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ON THE SOLVABILITY OF DUAL INTEGRAL EQUATIONS INVOLVING FOURIER TRANSFORM

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1. INTRODUCTION

Dual integral equations arise when integral transforms are used to solve mixed boundary value problems of mathematical physics and mechanics. Formal techniques for solving such integral equations have been extensively developed during the last three decades, but not so many efforts have been made to determine the conditions for the validity of various procedures [1, 2].

In [3] some results on dual series equations with general kernels have been obtained. The aim of the present paper is to consider existence and uniqueness problems for dual integral equations involving Fourier transform of generalized functions, which are a generalization of some equations encountered in mixed boundary value problems of mathematical physics and contact problems of elasticity [1, 2, 4].

2. FUNCTIONAL CLASSES

In this section we present definitions and auxiliary propositions from the theory of Sobolev spaces [5, 6]. Denote by $S = S(R)$ the space of quickly decreasing test functions and by S' the dual space of S, where $R = (-\infty, \infty)$. It is well-known that the direct and inverse Fourier transforms are defined on S by

$$\tilde{\varphi}(t) \equiv F[\varphi](t) = \int_{-\infty}^{\infty} \varphi(x)\, e^{ixt}\, dx,$$

$$\varphi(x) = F^{-1}[\tilde{\varphi}](x) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\varphi}(t)\, e^{-ixt}\, dt$$

and on S' by

$$\langle F^{\pm 1}[f], F^{\pm 1}[\varphi] \rangle \equiv (2\pi)^{\pm 1} \langle f, \varphi \rangle, \tag{1}$$

where $\varphi \in S$, $f \in S'$, $\langle f, \varphi \rangle$ denotes the value of the distribution $f \in S'$ on the test function $\bar{\varphi} \in S$.

Let $H_s(R)$ $(s \in R)$ be the Sobolev space defined as the set of all generalized functions $u \in S'$ the Fourier transforms $\tilde{u}(t)$ of which satisfy the condition [5, 6]

$$\| u \|_s^2 = \int_{-\infty}^{\infty} (1 + (t)^{2s} \mid \tilde{u}(t) \mid^2 dt < \infty. \tag{2}$$

The scalar product in $H_s(R)$ is given by the formula

$$(u, v)_s = \int_{-\infty}^{\infty} (1 + | t |)^{2s} \tilde{u}(t) \overline{\tilde{v}(t)} \, dt. \tag{3}$$

It is well-known [6] that in the case $s = m$, where m is a non-negative integer, the norm (2) is equivalent to the norm

$$\| u \|_m^{,2} = \sum_{k=0}^{m} \int_{-\infty}^{\infty} | D^k u(x) |^2 \, dx \ (D = id/dx) \tag{4}$$

and in the case $s \geqslant 0$ to that given by

$$\| u \|_s^{,,2} = \| u \|_0^{,2} + \iint_{-\infty}^{\infty} \frac{| \overset{[s]}{D}u(x) - \overset{[s]}{D}u(y) |^2}{| x - y |^{1+2s-2[s]}} \, dx \, dy, \tag{5}$$

where $[s]$ is the integral part of the number s.

LEMMA 1. *Let $\eta(x)$ be a function infinitely differentiable on R such that $|D^k \eta(x)| \leq C_k$ for $k = 0, 1, 2, \ldots$, where C_k are positive constants. Then for any $u \in H_s(R)$, $\eta(x)u$ belongs to $H_s(R)$.*

Proof. For non-negative integers s the above assertion is obvious in view of the equivalence of the norms (2) and (4), for non-negative numbers s this assertion follows from the equivalence of (2) and (5). Now suppose that $u \in H_{-s}(R)$, where $s \geqslant 0$. Taking account of the formula (1) and the fact that the set $C_0^{\infty}(R)$ of infinitely differentiable functions with a compact support in R is dense in $H_s(R)$ for all $s \in R$ with respect to the norm (2) [6] one can prove that

$$\int_{-\infty}^{\infty} F[\eta u] (t) \overline{F[v] (t)} \, dt = \int_{-\infty}^{\infty} F[u] (t) \overline{F[\eta v] (t)} \, dt$$

for any $v \in H_s(R)$.

It is known that $H_{-s}(R) = (H_s(R))^*$, where $(H_s(R))^*$ is the dual space of $H_s(R)$ [6]. Since $u \in H_{-s}(R)$ and $\overline{\eta} v \in H_s(R)$, then by the Buniakowsky inequality it is easy to see that ηu is a linear continuous functional on $H_s(R)$, i. e. $\eta u \in (H_s(R))^* = H_{-s}(R)$. The proof is thus complete.

Let ω be an open set in R. Denote by $\overset{0}{H}_s(\omega)$ the subspace of $H_s(R)$ consisting of all functions $u \in H_s(R)$ with support in $\overline{\omega}$. The norm in $\overset{0}{H}_s(\omega)$ is also defined by (2). Denote by p the restriction operator $pu = u/\omega$ and by $H_s(\omega)$ the space of functions pu with the norm

$$\|u\|_{H_s(\omega)} \equiv \inf_{l} \| lu \|,$$

where the infimum is taken over all possible extensions $lu \in H_s(R)$.

From [5—7] it follows that if ω is either a halfline or a finite integral, then the set $\overset{\infty}{C_0}(\omega)$ of infinitely differentiable functions with compact support n $\overline{\omega}$ is dense in $\overset{0}{H}_s(\omega)$ with respect to the norm of $H_s(R)$ for all $s \in R$.

The following lemma gives another condition under which this statement is also valid.

LEMMA 2. *Let* $\omega = \bigcup_{k=1}^{N} \omega_k$, *where* $\omega_k = (a_k, b_k)$,

$\overline{\omega}_k \cap \overline{\omega}_j = \phi, \quad k \neq j.$ *Then the set* $C_0^\infty(\omega)$ *is dense in* $\overset{0}{H}_s(\omega)$ *with respect to the norm of* $H_s(R)$ *for any* $s \in R$.

Proof. Let

$$\omega_n^\delta \equiv \{x : |x - y| < \delta, \quad y \in \omega_n\}; \; n = 1,2,\dots, N,$$

where δ is a positive number.

Let $\eta_n(x)$ be functions satisfying the assumptions of Lemma 1, and such that $\eta_n(x) = 1$ when $x \in \omega_n^\delta$; $\eta_n(x) = 0$ when $x \in R \setminus \omega_n^{2\delta}$. Here δ is assumed to be so small that $\overline{\omega}_n^{2\delta} \cap \overline{\omega}_{n+1}^{2\delta} = \phi$.

Putting

$$\eta(x) \equiv \eta_1(x) + \eta_2(x) + \dots + \eta_N(x), \; x \in R,$$

we see that $\eta(x) \in C^\infty(R)$ and $\eta(x) = 1$ in the neighbourhood of ω. Therefore, if $u \in \overset{0}{H}_s(\omega)$ then in the sense of distributions we have

$$u = u_1 + u_2 + \dots + u_N,$$

where $u_n = \eta_n u$; $n = 1,2, \dots, N$.

Obviously, supp $u_n \in \overline{\omega_n}$. Therefore in view of Lemma 1 the functions u_n belong to $H_s(\omega_n)$. Since ω_n is either a semiaxis or a finite integral, there exists for any u_n a function $\varphi_n \in C_0^\infty(\omega_n)$ such that

$$\| u_n - \varphi_n \|_s < \varepsilon/N,$$

where ε is an arbitrary positive, small enough number.

It is easy to see that the function

$$\varphi(x) \equiv \varphi_1(x) + \varphi_2(x) + \dots + \varphi_N(x)$$

approximates the function u with an error $\| u - \varphi \|_s < \varepsilon$. This completes the proof.

THEOREM 1. *For every $u \in \overset{0}{H}_s(\omega)$ and $f \in H_{-s}(\omega)$ the integral*

$$(f, u)_0 \equiv \int_{-\infty}^{\infty} \widetilde{lf}(t)\overline{\widetilde{u}(t)}\, dt. \tag{7}$$

where $\widetilde{u}(t) = F[u]$, exists and does not depend upon the choice of the extension lf. Therefore, this integral defines a linear continuous functional on $\overset{0}{H}_s(\omega)$.

Conversely, for every linear continuous functional $\phi(u)$ on $\overset{0}{H}_s(\omega)$ there exists a function $f \in H_{-s}(\omega)$ such that $\phi(u) = (f, u)_0$ and $\| \phi \| = \| f \|_{H_{-s}(\omega)}$.

Theorem 1 follows from Lemma 2 by an argument similar to that used in the proof of Lemma 4. 8 in [6].

By Theorem 1 we can identify $H_{-s}(\omega)$ with the dual space of $\overset{0}{H}_s(\omega)$, i. e.

$$H_{-s}(\omega) = (\overset{0}{H}_s(\omega))^*.$$

We shall need in the sequel the following class of functions. Let α be a real number. Denote by $\sigma_\alpha(R)$ the class of continuous functions $k(t)$, $t \in R$, such that $(1 + |t|)^{-\alpha} k(t) = O(1)$ when $|t| \to \infty$. By definition there is a positive constant C such that

$$|k(t)| \leqslant C(1 + |t|)^\alpha, \ t \in R.$$

We shall say that the function $k(t)$ belongs to the class $\sigma_\alpha^0(R)$ if $k(t) \in \sigma_\alpha(R)$ and $k(t) \geqslant 0$. Finally, the function $k(t)$ belongs to the class $\sigma_\alpha^\pm(R)$ if $k^{\pm 1}(t) \in \sigma_{\pm\alpha}(R)$ respectively. It follows that the function $k(t)$ belongs to the class $\sigma_\alpha^\pm(R)$ iff

$$C_1(1 + |t|)^\alpha \leqslant k(t) \leqslant C_2(1 + |t|)^\alpha, \ t \in R, \tag{8}$$

where C_1, C_2 are positive constants.

LEMMA 3. *Let $k(t) > 0$ and such that $(1 + |t|)^{-\alpha} k(t)$ is a bounded continuous function on R; suppose, moreover, that the positive limits of the function $(1 + |t|)^{-\alpha} k(t)$ exist when $t \to \pm\infty$. Then $k(t) \in \sigma_\alpha^\pm(R)$.*

Proof. Assume that

$$\lim_{t \to \pm\infty} (1 + |t|)^{-\alpha} k(t) = \lambda^{\pm}, \quad \lambda^{\pm} > 0.$$

Then for any positive small enough number $\varepsilon < \lambda^{\pm}$ there is a positive number R_0 such that

$$\lambda_- \leqslant (1 + |t|)^{-\alpha} k(t) \leqslant \lambda_+ , \; |t| \geqslant R_0 ,$$

where

$$\lambda_- = \min \{\lambda^{\pm} - \varepsilon\} , \; \lambda_+ = \max \{\lambda^{\pm} + \varepsilon\} .$$

Further, from the assumption it follows that the function $(1 + |t|)^{-\alpha} k(t)$ attains its greatest value M and smallest value m in the interval $[-R_0, R_0]$. Therefore, (8) holds with

$$C_1 = \min \{\lambda_-, m\}, \; C_2 = \max\{\lambda_+, M\}.$$

This completes the proof.

We now turn to the discussion of the solvability of dual integral equations.

3. EXISTENCE AND UNIQUENESS THEOREMS

1. In this section we shall consider the following dual integral equation

$$\begin{aligned} p F^{-1}[k(t)\tilde{u}(t)](x) &= f(x), \; x \in \omega, \\ p' F^{-1}[\tilde{u}(t)](x) &= g(x), \; x \in \omega'. \end{aligned} \tag{9}$$

Here $\omega' = R \setminus \omega$, $u = F^{-1}[\tilde{u}(t)] \in S'$ is a function to be found, $k(t)$ is a non-negative function (called the symbol of the given equation), $f \in \mathcal{D}'(\omega)$ and $g \in \mathcal{D}'(\omega')$ are given distributions on ω and ω' respectively; p and p' are restriction operators to ω and ω' respectively.

We shall investigate the dual equation (9) under the following conditions:

$$k(t) \in \overset{+}{\sigma}_{2\alpha} \, (\overset{0}{\sigma}_{2\alpha}) , \; f \in H_{-\alpha}(\omega) , \; g \in H_{\alpha}(\omega') \tag{10}$$

and we shall find the function u in the space $H_\alpha(R)$.

THEOREM 2. *(Uniqueness). Under the assumptions (10) the dual equation (9) has at most one solution in $H_\alpha(R)$.*

Proof. To prove the theorem it suffices to show that the homogeneous dual equation

$$\begin{aligned} p F^{-1}[k(t)\tilde{u}(t)](x) &= 0, \; x \in \omega, \\ p' F^{-1}[\tilde{u}(t)](x) = u(x) &= 0, \; x \in \omega' \end{aligned}$$

has only the trivial solution.

Since $u \in \overset{\circ}{H}_\alpha(\omega)$ the last equation can be rewritten as

$$(Ku)(x) = 0, \qquad x \in \omega, \tag{11}$$

where

$$(Ku)(x) \equiv p\, F^{-1}[k(t)\, \tilde{u}(t)](x). \tag{12}$$

Since $Ku \in H_{-\alpha}(\omega) = (\overset{\circ}{H}_\alpha(\omega))^*$ (see Theorem 1) we obtain from (7)

$$(Ku, u)_0 = \int_{-\infty}^{\infty} F[lKu](t)\, \overline{F[u](t)}\, dt,$$

where lKu is an arbitrary extension of Ku from ω onto R, $lKu \in H_{-\alpha}(R)$. Since the last integral does not depend upon the choice of lKu we can take

$$lKu = lpF^{-1}[k\tilde{u}](x) = F^{-1}[k\tilde{u}](x).$$

Then it is easy to see that

$$(Ku, u)_0 = \int_{-\infty}^{\infty} k(t)\, |\tilde{u}(t)|^2\, dt = 0$$

if the function $u = F^{-1}[\tilde{u}](x)$ satisfies the equation (11). From this it follows that $u = \tilde{u} = 0$ since $k(t) \geqslant 0$ ($k(t) \not\equiv 0$). Q.E.D.

LEMMA 4. *The dual integral equation (9) is equivalent to the following pseudo-differential equation*

$$p\, F^{-1}[k(t)\, \tilde{v}(t)](x) = f(x) - pF^{-1}[k(t)\, \widetilde{l'g}(t)](x), \tag{13}$$

where $v = F^{-1}[\tilde{v}] \in \overset{\circ}{H}_\alpha(\omega)$ *satisfies the condition*

$$v + l'g = u \in H_\alpha(R) \tag{14}$$

($l'_g \in H_\alpha(R)$ being an arbitrary extension of the function g from ω' onto R).

Proof. Assume that $u \in H_\alpha(R)$ satisfies the dual equation (9) and $l'g \in H_\alpha(R)$ is an arbitrary extension of the function $g \in H_\alpha(\omega')$. Taking $v = u - l'g$ we get $v \in \overset{\circ}{H}_\alpha(\omega)$. Putting (14) into the first equality in (9) we have (13). The right-hand side of (13) belongs to $H_{-\alpha}(w)$ in view of Lemma 4.4 in [6].

Conversely, assume that $v \in \overset{\circ}{H}_\alpha(\omega)$ satisfies the equation (13). Then obviously, the function u defined by (14) belongs to $H_\alpha(R)$. We shall prove that this function satisfies the dual equation (9) in the sense of distributions. Indeed, in transfering the second member in the right-hand side of (13) to the left-hand side and using (14) we obtain the first equality in (9). Further, from (14) it follows the second equality of this equation. Q.E.D.

Denote

$$h(x) = f(x) - \rho F^{-1} [k(t) \tilde{l}'_g(t)] (x). \tag{15}$$

Using (12) we can rewrite (13) as

$$(Kv)(x) = h(x), \quad x \in \omega. \tag{16}$$

Our purpose now is to establish the existence of solution of the equation (16) in the space $\overset{\circ}{H}_\alpha(\omega)$. To this end we shall consider the following cases.

2. The case $k(t) = k_+(t) \in \sigma^+_{2\alpha}(R)$. It is clear that in this case the norm in $H_\alpha(R)$ defined by (2) is equivalent to the norm defined by (cf. (8))

$$\| v \|^2_{k_+} = \int_{-\infty}^{\infty} k_+(t) \, |\tilde{v}(t)|^2 dt.$$

The norm $\| v \|_{k_+}$ can also be introduced by the following scalar product in $H_\alpha(R)$

$$(v, w)_{k_+} = \int_{-\infty}^{\infty} k_+(t) \tilde{v}(t) \overline{\tilde{w}(t)} \, dt.$$

We shall also write $K_+ v$ instead of Kv

THEOREM 3. *(Existence). If* $h \in H_{-\alpha}(\omega)$, $k(t) = k_+(t) \in \sigma^+_{2\alpha}(R)$ *then the equation (16) has a unique solution* $v \in \overset{\circ}{H}_\alpha(\omega)$.

Proof. By an argument similar to that used in the proof of the Theorem 2 we can show that

$$(K_+ v, w)_0 = \int_{-\infty}^{\infty} k_+(t) \tilde{v}(t) \overline{\tilde{w}(t)} \, dt = (v, w)_{k_+}$$

for arbitrary functions v and w belonging to $\overset{\circ}{H}_\alpha(\omega)$. Therefore, if $v \in \overset{\circ}{H}_\alpha(\omega)$ satisfies (16) then the following equality holds

$$(v, w)_{k_+} = (h, w)_0, \ \forall \, w \in \overset{\circ}{H}_\alpha(\omega). \tag{17}$$

We shall demonstrate that if (17) holds for any $w \in \overset{\circ}{H}_\alpha(\omega)$ then the function v will satisfy the equation (16) in the sense of $\mathcal{D}'(\omega)$. In fact, noting that (17) holds for $w = \varphi \in \overset{\circ}{C}{}^{\infty}(\omega)$ we get from (1) and (7)

$$(h, \varphi)_0 = \int_{-\infty}^{\infty} \widetilde{lh}(t)\, \overline{\tilde{\varphi}(t)}\, dt = 2\pi \langle lh, \varphi \rangle,$$

$$(v, \varphi)_{k_+} = \int_{-\infty}^{\infty} F[F^{-1}[k_+ \tilde{v}]](t)\, \overline{F[\varphi](t)}\, dt = 2\pi \langle F^{-1}[k_+ \tilde{v}], \varphi \rangle.$$

Hence

$$\langle F^{-1}[k_+ \tilde{v}], \varphi \rangle = \langle lh, \varphi \rangle,\ \varphi \in \overset{\circ}{C}{}^{\infty}(\omega),$$

i.e.

$$pF^{-1}[k_+ \tilde{v}](x) = plh(x) = h(x),\ x \in \omega.$$

We now return to the relation (17). Since $(h, w)_0$ is a linear continuous functional on the Hilbert space $\overset{\circ}{H}_{\alpha}(\omega)$, then by virtue of the Riesz theorem there exists a unique element $v_0 \in \overset{\circ}{H}_{\alpha}(\omega)$ such that

$$(h, w)_0 = (v_0, w)_{k_+},\ \forall\, w \in \overset{\circ}{H}_{\alpha}(\omega)$$

and moreover

$$\| v_0 \|_{k_+} \leqslant C \| h \|_{H_{-\alpha}(\omega)}, \tag{18}$$

where C is a positive constant.

Since (17) is equivalent to (16), the equation (16) has a unique solution $v = v_0 \in \overset{\circ}{H}_{\alpha}(\omega)$, and this completes the proof of Theorem 3.

Remark 1. It is easily seen that the operator K_+^{-1} from $H_{-\alpha}(\omega)$ into $\overset{\circ}{H}_{\alpha}(\omega)$ is bounded. This follows from Theorem 3 and inequality (18).

Remark 2. The solution u of the dual integral equation (9) expressed in terms of the solution v of the equation (13) by the formula (14) does not depend on the choice of the extension $l'g$. This fact follows from the uniqueness of solution of the dual equation (9). Hence, we can choose the extension $l'g$ such that

$$\| l'g \|_{\alpha} \leqslant 2 \| g \|_{H_{\alpha}(\omega)}.$$

In this case, from (14), (15) and (18) it is easy to obtain the following estimate

$$\| u \|_{\alpha} \leqslant C \left(\| f \|_{H_{-\alpha}(\omega)} + \| g \|_{H_{\alpha}(\omega')} \right),$$

where $C = \text{const} > 0$.

Therefore, the solution of the dual equation (9) depends continuously upon the functions given on the right-hand side.

3. The case, where $k(t) \in \overset{0}{\sigma}_{2\alpha}(R)$ and ω is bounded. Assume in addition that there is a function $k_+(t) \in \overset{+}{\sigma}_{2\alpha}$ such that

$$d(t) \equiv k(t) - k_+(t) \in \sigma_{2\alpha-\beta}\ (R),\ \beta > 0. \tag{19}$$

We now represent the operator K defined by (12) in the form $K = K_+ + D_0$, where

$$K_+ v = pF^{-1}[k_+ \tilde{v}],\ D_0 v = pF^{-1}[d\tilde{v}].$$

LEMMA 5. *Under these assumptions D_0 is a completely-continuous operator from $\overset{o}{H}_\alpha(\omega)$ into $H_{-\alpha}(\omega)$.*

Proof. Noting that the set $\overset{\infty}{C_0}(\omega)$ is dense in $\overset{o}{H}_\alpha(\omega)$ and taking account of (6) and (19), we can easily obtain the estimates

$$\| D_0 v \|_{\underset{\alpha+\beta}{H}(\omega)} \leqslant \| F^{-1}[d v] \|_{-\alpha+\beta} \leqslant C \| v \|_\alpha,$$
$$C = const > 0.$$

Therefore, the operator D_0 is continuous from $\overset{o}{H}_\alpha(\omega)$ into $H_{-\alpha+\beta}(\omega)$. On the other hand, in view of the boundedness of ω and $\beta > 0$, the immersion of $H_{-\alpha+\beta}(\omega)$ into $H_{-\alpha}(\omega)$ is completely continuous (cf. [5 — 7]). Q.E.D.

THEOREM 4 (*Existence*). *Under the above assumptions on w and k(t) (cf. (19)) for every $f \in H_{-\alpha}(\omega)$, $g \in H_\alpha(\omega)$ the dual equation (9) has a unique solution $u \in H_\alpha(R)$.*

Proof. According to Lemma 4 the dual equation (9) is equivalent to the equation (13) which is a Fredholm equation by virtue of Theorem 3, Remark 1 and Lemma 5. On the other hand, from Lemma 4 and Theorem 2 one can see that the equation (13) has a unique solution $v \in \overset{o}{H}_\alpha(\omega)$. Therefore, the dual equation (9) has also a unique solution $u \in H_\alpha(R)$. The proof is thus complete

By an analogous argument we can show that for $f \in H_{-\alpha+\beta}(\omega)$, $g \in H_{\alpha-\beta}(\omega')$ the equation

$$pF^{-1}[k\tilde{u}]\ (x) = f(x),\ x \in \omega,$$

$$p'F^{-1}[h\tilde{u}]\ (x) = g(x),\ x \in \omega'$$

has also a unique solution $u \in H_{\alpha+\beta}(R)$ in either of the following cases:

1) $k(t) \in \overset{+}{\sigma}_{2\alpha}(R)$, $h(t) \in \overset{+}{\sigma}_{2\beta}(R)$, ω is arbitrary;

2) $k'(t) \in \sigma^0_{2\alpha}(R)$, $h(t) \in \overset{+}{\sigma}_{2\beta}(R)$, ω is bounded;

3) $k(t) \in \overset{+}{\sigma}_{2\alpha}$, $h(t) \in \sigma^0_{2\beta}$, ω' is bounded under the corresponding conditions which are analogous to (19).

Finally, we also note that the cases when $k(t) \in \overset{+}{\sigma}_{\pm 1}$ $(\sigma^0_{\pm 1})$ occur frequently in practice. In these cases we must find the solution u of the dual equation (9) in spaces $H_{\mp 1/2}(R)$ and the solution v of the equation (13) in $\overset{0}{H}_{\pm 1/2}(\omega)$. From known results in [7] it follows that the function v in the cases $k(t) \in \sigma_{\pm 1}(R)$ can be represented as

$$v(x) = \rho^{\pm 1/2}(x)\, w(x) + \left(\frac{1 \mp 1}{2}\right) \psi(x), \tag{20}$$

where $w(x) \in L_2(\omega)$, $\psi(x) \in H_{-1/2}(\omega)$, $\rho(x) \in C^{\infty}(\omega)$, $\rho(x) \geqslant 0$ and $\rho(x)$ is equal to zero only on the boundary $\partial\omega$ of ω has the same order as the distance from $x \in \omega$ to $\partial\omega$.

Therefore, in these cases from (20) it follows that the function $v(x)$ in general has the singularity $\rho^{\pm 1/2}(x)$ in a neighbourhood of $\partial\omega$. This fact occurs when solving many mixed boundary value problems of mathematical physics and elasticity [1—4].

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