}, \dots, q^{a_u}; x \\ 1: q^{b_1}, q^{b_2}, \dots, q^{b_v} \end{matrix}\right]$$

$${}_{1+l}Y_{1+m}\left[\begin{matrix} -n+r: q^{\alpha_1}, q^{\alpha_2}, \dots, q^{\alpha_l}, \; y \\ 1 \quad : q^{\beta_1}, q^{\beta_2}, \dots q^{\beta_m} \end{matrix}\right] = \frac{(q^{a_1})_n (q^{a_2})_n \cdots (q^{a_u})_n}{(q^{b_1})_n (q^{b_2})_n \cdots (q^{b_v})_n}\; x_n$$

$${}_{1+v+l}Y_{1+u+m}\left[\begin{matrix} q^{-n}: q^{1-b_1-n}, q^{1-b_2-n}, \dots, q^{1-b_v-n}, q^{\alpha_1}, q^{\alpha_2}, \dots, q^{\alpha_l}; Z \\ q \;: q^{1-a_1-n}, q^{1-a_2-n}, \dots, q^{1-a_u-n}, q^{\beta_1}, q^{\beta_2}, \dots, q^{\beta_m}, q^{u-v} \end{matrix}\right] \qquad (2)$$

$$\text{where } Z = \frac{(-1)^{u-v}\; yq^{n(1+v-u)}}{xq^{a_1+a_2+\dots+a_u-b_1\dots-b_v}}.$$

In a subsequent paper, we propose to obtain more such results concerning mixed hypergeometric series. These results are expected to be of great importance in the study of the product of an ordinary polynomial and a q-polynomial.

REFERENCES

[1] R.P. Agarwal, and A. Verma, *Generalized basic hypergeometric series with unconnected bases*, Proc. Camb. Phil. Soc. 63 (1967), 727–734.

[2] R.P. Agarwal, *Fractional q-derivatives and q-integrals and certain hypergeometric functions*, Ganita, 27 (1976), 25-32.

[3] W.A. Al—Salam, Some *fractional q-integrals and q-derivatives*. Proc.Edinburgh Math. Soc.15 (1966), 135—140.

[4] W. Hahn, *Beitrage zur theorie der Heimeschen Reihen*. Math. Nachr. 2(1949), 340—379

[5] M.A. Khan, *An algebraic study of certain q-fractional integrals and q-derivatives*, The Mathematics Student Vol. XL, No.4 (1972) 442—446.

[6] M.A. Khan, *Certain fractional q-integrals and q-derivatives*, Nanta Mathematica, 7 (1974) 52 – 60.

[7] F. H. Jackson, *On basic double hypergeometric functions*, Quart J. Math. (Oxford) (1942), 70 – 80.

[3] H.L. Manocha and B. L. Sharma, *Some formulae by means of fractional derivatives*, Composition Math. 18 (3) (1967), 229 – 234.

[9] L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press (1966).

Received May 12, 1987

MATHEMATICS SECTION. Z. H. COLLEGE OF ENG. AND TECH.,
A.M. U. ALIGARH – 202001, U.P., INDIA.