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# A FAMILY OF SOLUTIONS OF THE PLANE FLOW PROBLEM

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## I. INTRODUCTION

In connection with an analysis of the familiar «Stokes paradox» the problem of plane flow passing an object has been discussed by various authors ([1], [3], [4], [6], [7]). This paradox consists in the fact that a solution of the homogeneous two-dimentional Stokes system which is equal to zero on the surface of the object and equal to a given constant different from zero at infinity had not been found. In fact, if the boundary of the object is smooth then, as it was proved in [1], this problem has no solution. It is then natural to ask what will occur if this boundary is not smooth.

In this paper, using the Fourier transform we shall find out a family of solution for a very particular case of the above mentioned problem, assuming that the object is an interval which is parallel to the speed of flow at infinity. Let

$$\Gamma = \{ (x, y) \in R^2 ; -l < x < l, y = 0\} \qquad (1)$$

and $G = R^2 \setminus \Gamma$. In the region $G$ we consider the following Stokes's system of viscous incompressible flow:

$$\Delta u + \frac{\partial p}{\partial x} = 0, \qquad (2)$$

$$\Delta v + \frac{\partial p}{\partial y} = 0, \qquad (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad (4)$$

with the following boundary conditions:

$$\lim_{|x|+|y|\to\infty} u(x, y) = a \quad , \quad \lim_{|x|+|y|\to\infty} v(x, y) = 0, \qquad (5)$$

$$u(x, y) \mid_{\Gamma} = 0 \quad , \quad v(x, y) \mid_{\Gamma} = 0, \qquad (6)$$

where $u = u(x, y)$, $v = v(x, y)$ are components of the speed of the flow and $p = p(x, y)$ is its pressure.

Our method for solving the problem (2) — (6) is as follows:

First of all we solve the system (2) — (4) in the domain $G$. For this, we try to find a solution in the whole $R^2$ of the system

$$\Delta u + \frac{\partial p}{\partial x} = f'(x)\,\delta(y) \,, \tag{7}$$

$$\Delta v + \frac{\partial p}{\partial y} = 0 \,, \tag{8}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \,, \tag{9}$$

where $f(x)$ is an unknown summable function on $R^1$ with supp $f \subset [-l, l]$ and $g(x)\,\delta(y)$ is a direct product of two distributions $g(x)$ and $\delta(y)$, i.e.

$$\langle g(x)\,\delta(y), \varphi(x,y) \rangle = \langle g(x), \varphi(x,0) \rangle \tag{10}$$

for $\varphi(x,y) \in S(R^2)$.

Then we define the function $f(x)$ satisfying the boundary conditions (5), (6). It is very interesting to note that in this case the problem (2) — (6) has more than one solution.

## 2. STUDY OF THE SYSTEM (7) — (9)

Let $\widehat{\varphi}(\xi) = (F\varphi)(\xi) = (2\pi)^{-\frac{n}{2}} \int e^{-i(z,\xi)}\, \varphi(z)\, dz$ be the Fourier transform of a function $\varphi(z) \in S(R^n)$. The Fourier transform $\widehat{h}(\xi)$ of a distribution $h(z) \in S'(R^n)$ is defined as a linear continuous functional on $S(R^n)$ such that

$$\langle \widehat{h}(\xi), \varphi(\xi) \rangle = \langle h(z), \widehat{\varphi}(z) \rangle .$$

We shall use the formulas:

$$\varphi(z) = F^{-1}\widehat{\varphi} = (2\pi)^{-\frac{n}{2}} \int e^{i(z,\xi)}\, \widehat{\varphi}(\xi)\, d\xi, \tag{11}$$

$$\widehat{\frac{\partial^{|\alpha|} \varphi}{\partial z^\alpha}}(\xi) = (i\xi)^\alpha\, \widehat{\varphi}(\xi), \tag{12}$$

$$\widehat{\varphi * \psi}(\xi) = (2\pi)^{\frac{n}{2}}\, \widehat{\varphi}(\xi)\, \widehat{\psi}(\xi), \tag{13}$$

$$\widehat{a}(\xi) = (2\pi)^{\frac{n}{2}}\, a\, \delta(\xi), \tag{14}$$

$$\widehat{g(x)\,\delta(y)}(\xi,\eta) = (2\pi)^{-\frac{1}{2}}\, \widehat{g}(\xi), \; z = (x,y) \in R^2, \; \xi = (\xi,\eta) \in R^2, \tag{15}$$

where $\varphi * \psi$ denotes a convolution of two functions $\varphi(z)$ and $\psi(z)$, $a(z) = a =$ const.

It is well known that in the case $n = 2$, for the functions $\frac{\xi\eta}{(\xi^2+\eta^2)^2}$ and $\frac{\xi^2}{\xi^2+\eta^2}$, the inverse Fourier transform exist in the sense of principae value of an integral (see [5], p. 124 — 128), and

$$F^{-1}\left(\frac{\xi\eta}{(\xi^2+\eta^2)^2}\right)=K_0(x,y), \tag{16}$$

$$F^{-1}\left(\frac{\xi^2}{\xi^2+\eta^2}\right)=2\pi^2\,\delta(x,y)+K_2(x,y), \tag{17}$$

where $K_m(x,y)$ is homogeneous function of degree $-m$,

$$K_m(x,y)\in C^\infty(R^2\setminus\{(0,0)\}),\ \int\limits_{x^2+y^2=1} K_m(x,y)\,d\omega=0.$$

PROPOSITION 1. *Let $f(x)$ be a summable function on $R^1$ with $\operatorname{supp} f\subset[-l,l]$; $K_0(x,y)$, $K_2(x,y)$ be defined by (16), (17) Then the system of the following functions;*

$$u(x,y)=a-\frac{1}{2\pi}\frac{\partial K_0(x,y)}{\partial y}*f(x)\delta(y), \tag{18}$$

$$v(x,y)=\frac{1}{2\pi}\frac{\partial K_0(x,y)}{\partial x}*f(x)\delta(y), \tag{19}$$

$$p(x,y)=\pi f(x)\delta(y)+\frac{1}{2\pi}K_2(x,y)*f(x)\delta(y), \tag{20}$$

*defines a solution of the system (7) — (9).*

*Proof*. The Fourier transform, applied to both sides of the system (7)—(9), yields

$$-(\xi^2+\eta^2)\,\widehat{u}(\xi,\eta)+i\xi\,\widehat{p}(\xi,\eta)=(2\pi)^{-\frac{1}{2}}\,i\xi\widehat{f}(\xi), \tag{21}$$

$$-(\xi^2+\eta^2)\,\widehat{v}(\xi,\eta)+i\eta\widehat{p}(\xi,\eta)=0, \tag{22}$$

$$i\xi\widehat{u}(\xi,\eta)+i\eta\widehat{v}(\xi,\eta)=0. \tag{23}$$

In view of (14) and of the fact that $h(\xi,\eta)\delta(\xi,\eta)=0$ for $h(\xi,\eta)\in C^\infty(R^2)$ and $h(0,0)=0$, the following functions $\widehat{u}(\xi,\eta)$, $\widehat{v}(\xi,\eta)$ satisfy (21), (22):

$$\widehat{u}(\xi,\eta)=\widehat{a}+\frac{i\xi}{\xi^2+\eta^2}\widehat{p}(\xi,\eta)-(2\pi)^{-\frac{1}{2}}\frac{i\xi}{\xi^2+\eta^2}\widehat{f}(\xi), \tag{24}$$

$$\widehat{v}(\xi,\eta)=\frac{i\eta}{\xi^2+\eta^2}\widehat{p}(\xi,\eta). \tag{25}$$

We obtain from (24). (25), that

$$i\xi\widehat{u}(\xi,\eta)+i\eta\widehat{v}(\xi,\eta)=-\widehat{p}(\xi,\eta)+(2\pi)^{-\frac{1}{2}}\frac{\xi^2}{\xi^2+\eta^3}\widehat{f}(\xi). \tag{26}$$

The equalities (26), (23) yield

$$\widehat{p}(\xi,\eta)=(2\pi)^{-\frac{1}{2}}\frac{\xi^2}{\xi^2+\eta^2}\widehat{f}(\xi). \tag{27}$$

By substituting (27) into (24) and (25) we have

$$\widehat{u}(\xi,\eta)=\widehat{a}-(2\pi)^{-\frac{1}{2}}\frac{i\xi\eta^2}{(\xi^2+\eta^2)^2}\widehat{f}(\xi), \tag{28}$$

$$\widehat{v}(\xi,\eta) = (2\pi)^{-\frac{1}{2}} \frac{i\xi^2\eta}{(\xi^2+\eta^2)^2} \widehat{f}(\xi). \tag{29}$$

Then (18) – (20) obviously follow from (27)—(29), (11) — (15), (16) and (17). The proof of the proposition is complete.

Since $K_m(x,y) \in C^\infty(R^2 \setminus \{(0,0)\})$ and $f(x)$ is summable, the functions $u(x,y)$, $v(x,y)$, $p(x,y)$, defined by (18) — (20), are infinitely differentiable out of $\Gamma$. Moreover, taking account of (10), we have

COROLLARY 1. *Provided $(x, y) \notin \Gamma$ the functions $u(x, y)$, $v(x, y)$, $p(x, y)$, defined by (18) — (20), can be represented as follows*

$$u(x, y) = a - \frac{1}{2\pi} \frac{\partial K_0(x, y)}{\partial y} \otimes f(x), \tag{30}$$

$$v(x, y) = \frac{1}{2\pi} \frac{\partial K_0(x, y)}{\partial x} \otimes f(x), \tag{31}$$

$$p(x, y) = \frac{1}{2\pi} K_2(x, y) \otimes f(x), \tag{32}$$

*where the symbol $\otimes$ denotes the convolution of functions in $x$ with fixed $y$.*

## 3. A FAMILY OF SOLUTIONS OF THE PLANE FLOW PROBLEM

The purpose of this section is to prove the following main result:

THEOREM. *The problem (2) — (6) has no unique solution. The solution of the homogeneous problem is*

$$u(x, y) = -\frac{\partial K_0(x, y)}{\partial y} \otimes \psi(x), \tag{33}$$

$$v(x, y) = \frac{\partial K_0(x, y)}{\partial x} \otimes \psi(x), \tag{34}$$

$$p(x, y) = K_2(x, y) \otimes \psi(x). \tag{35}$$

*This problem has a special solution as follows:*

$$u(x, y) = a - \frac{2a}{\pi} \frac{\partial K_0(x, y)}{\partial y} \otimes x\psi(x), \tag{36}$$

$$v(x, y) = \frac{2a}{\pi} \frac{\partial K_0(x, y)}{\partial x} \otimes x\psi(x), \tag{37}$$

$$p(x, y) = K_2(x, y) \otimes x\psi(x), \tag{38}$$

*where $K_0(x, y)$, $K_2(x, y)$ are defined by the formulas (16), (17) and $\psi(x)$ is given by:*

$$\psi(x) = \begin{cases} \dfrac{1}{\sqrt{l^2 - x^2}}, & |x| < l \\ 0, & |x| > l \end{cases}$$

We begin with some lemmas.

LEMMA 1. *Let $h(x,y) \in C^\infty(R^2 \setminus \{(0,0)\})$, $h(x,y)$ is homogeneous of degree $-1$, $\psi(x)$ be a summable function on $R^1$ with supp $\varphi \subset [-l, l]$. Then*

$$h(x, y) \otimes \varphi(x) \to 0 \quad \text{as } |x| + |y| \to \infty.$$

*Proof.* We have

$$h(x, y) \otimes \varphi(x) = \int_{-l}^{l} h(x-t, y)\, \varphi(t) dt.$$

Since $h(x, y)$ is homogeneous of degree $-1$ it is easy to see that

$$|h(x, y) \otimes \varphi(x)| \leqslant \sup_{-l \leqslant t \leqslant l} |h(x-t, y)| \cdot \int_{-l}^{l} |\varphi(t)|\, dt \leqslant$$

$$\leqslant C \sup_{-l \leqslant t \leqslant l} \frac{1}{\sqrt{(x-t)^2 + y^2}},$$

from which the assertion of the lemma follows.

COROLLARY 2. *Let the functions $u(x, y)$, $v(x, y)$ be given by (18), (19), where $K_0(x, y)$ is defined by (16) and $f(x)$ is any summable function on $R^1$ with supp $f \subset [-l, l]$. Then*

1) $\lim\limits_{|x|+|y| \to \infty} u(x, y) = a.$

2) $\lim\limits_{|x|+|y| \to \infty} v(x, y) = 0.$

*Proof.* This is an easy consequence of the Lemma 1 and the fact that the functions $\dfrac{\partial K_0(x, y)}{\partial y}$ and $\dfrac{\partial K_0(x, y)}{\partial x}$ are homogeneous of degree $-1$.

LEMMA 2. *Let $K_0(x, y)$ be defined by (16). Then the function $\dfrac{\partial K_0(x, y)}{\partial x}$, which is infinitely differentiable in $x$ when $y \neq 0$ and fixed, converges to zero as $y \to 0$ in the topology of the space $S'(R^1_x)$.*

*Proof.* It is sufficient to verify that for any $\varphi(x) \in S(R^1)$

$$\left\langle \frac{\partial K_0(x, y)}{\partial x}, \varphi(x) \right\rangle \to 0 \text{ as } y \to 0. \tag{39}$$

It follows from (16) and by definition of Fourier transform that

$$\overline{\frac{\partial K_0(., y)}{\partial x}}(\xi) = i(2\pi)^{-\frac{1}{2}} \int \frac{\xi^2 \eta}{(\xi^2 + \eta^2)^2} e^{iy\eta}\, d\eta.$$

We have

$$\left\langle \frac{\partial K_0(x, y)}{\partial x}, \varphi(x) \right\rangle = \left\langle \frac{\partial \widehat{K_0(., y)}}{\partial x}(\xi), (F^{-1}\varphi)(\xi) \right\rangle.$$

Therefore

$$\left\langle \frac{\partial K_0(x, y)}{\partial x}, \varphi(x) \right\rangle = i\,(2\pi)^{-\frac{1}{2}} \int\left[\int \frac{\xi^2\eta}{(\xi^2+\eta^2)^2} \; e^{iy\eta} d\eta\right](F^{-1}\varphi)\,(\xi)\, d\xi. \tag{40}$$

It is easy to verify that

$$\left| \frac{\xi^2\eta}{(\xi^2+\eta^2)^2} \; e^{-iy\eta}\right| \leqslant \frac{\xi^2\,|\eta|}{(\xi^2+\eta^2)^2}, \tag{41}$$

$$\int \frac{\xi^2|\eta|}{(\xi^2+\eta^2)^2}\, d\eta = \int \frac{|\eta| d\eta}{(1+\eta^2)^2} < \infty, \quad \text{for } \xi \neq 0 \tag{42}$$

$$\int \frac{\xi^2\eta}{(\xi^2+\eta^2)^2}\, d\eta = \int \frac{\eta d\eta}{(1+\eta^2)^2} = 0, \quad \text{for } \xi \neq 0. \tag{43}$$

In view of (41), (42) and since $(F^{-1}\varphi)(\xi)$ belongs to the Schwartz class $S(R^1)$ we can apply the Lebesgue theorem to pass to the limit in (40) as $y \to 0$. Then the assertion (39) follows from (43).

COROLLARY 3. *Let $v(x, y)$ be defined by (31). For any summable function $f(x)$ on $R^l$ with supp $f \subset [-l, l]$ we have*

$$\lim_{y\to 0} v(x, y) = 0,$$

*where the limit is taken in $S'(R^1_x)$. Therefore the boundary condition $v(x,y)|_\Gamma = 0$ is always satisfied.*

LEMMA 3. *Let $K_0(x, y)$ be defined by (16). Then the function $\frac{\partial K_0(x, y)}{\partial y}$, which is infinitely differentiable in $x$ for any fixed $y \neq 0$, converges to $-\frac{1}{2}$ p.v. $\left(\frac{1}{x}\right)$ in the topology of the space $S'(R^1_x)$.*

*Proof*. It is sufficient to prove that for any $\varphi(x) \in S(R^1)$

$$\left\langle \frac{\partial K_0(x, y)}{\partial y}, \varphi(x) \right\rangle \xrightarrow[y\to 0]{} -\frac{1}{2}\left\langle p.\,v.\left(\frac{1}{x}\right), \varphi(x) \right\rangle. \tag{44}$$

We have (see [5])

$$\widehat{p.\,v.\left(\frac{1}{x}\right)}(\xi) = -i\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \text{sign}\,\xi.$$

It follows from (16) that

$$\frac{\partial \widehat{K_0(.,\, y)}}{\partial y}(\xi) = i\,(2\pi)^{-\frac{1}{2}} \int \frac{\xi\eta^2}{(\xi^2+\eta^2)^2} \; e^{iy\eta} d\eta. \tag{45}$$

By definition we obtain

$$\left\langle \frac{\partial K_0(x,y)}{\partial y} + \frac{1}{2}\, p.\,v.\left(\frac{1}{x}\right), \varphi(x) \right\rangle =$$

$$= i(2\pi)^{-\frac{1}{2}} \int\left[\int \frac{\xi\eta^2}{(\xi^2+\eta^2)^2} \; e^{iy\eta} d\eta - \frac{\pi}{2}\; \text{sign}\,\xi\right](F^{-1}\varphi)\,(\xi)\, d\xi =$$

$$= i(2\pi)^{-\frac{1}{2}} \int\left[\frac{\xi\eta^2}{(\xi^2+\eta^2)^2}\; (e^{iy\eta} - 1)\, d\eta\right](F^{-1}\varphi)\,(\xi) d\xi, \tag{46}$$

where we have used the inequality

$$\int \frac{\xi\eta^2}{(\xi^2+\eta^2)^2}\,d\eta = \text{sign}\xi \int \frac{\eta^2 d\eta}{(1+\eta^2)^2} = \frac{\pi}{2}\,\text{sign}\,\xi,\ \xi \neq 0.$$

As in the proof of Lemma 3, the Lebesgue theorem can be applied for (46) as $y \to 0$, and the desired assertion (44) follows.

COROLLARY 4. *Let $u(x, y)$ be defined by (30), where $f(x)$ is any summable function on $R^1$ with supp $f \subset [-l, l]$. Then we have*

$$\lim_{y\to 0} u(x, y) = a + \frac{1}{4\pi}\int_{-l}^{l} \frac{f(t)}{x-t}\,dt, \tag{47}$$

*where the limit is taken in $S'(R^1_x)$.*

*Proof of the theorem.* In view of Corollaries 2 and 3 we have to define $f(x)$ satisfying the last condition

$$u(x,y)\mid_{\Gamma} = 0. \tag{6'}$$

It follows from (6') and (47) that

$$\frac{1}{\pi}\int_{-l}^{l} \frac{f(t)}{t-x}\,dt = 4a,\ -l < x < l. \tag{48}$$

This singular integral equation is well solved (see [2], p.446).
The homogeneous equation corresponding $a = 0$ has the solution

$$f(x) = \frac{1}{\sqrt{l^2-x^2}},\ -l < x < l, \tag{49}$$

while the nonhomogeneous equation corresponding to $a \neq 0$ has the solution

$$f(x) = 4a\,\frac{x}{\sqrt{l^2-x^2}},\ -l < x < l. \tag{50}$$

The assertions of the theorem follow now from (30) — (32), (49) and (50). The proof of the theorem is thus complete.

REFERENCES


[1] K. Finn and W. Noll, *On the uniqueness and non-existence of stokes flow*, Arch. Rat. Mech. Analysis 1 (1957), 95 — 106.

[2] Gakhov, *Boundary problems*, Nauka, Moscow, 1977 (in Russian).

[3] P. Grisvard, *Boundary value problem in non-smooth domains*, Université de Nice, France, 1981.

[4] O.A. Ladyzhenskaya *Mathematical problems of the Dynamics of a viscous incompressible liquid*, Nauka, Moscow, 1970 (in Russian).

[5] Z. Mizohata, *Partial differential equations*, Mir, Moscow. 1977 (in Russian).

[6] B.V. Rusanov, *Slow nonequilibrium flow of a viscous fluid passing a circular cylinder*, Dokl. Acad. Nauk USSR, 89 (1953). 983 – 986 (in Russian).

[7] T.S. Varjanskaya and L.A. Chudov, *Plane flow of a viscous incompressible fluid near by a sharp front edge of a plate with zero angle of attact*, Proceedings of Computer Center of Moscow University, 7 (1967), 28 – 54 (in Russian).

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