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BANACH SPACES OF D.C. FUNCTIONS AND QUASIDIFFERENTIABLE FUNCTIONS

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In recent years, quasidifferentiable functions (q.d. functions) and functions that are representable as differences of convex functions (d.c. functions) have emerged as natural tools in the study of many nondifferentiable optimization problems.

In this paper we shall be concerned with some properties of the spaces of these functions.

In the first section we introduce two definitions of quasidifferentiability. These definitions extend the concept of quasidifferentiability to functions on a nonnecessarily open set. They will include as special cases both Demyanov-Rubinov's and Shapiro's definitions of q.d. functions.

The second section is devoted to properties such as continuity, Lipschitz property, integral representability of directionally differentiable functions. The results obtained in this section will play an important role in the study of the space of q.d. functions.

The basic results of the paper are presented in Section 3, where we prove some theorems about Banach spaces of d.c. functions and q.d. functions.

In the final section, we consider q.d. and d.c. functions on [0,1]. Due to the special structure of [0,1], these functions have a number of interesting properties.

1. D.C.H. AND Q.D. FUNCTIONS

We shall consider finite functions defined on a nonempty cone or on a nonempty set of R_n. Denote by $\|\cdot\|_X$ the norm in the space X (the subscript will be omitted in the case $X = R_n$). We assume that the reader is familiar with the concepts of Convex Analysis [7].

Let $K \subset R_n$ be a nonempty convex cone.

DEFINITION 1.1. *A function $h: K \to R_1$ is called a d.c.h. (K) function if it can be represented as the difference of two convex positively homogeneous functions defined on K.*

When no confusion can arise we write d.c.h. functions.

DEFINITION 1.2. *A d.c.h. (K) function is called total d.c.h. (K) if it can be extended to a d.c.h. (R_n) function.*

To motivate this concept of total d.c.h. (K) functions, consider the following example. Let $K = \{ u = (u_1, u_2) \in R_2,\ 0 < u_1 < \infty, 0 < u_2 < \infty \}$ and $h(u) = - u_1 (1 + u_1 / u_2)^{-1/2}$ for each $u \in K$. Clearly, $h(u)$ is d.c.h. (K). However, using Theorems 24.7, 25.5 [7] we can easily prove that this function cannot be extended to a d.c.h. (R_2) function.

A class of functions closely related to d.c.h. functions consists of q.d. functions which we are going to define.

Let Ω be a nonempty subset of R_n. A vector $u \in R_n$ is said to be a feasible direction of Ω at x if there exists a number $\lambda > 0$ such that $[x, x + \lambda u] \subset \Omega$, where $[x, y]$ denotes the line segment joining x and y. The set of feasible direction of Ω at x is a cone denoted by K_x.

DEFINITION 1.3. *A function $f: \Omega \to R_1$ is said to be directionally differentiable at $x \in \Omega$ if the directional derivative*

$$f'(x, u) = \lim_{\lambda \downarrow 0} \frac{f(x + \lambda u) - f(x)}{\lambda}$$

exists for every $u \in K_x$.

We are now in a position to define the concept of quasidifferentiability. Notice that in previously published works there have been two approaches to the definition of this concept for the case when Ω is an open subset of R_n. The first approach uses sup- and super-differentials, the second uses d.c.h. functions.

DEFINITION 1.4. *A function f is called quasidifferentiable (q.d.) at $x \in \Omega$ if it is directionally differentiable at x and there exist two nonempty convex compact subsets $\underline{\partial} f(x)$, $\overline{\partial} f(x)$ of R_n such that*

$$f'(x,u) = \max_{v \in \underline{\partial} f(x)} (v,u) + \min_{w \in \overline{\partial} f(x)} (w,u) \tag{1.1}$$

for all $u \in K_x$.

Following Demyanov-Rubinov [2] we call $\underline{\partial} f(x)$, $\overline{\partial} f(x)$ the sub- and super-differentials of f at x, respectively. The pair of sets $\{\underline{\partial} f(x), \overline{\partial} f(x)\}$ is called the quasidifferential of f at x. Observe that the quasidifferential of f at x is not unique.

Let us now give another concept of quasidifferentiability.

DEFINITION 1.5. *Let $x \in \Omega$ be a point such that K_x is nonempty and convex. A function $f : \Omega \to R_1$ is said to be quasidifferentiable at x if it is directionally differentiable at x and $f'(x,.)$ is total d.c.h. (K_x).*

Using the technique of Convex Anlysis and Corollaries 13.2.1, 10.1.1 in [7] one can easily prove the following

THEOREM 1.1. *Suppose that the cone K_x is nonempty, convex. Then Definitions 1.4 and 1.5 are equivalent.*

Remark 1.1. The notion of quasidifferentiability was introduced by Demyanov and Rubinov [2], who called a directionally differentiable function on an open set of R_n quasidifferentiable at x if (1.1) holds for each $u \in R_n$. Later, Shapiro [8] defined a directionally differentiable function on R_n to be quasidifferentiable at x if $f'(x,.)$ belongs to the space DSL (R_n) of all functions that are representable as differences of sublinear functions. Note that in both cases the set Ω on which the function is defined is open so that $K_x = R_n$ for every $x \in \Omega$. Hence, our Definitions 1.4 — 1.5 extend the concept to the general case where Ω may not be open.

Remark 1.2. D. c. h. (R_n) functions have been studied by many authors However, to our knowledge the notion of total d. c. h. (K) function is first used in this paper.

2. DIRECTIONALLY DIFFERENTIABLE FUNCTIONS

As q.d. functions must be directionally differentiable, it is natural to study directionally differentiable functions in more detail.

From classical analysis the following result can easily be proved.

THEOREM. 2. 1. *Let there be given a function $f : [0, 1] \to R_1$. Suppose that f is directionally differentiable on $[0, 1]$ and*

$$L = max \{ \sup_{x \in [0,1)} | f'(x, 1) | ; \sup_{x \in (0,1]} | f'(x, -1) | \} < \infty. \tag{2.1}$$

Then: 1) *f is Lipschitz with a Lipschitz constant equal to* L.

2) *f is continuous on* $[0,1]$.

3) *The integral representation*

$$f(x) = f(0) + \int_0^x f'(\gamma,1)\, d\gamma \tag{2.2}$$

holds.

4) $f'(x, 1)$, $f'(x, -1)$ *are identical almost everywhere on* $[0, 1]$.

Consider now a subset Ω of R_n with the following property.

PROPERTY 2.1. *For every* $x \in \Omega$ *there exists an* ε*-neighbourhood* $B(x, \varepsilon) = \{ y : \| x - y \| \leqslant \varepsilon \}$ *of* x $(\varepsilon > 0)$ *such that* $B(x, \varepsilon) \cap \Omega \neq \{ x \}$ *and for all* $y \in B(x, \varepsilon) \cap \Omega$ *the inclusion* $[x, y] \subset B(x, \varepsilon) \cap \Omega$ *holds.*

Evidently, the family of sets with Property 2.1 includes open sets, convex sets and is closed under the operation of union and intersection over finite number of sets.

For any $x \in \Omega$ and $u \in K_x$, $\| u \| = 1$, denote

$$R(x, u) = \sup \{ \lambda > 0 : \quad [x, x + \lambda u] \subset \Omega \}.$$

Let f be directionally differentiable on Ω. For every $\theta \in (0, R(x, u))$ set

$$\alpha(x, u, \theta) = \max \{ \sup_{\lambda \in [0, \theta)} |f'(x + \lambda u, u)|, \sup_{\lambda \in (0, \theta]} |f'(x + \lambda u, -u)| \}. \tag{2.3}$$

THEOREM 2.2 *If*

$$\alpha(x, u, \theta) < \infty \tag{2.4}$$

then for each $y \in [x, x + \theta u]$ *we have the integral representation*

$$f(y) = f(x) + \int_0^{\| y - x \|} f'(x + \mu u, u) d\mu \tag{2.5}$$

Proof. This follows immediately by applying Theorem 2.1 to the function $h(\lambda) = f(x + \lambda u)$.

THEOREM 2.3 *Suppose that*

$$\beta(x) = \sup_{u \in K_x, \ \| u \| = 1} \ \sup_{\theta \in (0, R(x, u))} \alpha(x, u, \theta) < \infty. \tag{2.6}$$

Then: 1) *f is locally Lipschitz at every* $x \in \operatorname{int} \Omega$.

2) *If* $[x, y] \subset \Omega$ *then* $| f(y) - f(x) | \leqslant \beta(x) \ \| y - x \|$ *and, therefore f is continuous at* x.

3) *If we assume, in addition, that* Ω *is convex and*

$$L = \sup_{x \in \Omega} \beta(x) < \infty \tag{2.7}$$

then f is Lipschitz on Ω *with Lipschitz constant L. Moreover f can be approximated, as closely as desired, by a difference of two convex functions.*

Proof. The proof is immediate from Theorem 2.2 and the theorem about approximation of continuous functions by differences of convex functions.

Note that all the conditions (2.4), (2.6) and (2.7) are fulfilled if

$$\sup_{x \in \Omega,\ u \in K_x,\ \| u \| = 1} | f'(x, u) | < \infty, \tag{2.8}$$

If a q.d. function f satisfies (2.8) then f has all the properties stated in the just established theorem.

3. BANACE SPACES OF D. C. H. FUNCTIONS AND Q. D. FUNCTIONS

The problem we are concerned with in this fundamental section is to study Banach spaces of d. c. h. (K), total d. c. h. (K) and q. d. functions, respectively.

I. We begin by considering d. c. h. (K) functions

Let $K \subset R_n$ be a convex cone and $h(u)$ be a d. c. h. (K) function having at least one representation

$$h(u) = g^1(u) - g^2(u) \qquad \forall u \in K \tag{3.1}$$

such that

$$\sup_{u \in K, \|u\|=1} |g^1(u)| + \sup_{u \in K, \|u\| = 1} |g^2(u)| < \infty. \tag{3.2}$$

From now on by DCH (K) we shall always mean the family of d. c. h. (K) function for which there exists at least one representation (3.1) satisfying (3.2). It is easily seen that DCH (K) is a normed space with the norm

$$\|h\|_{DCH(K)} = \inf_{h=g^1 - g^2} \{ \sup_{u\in K, \|u\| = 1} |g^1(u)| + \sup_{u \in K, \|u\| = 1} |g^2(u)|\}. \tag{3.3}$$

In addition, we shall show that this normed space is a Banach space. To this end, we need two lemmas

LEMMA 3.1. *A normed space X is complete if and only if any series* $\sum_{i=1}^{\infty} h_i$ *with* $\sum_{i=1}^{\infty} \| h_i \|_X < \infty$ *converges. (The convergence of a series* $\sum_{i=1}^{\infty} h_i$ *means the existence of an element* $h \in X$ *such that*

$$\lim_{m \to \infty} \|h - \sum_{i=1}^{m} h_i \|_X = 0)$$

LEMMA 3. 2. *Let* $K \subset R_n$ *be a convex cone and* $h(u)$, $h_m(u)$, $m=1, 2,\ldots$ *be positively homogeneous functions from* K *to* R_1. *Assume, in addition, that each* $h_m(u)$ *is convex and*

$$\lim_{m\to\infty} \sup_{u\in K,\|u\|=1} | h(u) - h_m(u) | = 0 \tag{3.4}$$

Then $h(u)$ is convex.

The proof of these lemmas is simple and is omitted. Note that from (3.3) we can derive the inequality

$$\sup_{u\in K,\|u\|=1} | h(u) | \leqslant \| h \|_{DCH(K)} \tag{3.5}$$

which will be used later.

Now one of our main results is the following

THEOREM 3.1. *The linear space $DCH(K)$ endowed with the norm (3.3) is a Banach space.*

Proof. Let $\sum_{i=1}^{\infty} \|h_i\|_{DCH(K)}$ be a convergent series. By Lemma 3.1, it suffices to verify the convergence of $\sum_{i=1}^{\infty} h_i$. Suppose that $\|h_i\|_{DCH(K)} \neq 0$ for $i = 1,2,...,$

Fix a representation

$$h_i(u) = g_i^1(u) - g_i^2(u) \qquad \forall u \in k,\ i = 1,2,...$$

such that

$$\sup_{u\in\|K,\|u\|=1} | g_i^1(u) | + \sup_{u\in K,\|u\|=1} | g_i^2(u) | \leqslant 2 \|h_i\|_{DCH(K)}. \tag{3.6}$$

Setting

$$S_m^1(u) = \sum_{l=1}^{m} g_i^1(u),\ S_m^2(v) = \sum_{i=1}^{m} g_i^2(u), S_m(u) = S_m^1(u) - S_m^2(u),$$

we get

$$\sup_{u\in K,\|u\|=1} | S_{m+p}^1(u) - S_m^1(u) | + \sup_{u\in K,\|u\|=1} | S_{m+p}^2(u) - S_m^2(u) | \leqslant$$

$$\leqslant 2 \sum_{i=m+1}^{m+p} \|h_i\|_{DCH(K)}. \tag{3.7}$$

Consequently, for every $u \in K, \|u\| = 1$ there exist numbers $g^1(u)$, $g^2(u)$ such that

$$g^1(u) = \lim_{i\to\infty} S_i^1(u), \quad g^2(u) = \lim_{i\to\infty} S_i^2(u)$$

By letting $p \to \infty$ in (3. 7) we obtain

$$\sup_{u\in K. \| u \| =1} | g^1(u) - S_m^1(u) | + \sup_{u\in K, \| u \| =1} | g^2(u) - S_m^2(u) | \leqslant$$

$$\leqslant 2 \sum_{i=m+1}^{\infty} \| h_i \|_{DCH(K)}. \tag{3.8}$$

Replacing m by $m+p$ in (3.8) where p is an arbitrary positive integer yields

$$\sup_{u\in K,\ \|u\|=1} |(g^1(u)-S^1_m(u))-(S^1_{m+p}(u)-S^1_m(u))| + \sup_{u\in K,\ \|u\|=1} |(g^2(u)-$$

$$-S^2_m(u))-(S^2_{m+p}(u)-S^2_m(u))| \leqslant 2\sum_{i=m+p+1}^{\infty} \|h_i\|_{DCH(K)} \tag{3.9}$$

Now extend $g^1(u)$, $g^2(u)$ to all K by setting

$$g^1(u)=\|u\|\, g^1\left(\frac{u}{\|u\|}\right),\ g^2(u)=\|u\|\, g^2\left(\frac{u}{\|u\|}\right)\ \forall u\in K\setminus\{0\}$$

$$g^1(0)=g^2(0)=0 \qquad \text{if} \quad 0\in K$$

From (3.9) and Lemma 3.2 applied to $g^1(u)$, $g^2(u)$, $g^1(u)-S^1_m(u)$, $g^2(u)-S^2_m(u)$ it follows immediately that these functions are convex. Therefore, the function

$$h(u) = g^1(u) - g^2(u)$$

is d.c.h. (K).. By direct computation we can check that (3.2) holds. Finally, combining (3.3) and (3.8) yields

$$\lim_{m\to\infty} \|h-S_m\|_{DCH(K)} \leqslant \lim_{m\to\infty} (2\sum_{i=m+1}^{\infty} \|h_i\|_{DCH(K)}) = 0$$

which shows the convergence of $\sum_{i=1}^{\infty} h_i$. The proof is complete.

Observe that (3.2) is automatically fulfilled if $K=R_n$. As an immediate consequence of Theorem 3.1 we obtain

COROLLARY 3.1. *The linear space $DCH(R_n)$ endowed with the norm*

$$\|h\|_{DCH(R_n)} = \inf_{h=g^1-g^2} \left\{ \max_{\|u\|=1} |g^1(u)| + \max_{\|u\|=1} |g^2(u)| \right\}$$

is a Banach space.

Remark 3.1. In [8] and [9] the $DCH(R_n)$ and $DSL(R_n)$ have been considered with the sup-norm

$$\|h\|_{DCH(R_n)} \ (\text{or } \|h\|_{DSL(R_n)}) = \sup_{\|u\|=1} |h(u)|$$

As far as we know, the completeness of these spaces has not been established yet.

Remark 3.2. Nguyen Dinh Dan [6] considered the family $DC(\Omega)$ of functions representable as differences of convex functions on a convex compact set. He proved that $DC(\Omega)$ is a Banach space with the norm

$$\|f\|_{DC(\Omega)} = \inf_{f=f^1-f^2} \max_{x\in\Omega} \{\max \{|f^1(x)|, |f^2(x)|\}\}. \tag{3.10}$$

II. Our next task is to find a norm in $\widetilde{DCH}(K)$ (the linear space of total $DCH(K)$ functions) that makes it a Banach space.

We first consider an interesting subspace of $DCH(R_n)$. Let $K \subset R_n$ be a nonempty convex cone, $K \not\equiv R_n$. Denote

$$DCH_K(R_n) = \{ h \in DCH(R_n) : h(u) = 0 \ \forall\, u \in K \}.$$

It is easily seen that $DCH_K(R_n)$ is nontrivial. Indeed, since the function

$$k(u) = \inf_{v\in A} \|v-u\| \qquad \forall\, u \in R_n$$

is convex (respectively, positively homogeneous) whenever $A \subset R_n$ is a convex set (respectively, a cone), the function

$$h(u) = \inf_{v\in K} \|v-u\| \qquad \forall\, u \in R_n$$

belongs to $DCH_K(R_n)$ and $h(u) \not\equiv 0$.

Noting that $DCH_K(R_n)$ is closed, we can consider the quotient-space

$$DCH(R_n) / _{DCH_K(R_n)} = \left\{ \widetilde{h} : \widetilde{h} = \left\{ \overline{h}, \overline{k}, \ldots \in DCH(R_n) : \overline{h} - \overline{k} \in DCH_K(R_n) \right\} \right.$$

As is known, this quotient-space is a Banach space with the norm

$$\|\widetilde{h}\|_{DCH(R_n) / DCH_K(R_n)} = \inf_{\overline{h}\in\widetilde{h}} \|\overline{h}\|_{DCH(R_n)}.$$

Now we define a norm in DCH (K) in the following way. Let us associate with every $h \in \widetilde{DCH}(K)$ a vector $h \in DCH(R_n) / DCH_K(R_n)$ such that

$$\overline{h}(u) = h(u) \qquad \forall u \in K,\ \forall \overline{h} \in \widetilde{h}$$

We then write $h \leftrightarrow \widetilde{h}$. It is easy to prove that « $\leftrightarrow$ » is a linear one-to-one correspondence. Now for every $h \in \widetilde{DCH}(K)$ define

$$\|h\|_{\widetilde{DCH}(K)} = \|\widetilde{h}\|_{DCH(R_n) / DCH_K(R_n)} \tag{3.11}$$

if $h \leftrightarrow \widetilde{h}$. Then we have

THEOREM 3.2. *The linear space $\overleftrightarrow{DCH}(K)$ of total d.c.h. (K) functions endowed with the norm (3.11) is a Banach space.*

III. We now turn to the study of the space of q.d. functions and its relation to the classical Banach spaces C_Ω, $C_\Omega^{(1)}$.

Let $\Omega \subset R_n$ be a convex compact set (some of the results below remain valid if the compact set Ω has Property 2.1 and K_x is convex for each $x \in \Omega$).

Denote by $qC_\Omega^{(1)}$ the space consisting of q.d. functions that satisfy

$$\sup_{x\in\Omega} \| f'(x,.) \|_{\overleftrightarrow{DCH}(K_x)} < \infty. \tag{3.12}$$

Combining (3.12), (3.5) yields (2.10). Therefore, by virtue of Theorems 2.2, 2.3, f is continuous on Ω and for every $x, y \in \Omega$ we have

$$f(y) = f(x) + \int_0^{\|y-x\|} f'(x + \eta u, u)\, d\eta, \quad u = \frac{y-x}{\|y-x\|}.$$

Note that from (3.12), (3.5) we can derive the inequality

$$|f'(x + \eta u, u)| \leqslant \sup_{x\in\Omega} \| f'(x,.) \|_{\overleftrightarrow{DCH}(K_x)} < \infty \tag{3.13}$$

which will be used later.

The main result of this paper is the following

THEOREM 3.3. *$qC_\Omega^{(1)}$ is a Banach space with the norm*

$$\| f \|_{qC_\Omega^{(1)}} = \| f \|_{C_\Omega} + \sup_{x\in\Omega} \| f'(x,.) \|_{\overleftrightarrow{DCH}(K_x)} \tag{3.14}$$

Proof. It is easily seen that $qC_\Omega^{(1)}$ is a normed space. By virtue of Lemma 3.1 we need only show that for every convergent series $\sum_{i=1}^{\infty} \| f_i \|_{qC_\Omega^{(1)}}$ we can find a function $f \in qC_\Omega^{(1)}$ such that

$$\lim_{m\to\infty} \| f - S_m \|_{qC_\Omega^{(1)}} = 0 \tag{3.15}$$

where $S_m = \sum_{i=1}^{m} f_i$.

It is clear that

$$\sum_{i=1}^{\infty} \sup_{x\in\Omega} \| f_i'(x,.) \|_{\overleftrightarrow{DCH}(K_x)} < \infty.$$

This, together with Lemma 3.1, implies the existence of functions $h(x,.)$ which are limits of $\sum_{i=1}^{\infty} f'_i(x,.)$ in $\widetilde{DCH}(K_x)$. By an argument similar to that used for the proof of Theorem 3.1 we get

$$\sup_{x\in\Omega} \| h(x,.)\|_{\widetilde{DCH}(K_x)} \leqslant \sum_{i=1}^{\infty} \sup_{x\in\Omega} \| f'_i(x,.)\|_{\widetilde{DCH}(K_x)} < \infty \tag{3.16}$$

and

$$\sup_{x\in\Omega} \| h(x,.) - S'_m\|_{\widetilde{DCH}(K_x)} \leqslant \sum_{i=m+1}^{\infty} \sup_{x\in\Omega} \| f'_i(x,.)\|_{\widetilde{DCH}(K_x)}. \tag{3.17}$$

Let us fix a point $x_0 \in \Omega$ and define

$$f(x_0) = \sum_{i=1}^{\infty} fi(x_0)$$

$$f(y) = f(x_0) + \int_0^{\|y-x_0\|} h\left(x_0 + \eta \frac{y-x_0}{\|y-x_0\|}, \frac{y-x_0}{\|y-x_0\|}\right) d\eta. \tag{3.18}$$

We have to show that $f \in qC^{(1)}_{\Omega}$ and that (3.15) holds. First we prove that the representation (3.18) does not depend on the choice of x_0, i.e. for all $x,y \in \Omega$

$$f(y) = f(x) + \int_0^{\|y-x\|} h\left(x + \eta \frac{y-x}{\|y-x\|}, \frac{y-x}{\|y-x\|}\right) d\eta \tag{3.19}$$

Let x,y be given. Denote

$$u_1 = \frac{x-x_0}{\|x-x_0\|},\ u_2 = \frac{y-x_0}{\|y-x_0\|},\ u = \frac{y-x}{\|y-x\|} .$$

Let $\varepsilon > 0$ be an arbitrary number. By virtue of (3.5), (3.17) we can choose a positive integer m_0 such that

$$\sup_{x\in\Omega,\ \|u\|=1} | h(x,u) - S'_{m_0}(x,u)| \leqslant \frac{\varepsilon}{3\mathrm{diam}\Omega}$$

where diam Ω stands for the diameter of Ω.
Therefore taking account of the equality

$$\int_0^{\|y-x\|} S'_m(x+\eta u, u)\, d\eta = \int_0^{\|y-x_0\|} S'_m(x_0+\eta u_2, u_2)\, d\eta - \int_0^{\|x-x_0\|} S'_m(x_0+\eta u_1, u_1) d\eta$$

we get

$$\left|\int_0^{\|y-x\|} S'_m(x+\eta u, u)\,d\eta - \int_0^{\|y-x_0\|} h(x_0+\eta u_2, u_2)\,d\eta + \int_0^{\|x-x_0\|} h(x_0+\eta u_1, u_1)\,d\eta\right| \leqslant$$

$$\leqslant \frac{3\varepsilon\,\mathrm{diam}\Omega}{3diam\Omega} = \varepsilon$$

Since $\varepsilon > 0$ is arbitrary it follows that

$$\int_0^{\|y-x\|} h(x+\eta u, u)\,d\eta = \int_0^{\|y-x_0\|} h(x_0+\eta u_2, u_2)\,d\eta - \int_0^{\|x-x_0\|} h(x_0+\eta u_2, u_1)\,d\eta$$

i.e.

$$f(y) = f(x) + \int_0^{\|y-x\|} h(x+\eta u, u)d\eta,$$

as desired.

Our next step is to prove that

$$f'(x,u) = h(x,u) \qquad \forall u \in K_x, \|u\| = 1 \tag{3.20}$$

for every $x \in \Omega$. To this end, for the chosen number m_0 we take $\delta > 0$ such that

$$\frac{1}{\lambda}\left|\int_0^{\lambda} S'_{m_0}(x+\eta u, u)d\eta - S'_{m_0}(x,u)\right| < \varepsilon$$

for any $\lambda \in (0, \delta)$. Taking (3.19) into account we have

$$\left|\frac{1}{\lambda}(f(x+\lambda u)-f(x)) - h(x,u)\right| = \left|\frac{1}{\lambda}\int_0^{\lambda} h(x+\eta u,u)d\eta - h(x,u)\right| \leqslant \left|\frac{1}{\lambda}\int_0^{\lambda} h(x+\eta u,u)d\eta -\right.$$

$$\left.-\frac{1}{\lambda}\int_0^{\lambda} S'_{m_0}(x+\eta u,u)d\eta\right| + \left|h(x,u) - S'_{m_0}(x,u)\right| + \varepsilon \leqslant \frac{2\varepsilon}{3diam\Omega} + \varepsilon.$$

Hence (3.20) follows. Thus $f(x)$ is directionally differentiable and $f'(x,.)\in\widetilde{DCH}(K_x)$. From (3.16) we have $f \in qC^{(1)}_{\Omega}$. It is now easy to check that (3.15) follows from (3.13), (3.17) and (3.18). This completes the proof of Theorem 3.3.

Let C_{Ω}, $C^{(1)}_{\Omega}$ denote the classical Banach spaces of continuous and continuously differentiable functions on the compact set $\Omega \subset R_n$. Assuming, as usual, that Ω is convex compact we can state the following relation between $qC^{(1)}_{\Omega}$ and C_{Ω}, $C^{(1)}_{\Omega}$.

THEOREM 3.4. 1. $C_\Omega^{(1)} \subset qC_\Omega^{(1)}$

2. If $\mathrm{int}\,\Omega \neq \varnothing$ *and* $\Omega = \mathrm{int}\,\Omega \cup$ *boundary* Ω *then for any* $f \in C_\Omega^{(1)}$ *we have*

$$\| f \|_{qC_\Omega^{(1)}} = \| f \|_{C_\Omega^{(1)}}. \tag{3.21}$$

Proof. The first assertion being immediate, we need only verify (3.21). Let $f \in C_\Omega^{(1)}$. Since $f'(x, \acute{o})$ can be represented as

$$f'(x, u) = (f'(x), u) - 0,$$

it follows that

$$\| f'(x,.) \|_{\overleftarrow{DCH(K_x)}} \leqslant \sup_{u \in K_x, \|u\| = 1} | (f'(x), u) | \leqslant \| f'(x) \| .$$

Therefore

$$\| f \|_{qC_\Omega^{(1)}} \leqslant \| f \|_{C_\Omega^{(1)}},$$

To establish the converse inequality we take an arbitrary vector $x \in \mathrm{int}\ \Omega$ and denote $u_0 = \dfrac{f'(x)}{\| f'(x) \|}$. Then we have $\| f'(x) \| = \left(f'(x), \dfrac{f'(x)}{\| f'(x) \|} \right) =$ $= (f'(x), u_0) = f'(x, u_0) \leqslant \sup\limits_{x \in int \Omega} \| f'(x,.) \|_{DCH(R_n)} \leqslant \sup\limits_{x \in \Omega} \| f'(x,.) \|_{\overleftarrow{DCH(K_x)}}$

i. e.

$$\max_{x \in \Omega} \| f'(x) \| = \sup_{x \in int \Omega} \| f'(x) \| \leqslant \sup_{x \in \Omega} \| f'(x,.) \|_{\overleftarrow{DCH(K_x)}} .$$

Therefore

$$\| f \|_{C_\Omega^{(1)}} \leqslant \| f \|_{qC_\Omega^{(1)}}$$

which together with $\| f \|_{qC_\Omega^{(1)}} \leqslant \| f \|_{C_\Omega^{(1)}}$ implies (3. 21).

The theorem follows.

THEOREM 3. 5. $qC_\Omega^{(1)}$ *is dense in* C_Ω

Proof. It is evident that $qC_\Omega^{(1)} \subset C_\Omega$. That $qC_\Omega^{(1)}$ is dense in C_Ω follows from Stone-Veierstrass' Theorem applied to $qC_\Omega^{(1)}$. The proof is thus complete.

Following [5] we say that a Banach space E_1 is normally embedded into a Banach space E_0 if $E_1 \subset E_0$ and for every $f \in E_1$ we have

$$\| f \|_{E_0} \leqslant \| f \|_{E_1}.$$

From Theorem 3. 5 and the fact that

$$\|f\|_{C_\Omega} \leqslant \|f\|_{qC_\Omega^{(1)}}$$

We derive the fol owing

COROLLARY 3. 5. 1. *The Banach space* $qC_\Omega^{(1)}$ *is normally embedded into* C_Ω.

4. Q. D. AND D. C. FUNCTIONS ON [0,1].

This final section is devoted to the spaces of q. d. and d. c. functions on [0,1]. The simple structure of [0,1] allows us to establish some interesting facts about these spaces. Along the way shall illustrate the results obtained in §3.

1. We shall first consider the space of q. d. functions on [0,1]. Observe that both Definitions 1.4 and 1.5 can be used. Using for instance Definition 1.4 we can easily establish the following criterion for quasidifferentiability of directionally differentiable functions.

Let f be a function on [0,1].

PROPOSITION 4. 1. *Suppose that f is directionally differentiable at x. Then f is q. d. at x if and only if $f'(x,1)$, $f'(x,-1)$ are finite (at $x=0$ and $x=1$ only $f'(0,1)$ and $f'(1,-1)$ are considered, respectively).*

Further, a concept of k-times q. d. functions can be defined in the following way. Let f be quasidifferentiable on [0,1]. By setting $f'(0,-1)=f'(0,1)$, $f'(1,1)=f'(1,-1)$ we obtain two functions $f'(.,1)$, $f'(.,-1)$ which are defined on the whole interval [0, 1].

DEFINITION 4. 1. *A q. d. function f on [0,1] is called twice quasidifferentiable at x if $f'(.,1)$, $f'(.,-1)$ are quasidifferentiable at x.*

Suppose now that $f(x)$ is twice-quasidifferentiable on [0, 1] and that the q. d. functions $f,(x,1)$, $f'(x,-1)$ satisfy the inequality (2.1). By repeated application of Theorem 2. 1 one can establish the interesting fact that if the directional derivatives of a twice-quasidifferentiable on [0,1] function f satisfy an inequality of the type (2. 1) then as a matter of fact, f is continuously differentiable, with a q. d. derivative.

The concept of k—times quasidifferentiability can be defined in the same way.

Turning now to the space $q\,C^{(1)}[0,1]$ of q. d. functions on [0, 1] that satisfy

$$\sup_{x\in[0,1],\, u=\pm 1} |f'(x,u)| < \infty,$$

we can show by a straightforward calculation that the norm (3.14) becomes

$$\|f\|_{qC^{(1)}_{[0,1]}} = \|f\|_{C_{[0,1]}} + \sup_{x\in[0,1]} \max \{ | f'(x, 1) | , | f'(x,-1) | \}$$

(recall that $f'(0,-1) = f'(0,1)$, $f'(1,1) = f'(1,-1)$).

Therefore it is evident that for ever y $f \in C^{(1)}_{[0,1]}$ we have

$$\|f\|_{qC^{(1)}_{[0,1]}} = \|f\|_{C^{(1)}_{[0,1]}} = \|f\|_{C_{[0,1]}} + \|f'\|_{C_{[0,1]}} .$$

2. We conclude by some remarks on d. c. functions on [0,1]. In recent years, d. c. functions have attracted much attention from researchers in connection problems of global optimization. We refer the interested reader to [4] for a detailed treatment of these functions. Let us recall only the following

PROPOSITION 4.2. [3] *A function $f: (a,b)\subset(-\infty, \infty)\to(-\infty, \infty)$ is d.c. on (a,b) if and only if*

1) f has directional derivatives $f'(x, 1), f'(x, -1)$ (where these derivatives are meaningful)

2) $f'(x, 1)$, $f'(x, -1)$ are of bounded variation over all of $[c, d] \subset (a, b)$.

Now let f be a d.c. function on [0, 1] and set $f'(0, -1) = f'(0,1)$, $f'(1, 1) = f'(1, -1)$. Observe that, although $f'(x, 1)$, $f'(x, -1)$ belong to $V_{[c,d]}$ for all $[c, d] \subset (0, 1)$, $f'(x,1)$, $f'(x, -1)$ may not belong to $V_{[0,1]}$ (where $V_{[c,d]}$ denotes the Banach space of functions of bounded variation on $[c, d]$). This occurs for example for the function

$$f(x) = \sqrt{1-(1-x)^2} + \sqrt{1-x^2} \qquad x \in [0, 1].$$

It is a simple matter to prove the following

PROPOSITION 4.3. *Let f be a d.c. function on $[0,1]$. Then f can be extended to a d.c. function on $(-\infty, \infty)$ if and only if $f'(x,1)$, $f'(x, -1)$ belong to $V_{[0,1]}$.*

As is well-known, for these functions, $f'(x,1)$, $f'(x,-1)$ are identical almost everywhere. This fact together with the equality $f'(0,-1) = f'(0,1)$ gives, by virtue of Theorem 24.1 in [7] :

$$\| f'(\cdot, 1) \|_{V_{[0,1]}} = \| f'(\cdot, -1) \|_{V_{[0,1]}}.$$

Observe, moreover, that

$$\max \{ \sup_{x\in[0,1]} | f'(x,1) |, \sup_{x\in[0,1]} | f'(x, -1) |\} \leqslant \bigvee_0^1 f'(\cdot,1) \leqslant \| f'(\cdot, 1) \|_{V_{[0,1]}} .$$

Therefore applying Theorem 2.1 we obtain

$$f(x) = f(0) + \int_0^x f'(\lambda, 1)\, d\lambda. \tag{4.1}$$

From this and the completeness of $V_{[0,1]}$ we can derive

THEOREM 4.1. *The family $DC_{[0,1]}$ of d.c. functions whose directional derivatives $f'(x, 1)$, $f'(x, -1)$ belong to $V_{[0,1]}$ is a Banach space with the norm*

$$\|f\|_{DC_{[0,1]}} = |f(0)| + \|f'(\cdot, 1)\|_{V_{[0,1]}} . \tag{4.2}$$

Remark 4.1. The norm (4.2) is different from the norm (3.10) applied to $\Omega = [0, 1]$.

Finally, it is worth mentioning a close relation between the space of d.c. functions and that of functions of bounded variation. Let

$$V^o_{[0,1]} = \{f \in V_{[0,1]} : mes\,\{x : f(x) \neq 0\} = 0\},$$

$$\widetilde{V}_{[0,1]} = V_{[0,1]} / V^o_{[0,1]},$$

$$DC^o_{[0,1]} = \{f \in DC_{[0,1]} : f(0) = 0\}.$$

If we define a correspondence « ↔ » between $\widetilde{V}_{[0,1]}$ and $DC^o_{[0,1]}$ by setting $f \leftrightarrow g$ if $f \in DC^o_{[0,1]}$, $\widetilde{g} = \{g, h, \dots : g - h \in V^o_{[0,1]}\} \in \widetilde{V}_{[0,1]}$ and $f'(x,1) \in \widetilde{g}$, then it is easy to prove.

PROPOSITION 4.4. *The just described correspondence between $DC^o_{[0,1]}$ and $\widetilde{V}_{[0,1]}$ is an isomorphism.*

On the other hand, if we consider $DC_{[0,1]}$ and $V_{[0,1]}$ as Banach spaces ordered by cones $K_{DC_{[0,1]}} = \{f \in DC_{[0,1]} : f(x) \geqslant 0, \forall x \in [0,1]\}$, $K_{V_{[0,1]}} = \{f \in V_{[0,1]} : f(x) \geqslant 0, \forall x \in [0,1]\}$ then in the terminology of the theory of ordered Banach spaces [1], the cone $K_{V_{[0,1]}}$ is d-extremal but is not normal [10]. As one might expect, so is the cone $K_{DC_{[0,1]}}$. Thus between $DC_{[0,1]}$ and $V_{[0,1]}$ there exists a close relationship.

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