APPROXIMATELY SOLVING CAUCHY PROBLEM FOR THE WAVE EQUATION BY THE METHOD OF DIFFERENTIAL OPERATORS OF ININFITE ORDER

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In recent years the theory of differential operators of infinite order (DOIO) has various fruitful applications to partial differential equations in general and to Mathematical Physics in particular [1, 2, 3, 4]. The foundation of these applications is a nonformal algebra of DOIO, which are generated from entire functions. The nonformality of this algebra is obtained by considering DOIO in corresponding Sobolev spaces of infinite order. Observe that the type of the partial differential equation plays no role if this equation is considered in the Sobolev space of infinite order [1, 2, 4]. However, we emphasize that for those problems which are correct in the spaces of finite smoothness, both the spaces of infinite order and DOIO play an intermediate role and provide useful tools for the investigation of the initial problem. At the same time, the introduction of the spaces of infinite order for those problems which are ill-posed in the usual sense, plays a decisive role in this method: problems which are ill-posed in the classical sense are correct in these spaces.

The theory of DOIO opens new possibilities for the solution of approximate partial differential equations. This idea is based on a very simple but in no way trivial fact, namely: the spaces of the type H^{∞} [1], W^{∞} [4] are dense in the Sobolev spaces of finite order in which we are seeking the solutions of initial and boundary value problems for PDE.

In this paper we shall consider the Cauchy problem for the wave equation and present a new method for finding an approximate solution of it by the technique of DOIO.

In Section 1 we review some general qualitative properties of the solution of the Cauchy problem for the wave equation. The main result of Section 2 is Theorem 2.3, which asserts that for all $f \in C^m(\mathbb{R}^n)$ there exists a sequence of entire functions of exponential type $f_k \in SЖ_{v_k \infty}(R^n)$ [5], $(v_k > 0)$, such that

$$\| f_k - f \|_{C^m(K)} \to 0, \; k \to \infty,$$

for every compact set $K \subset R^n$. In Section 3 we consider the approximation ö solution of the Cauchy problem by the technique of DOIO. We show that by this method one can obtain an approximate solution with arbitrary accuracy.

1. CAUCHY PROBLEM FOR THE WAVE EQUATION

Let us consider the Cauchy problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f(x,t),\ x \in \mathbf{R}^n,\ t > 0, \tag{1.1}$$

$$u(0,x) = \varphi(x), \tag{1.2}$$

$$\frac{\partial u(0,x)}{\partial t} = \psi(x), \tag{1.3}$$

where Δ is the Laplace operator in $\mathbf{R}^n$, $f(x,t)$, $\varphi(x)$, $\psi(x)$ are given functions, $(x,t) = (x_1,\dots,x_n,t)$, $x \in \mathbf{R}^n$, $t \in \mathbf{R}^1$. We denote the cone $\{(x,t) : |x-X| < T-t,\ t^0 < t < T\}$ by K_{X,T,t^0} and its base by D_{X,T,t^0}. Thus, $D_{X,T,t^0} = \{(x,t) : |x-X| < T-t,\ t = t^0\}$. For simplicity, let us denote the sets

$$\{(x,t) : x \in \mathbf{R}^n,\ t > t^0\},\ \{(x,t) : x \in \mathbf{R}^n,\ t \geqslant t^0\},$$

$$\{(x,t) : x \in \mathbf{R}^n,\ t^0 \leqslant t \leqslant t^1\},$$

by $\{t > t^0\}$, $\{t \geqslant t^0\}$, $\{t^0 \leqslant t \leqslant t^1\}$, respectively and the space $C^k(\{t > t^0\})$, $C^k(\{t \geqslant t^0\})$... by $C^k(t > t^0)$, $C^k(t \geqslant t^0)$..., respectively.

A function $u(x,t) \in C^2(t > 0) \cap C^1(t \geqslant 0)$ is called the classical solution of problem (1.1)—(1.3) in the half-space $\{t > 0\}$ if for any $x \in \mathbf{R}^n$, $t > 0$, $u(x,t)$ satisfies (1.1) and when $t = 0$, $u(x,t)$ satisfies (1.2) and (1.3). For convenience we consider only the case $f(x,t) \equiv 0$.

THEOREM 1.1 [6] *If $\varphi \in C^{m+3}(\mathbf{R}^n)$, $\psi \in C^{m+2}(\mathbf{R}^n)$, where $m = \max\left(\left[\frac{n}{2}\right] - 1, 0\right)$, then the Cauchy problem (1.1)—(1.3) has a unique classical solution in the half-space $\{t>0\}$. Moreover, for any $(X,T) \in \{t \geqslant 0\}$ the following estimate is valid*

$$\|u\|_{C(\overline{K}_{X,T,0})} \leqq C\left\{ \|\varphi\|_{C^{m+1}(\overline{D}_{X,T,0})} + \|\psi\|_{C^m(\overline{D}_{X,T,0})} \right\}, \tag{1.4}$$

where the constant C depends only on T.

When $n = 1$ we have

THEOREM 1.2 [6]. *If $\varphi \in C^2(\mathbf{R}^1)$, $\psi \in C^1(\mathbf{R}^1)$, then the function $u\ (x, t)$ given by the D'Alambert formula*

$$u(x, t) = \frac{\varphi(x+t) + \varphi(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi)\, d\xi \tag{1.5}$$

is the classical solution of the problem (1.1) — (1.3). *Moreover, for any* $(X, T) \in \{t > 0\}$ *the following estimate is valid*

$$\| u \|_{C(\overline{K}_{X,T,0})} \leqslant \| \varphi \|_{C(\overline{D}_{X,T,0})} + T \| \psi \|_{C(\overline{D}_{X,T,0})} \,. \tag{1.6}$$

THEOREM 1.3 [6] (Generalized solution). *If $\varphi \in H^1(\,|x| < R)$, $\psi \in L_2(|x| < R)$ for every $R > 0$, then there exists a generalized solution of the Cauchy problem (1.1) — (1.3) in the cylinder $\Pi_T = \{x \in \mathbf{R}^n,\ 0 < t < T\}$.*

For the notation of generalized solution of the Cauchy problem (1.1)—(1.3), the reader is referred to [6, p. 325].

2. APPROXIMATION OF SMOOTH FUNCTIONS BY ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

We use the terminology and notation in [5]. Let $f(x)$ be a function defined in $\mathbf{R}^n$, $h \in \mathbf{R}^n$,

$$\Delta_h f(x) = f(x+h) - f(x).$$

$$\Delta_h^k f(x) = \Delta_h^{k-1} \Delta f(x) \quad k \in N,$$

$$\omega_h^k (f, \delta)_p = \sup_{|t| \leqslant \delta} \| \Delta_{th}^k f(x) \|_{L_p(\mathbf{R}^n)},$$

$$f_h^m = \sum_{|\alpha| = m} D^\alpha f h^\alpha, \quad h^\alpha = h_1^{\alpha_1}, \dots h_n^{\alpha_n},$$

$$\Omega^k (f^m, \delta)_p = \sup_{|h| = 1} \omega_h^k (f_h^m, \delta)_p$$

(If $k = 1$, then the superscript k is dropped).

Let φ be a non-negative even function of a single variable of exponential type 1. For example,

$$\varphi = C \left(\frac{\sin t/\lambda}{t} \right)^\lambda,$$

where the constant C and the number λ are chosen so that

$$\int_{\mathbf{R}^n} \varphi(\,|x|\,)\, dx = 1.$$

For $f \in L_p(\mathbf{R}^n)$, we set

$$\varphi_\sigma(x) = \varphi_\sigma(f, x) = \int_{\mathbf{R}^n} \varphi(|t|) \left\{(-1)^m \Delta_{t/\sigma}^{m+1} f(x) + f(x)\right\} dt,$$

$$f(x) - \varphi_\sigma(x) = (-1)^m \int_{\mathbf{R}^n} \varphi(|t|) \Delta_{t/\sigma}^{m+1} f(x)\, dt,$$

$\sigma > 0,\ m \in \mathbf{N}.$

LEMMA 2.1 ([5, p. 185]) *Let $f(x) \in W_p^m(\mathbf{R}^n)$, $1 \leqslant p \leqslant +\infty$. Then the following estimates hold:*

a) $$\| f - \varphi_\sigma \|_{L_p(\mathbf{R}^n)} \leqslant \frac{C}{\sigma^m} \Omega\left(f^m, \frac{1}{\sigma}\right)_p,$$

b) *For* $\alpha \in \mathbf{Z}_+^n$, $|\alpha| \leqslant m$

$$\| D^\alpha f - D^\alpha \varphi_\sigma \|_{L_p(\mathbf{R}^n)} \leqslant \frac{C}{\sigma^{m-|\alpha|}} \sum_{|\nu|=m} \Omega(f^{(\nu)}, 1/\sigma)_p$$

bd) **Remark 2.2.**

$$\Omega\left(f^m, \frac{1}{\sigma}\right)_p \to 0,\ \sigma \to \infty,$$

$$\Omega\left(f^{(\nu)}, \frac{1}{\sigma}\right)_p \to 0,\ \sigma \to \infty.$$

THEOREM 2.3. *Let $f(x) \in C^m(\mathbf{R}^n)$. Then there exists a sequence of entire functions ψ_k of exponential type ν_k ($\psi_k \in SЖ_{\nu_k}^{\infty}(\mathbf{R}^n)$), such that for any compact $K \subset \mathbf{R}^n$ we have*

$$\| f - \psi_k \|_{C^m(K)} \to 0\quad k \to \infty.$$

Proof. We introduce the following notations. For $f \in C^m(\mathbf{R}^n)$ we set

$$f_k(x) = \begin{cases} f(x), & |x| \leqslant k, \\ 0, & |x| > k, \end{cases}$$

$$\varphi_{\sigma,k}(x) = \varphi_\sigma(f_k, x) =$$

$$= \int_{R^n} \varphi(|t|) \left\{(-1)^m \Delta_{t/\sigma}^{m+1} f_k(x) + f_k(x)\right\} dt,$$

$$Q_k = \{x \in \mathbf{R}^n,\ |x| \leqslant k\}.$$

We invoke Lemma 2.1 to deduce that for $\alpha \in Z_+^n$, $|\alpha| \leqslant m$

$$\| D^\alpha f - D^\alpha \varphi_{\sigma,k} \|_{L_\infty(Q_k)} = \| D^\alpha f_k - D^\alpha \varphi_{\sigma,k} \|_{L_\infty(Q_k)} \leqslant$$

$$\leqslant \| D^\alpha f_k - D^\alpha \varphi_{\sigma,k} \|_{L_\infty(\mathbb{R}^n)} \leqslant \frac{C}{\sigma^{m-|\alpha|}} \Omega(D^\alpha f_k, 1/\sigma) \tag{2.1}$$

We select $\sigma = \sigma_k$ such that

$$\max_{|\alpha| \leqslant m} \frac{C\,\Omega(D^\alpha f_k, 1/\sigma_k)}{\sigma_k^{m-|\alpha|}} \leqslant 1/k.$$

Then, it follows readily from (2.1) that

$$\| D^\alpha f - D^\alpha \psi_k \|_{L_\infty(Q_k)} \leqslant 1/k, \tag{2.2}$$

where $\psi_k = \varphi_{\sigma_k, k}$.

Let K be an arbitrary compact in $\mathbb{R}^n$. Obviously, there exists a number k_0 such that $\forall\ k > k_0 : K \subset Q_k$. Then for every $k \geqslant k_0$ we have from (2.2)

$$\| f - \psi_k \|_{C^m(K)} \leqslant \| f - \psi_k \|_{C^m(Q_k)} \leqslant 1/k.$$

Hence

$$\| f - \psi_k \|_{C^m(K)} \to 0, \text{ when } k \to \infty.$$

This completes the proof.

3. APPROXIMATE SOLUTION OF THE CAUCHY PROBLEM OF THE WAVE EQUATION

Now, let us consider the Cauchy problem (1. 1) — (1. 3) with the following initial conditions

$$\varphi \in C^{m+3}(\mathbb{R}^n), \psi \in C^{m+2}(\mathbb{R}^n),$$

where $m = \max\left(\left[\frac{n}{2}\right] - 1, 0\right)$. Based on the fact that $\bigcup_{\nu>0} SЖ_{\nu\infty}$ is dense in $C^m(\mathbb{R}^n)$ (Theorem 2. 3) one can approximately solve the problem (1.1) — (1.3) by the method of DOIO.

First, Theorem 2. 3 shows that there exist sequences $\{\varphi_k\}$, $\{\psi_k\}$ from $\bigcup_{\nu>0} SЖ_{\nu\infty}$ such that for every compact $K \subset \mathbb{R}^n$ we have

$$\| \varphi_k - \varphi \|_{C^{m+3}(K)} \to 0,\ k \to \infty, \tag{3.1}$$

$$\| \psi_k - \psi \|_{C^{m+2}(K)} \to 0,\ k \to \infty, \tag{3.2}$$

THEOREM 3.1. *An approximate solution of (1. 1) — (1. 3) can be obtained in the form*

$$u_k(x, t) = \sum_{l=0}^{\infty} \frac{t^{2l}}{(2l)!} \Delta^l \varphi_k(x) +$$

$$+ \sum_{l=0}^{\infty} \frac{t^{2l+1}}{(2l+1)!} \Delta^l \psi_k(x), \tag{3.3}$$

where $\triangle$ *is the Laplace operator. Moreover, for all* $(X, T) \in \{t > 0\}$ *the following estimate holds*

$$\| u(x, t) - u_k(x, t) \|_{C(\bar{K}_{X, T, 0})} \leqslant$$

$$C(\| \varphi_k - \varphi \|_{C^{m+1}(\bar{D}_{X, T, 0})} + \| \psi_k - \psi \|_{C^m(\bar{D}_{X, T, 0})}), \tag{3.4}$$

where the constant C depends only on T.

Proof. Putting $\xi = i\sqrt{\Delta}$ we have from (1. 1)

$$\frac{\partial^2 u}{\partial t^2} + \xi^2 u = 0.$$

Solving this equation as an ordinary differential equation in t we get the formula

$$u(x, t) = e^{it\xi} C_1(x) + e^{-it\xi} C_2(x),$$

where the functions $C_1(x)$, $C_2(x)$ are arbitrary. In order to determine these functions we use the initial conditions (1. 2) and (1.3) This yields

$$C_1(x) + C_2(x) = \varphi(x),$$

$$i\xi C_1(x) + (-it\xi) C_2(x) = \psi(x).$$

A direct computation shows that

$$u(t, x) = \frac{e^{it\xi} + e^{-it\xi}}{2} \varphi(x) + \frac{e^{it\xi} - e^{-it\xi}}{2i\xi} \psi(x).$$

Taking account of the relation $\xi = i\sqrt{\triangle}$, we find that the formal desired solution has the form

$$u(x, t) = \operatorname{ch} t\sqrt{\Delta}\, \varphi(x) + \frac{\operatorname{sh} t\sqrt{\Delta}}{\sqrt{\Delta}} \psi(x), \tag{3.5}$$

or, equivalently,

$$u(x, t) = \sum_{l=0}^{\infty} \frac{t^{2l}}{(2l)!} \Delta^l \varphi(x) +$$

$$+ \sum_{l=0}^{\infty} \frac{t^{2l+1}}{(2l+1)!} \Delta^l \psi(x). \tag{3.6}$$

From the results obtained in [4] we see that when the initial functions φ and ψ belong to $S Ж_{\nu,\infty} \subset W^{\infty}$, the formulas (3.5) and (3.6) have a nonformal sense and give the classical solution of (1.1) — (1.3). Therefore, the function $u_k(x, t)$ defined by the following series

$$u_k(x, t) = \sum_{l=0}^{\infty} \frac{t^{2l}}{(2l)!} \Delta^l \varphi_k(x) +$$

$$+ \sum_{l=0}^{\infty} \frac{t^{2l+1}}{(2l+1)!} \Delta \psi_k(x), \tag{3.7}$$

is a unique classical solution of the Cauchy problem

$$\frac{\partial^2 u_k}{\partial t^2} - \Delta u_k = 0, \tag{3.8}$$

$$u_k(0, x) = \varphi_k(x) \quad , \tag{3.9}$$

$$\frac{\partial u_k}{\partial t}(0, x) = \psi_k(x). \tag{3.10}$$

To complete the proof it remains to show that $u_k(x, t)$ is an approximate solution of the problem (1. 1) — (1. 3). But we remark that by virtue of Theorem 1.1, for all $(X, T) \in \{t > 0\}$, the following inequality holds

$$\| u_k - u \|_{C^m(\bar{K}_{X,T,0})} \leq C(\| \varphi_k - \varphi \|_{C^{m+1}(\bar{D}_{X,T,0})} + \| \psi_k - \psi \|_{C^m(\bar{D}_{X,T,0})}).$$

Consequently, (3.1) and (3.2) show that

$$\| u_k - u \|_{C^m(\bar{K}_{X,T,0})} \to 0, \quad k \to \infty.$$

This completes the proof.

From now on, we shall restrict ourselves to the case $n = 1$. Let us consider the problem (1.1) — (1.3) under the hypothesis that

$$\varphi \in C^2(\mathbf{R}^1), \ \varphi \in C^1(\mathbf{R}^1) \tag{3.11}$$

By Theorem 2.3 there exist sequences φ_k, φ_k from $\bigcup_{\nu>0} S Ж_{\nu,\infty}(\mathbf{R}^1)$ such that for every compact $K \subset \mathbf{R}^n$

$$\| \varphi - \varphi_k \|_{C^2(K)} \to 0, \| \psi - \psi_k \|_{C^1(K)} \to 0, \ k \to \infty. \tag{3.12}$$

In view of Theorem 1.2, there exist solutions of the problems (1.1) — (1.3) and (3.8) — (3.10), and they are given by the D'Alambert formula

$$u(x, t) = \frac{\varphi(x+t) + \varphi(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \psi(\xi)\, d\xi, \tag{3.13}$$

$$u_k(x, t) = \frac{\varphi_k(x+t) + \varphi_k(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \psi_k(\xi)\, d\xi. \tag{3.14}$$

From (3.11), (3.13), (3.14) it is easily seen that $u(t, x) \in C^2$ $(t \geqslant 0)$ and for all $(X, T) \in \{t \geqslant 0\}$, the following estimates hold:

$$\| u - u_k \|_{C(\bar{K}_{X,T,0})} \leqslant \| \varphi - \varphi_k \|_{C(\bar{D}_{X,T,0})} + T \| \psi - \psi_k \|_{C(\bar{D}_{X,T,0})},$$

$$\left\| \frac{\partial u}{\partial t} - \frac{\partial u_k}{\partial t} \right\|_{C(\bar{K}_{X,T,0})} \leqslant \| \varphi - \varphi_k \|_{C(\bar{D}_{X,T,0})} + \| \psi - \psi_k \|_{C(\bar{D}_{X,T,0})},$$

$$\left\| \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_k}{\partial t^2} \right\|_{C(\bar{K}_{X,T,0})} = \left\| \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_k}{\partial x^2} \right\|_{C(\bar{K}_{X,T,0})} \leqslant$$
$$\| \varphi - \varphi_k \|_{C^2(\bar{D}_{X,T,0})} + \| \psi - \psi_k \|_{C^1(\bar{D}_{X,T,0})},$$

$$\left\| \frac{\partial u}{\partial x} - \frac{\partial u_k}{\partial x} \right\|_{C(\bar{K}_{X,T,0})} \leqslant \| \varphi - \varphi_k \|_{C^1(\bar{D}_{X,T,0})} + \| \psi - \psi_k \|_{C(\bar{D}_{X,T,0})}$$

Noting that

$$\| f \|_{C^m(K)} = \max_{|\alpha| \leqslant m} \| D^\alpha f \|_{C(K)},$$

we have

$$\| u - u_k \|_{C^2(\bar{K}_{X,T,0})} \leqslant \| \varphi - \varphi_k \|_{C^2(\bar{D}_{X,T,0})} + \tag{3.15}$$
$$+ (T+1) \| \psi - \psi_k \|_{C^1(\bar{D}_{X,T,0})}.$$

Hence, from (3. 12) and (3. 15) it follows that u_k converges to $u(x)$ uniformly on every compact in $\{t > 0\}$. We invoke Theorem 4 in [4] to deduce that the solution of the Cauchy problem (1.1) — (1.3) belongs to $C^2(\mathbf{R}^1_+, W^\infty(\mathbf{R}^1))$ if $\varphi, \Psi \in W^\infty(\mathbf{R}^1)$ and is given by the formula

$$u(x, t) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} \varphi^{(2k)}(x) +$$
$$+ \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \psi^{(2k)}(x), \ t > 0, \ x \in \mathbf{R}^1. \tag{3.16}$$

Now, let $\varphi, \psi \in Ж_{\nu\infty}(\mathbf{R}^1)$. Put

$$A = \| \varphi \|_{L_\infty(\mathbf{R}^1)}, \ B = \| \psi \|_{L_\infty(\mathbf{R}^1)}.$$

Taking the results of [5, p. 137] into account we get

$$\| \varphi^{(k)} \|_{L_\infty(\mathbf{R}^1)} \leqslant \nu^k . A,$$

$$\| \psi^{(k)} \|_{L_\infty(\mathbf{R}^1)} \leqslant \nu^k . B.$$

We show that in this case the approximate solution of (1.1) — (1.3) may be represented as

$$u_N(x, t) = \sum_{k=0}^{N} \frac{t^{2k}}{(2k)!} \varphi^{(2k)}(x) + + \sum_{k=0}^{N} \frac{t^{2k+1}}{(2k+1)!} \psi^{(2k)}(x). \tag{3.17}$$

Indeed, upon simple computation, we get

$$| u(x, t) - u_N(x, t) | \leqslant A . \sum_{k>N} \frac{(t\nu)^{2k}}{(2k)!} + + \frac{B}{\nu} \sum_{k>N} \frac{(t\nu)^{2k+1}}{(2k+1)!} \leqslant sh(t\nu) . \frac{(t\nu)^{2N+1}}{(2N+1)!} \left(A + \frac{Bt}{2N+1} \right).$$

Therefore

$$\| u_N(x, t) - u(x, t) \|_{C(\bar{K})} \leqslant sh(T\nu) \frac{(T\nu)^{2N+1}}{(2N+1)!} \left(A + \frac{BT}{2N+1} \right), \tag{3.18}$$

where $K = \{ 0 < t < T \}$.

It follows from the above that:

THEOREM 3. 2. *Let $\varphi \in C^2(\mathbf{R}^1)$, $\Psi \in C^1(\mathbf{R}^1)$ and let φ_ν, Ψ_ν be their approximations that satisfy (3.12). Then an approximate solution of (1.1) — (1.3) can be obtained in the form (3.17), with the estimate (3.18) being valid for every $K = \{ 0 < t < T \}$.*

We conclude this paper with the following comment. Using the Theorem 1. 3 and Lemma 2. 1 and the above method we can find the approximate generalized solution of the Cauchy problem (1.1) — (1.3) in the form (3. 3).

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