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DECOMPOSITIONS AND LIMITS FOR MARTINGALE-LIKE SEQUENCES IN BANACH SPACES

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0. INTRODUCTION

The notion of games which become fairer with time (see definition in the next section) was first introduced by Blake (1970) who proved that every real-valued game which becomes fairer with time (X_n), converges in L^1, provided $\|X_n\| \leqq X$ a. e. for some $X \in L^1_R$ and for all $n \in N$. Three years later, Mucci (1973) and Subramanian (1973) extended (independently) the above-mentioned result to the real-valued uniformly integrable case. In the present note we prove first that every L^1-bounded real-valued game fairer with time converges in probability, using the structure results of Talagrand (1985) for martingales in the limit. Further, Neveu (1972) proved that every L^1-bounded Banach space-valued martingale converges (strongly) to zero a. e. if it converges scalarly to zero a. e. In this note we extend also this wellknown result of Neveu to Banach space-valued games which become fairer with time. Finally, using this extension and the structure results of Talagrand (1985) we show that every L^1-bounded Banach space-valued game fairer with time (X_n) can be written in a unique form: $X_n = M_n + P_n$, where (M_n) is a uniformly integrable martingale and (P_n) is a game fairer with time that goes to zero in probability. Hence, every such a game fairer with time taking values in a Banach space having the Radon-Nikodym property, converges in probability. Thus, the main convergence problem for games which become fairer with time is solved.

1. DEFINITIONS AND PRELIMINARIES

Throughout this paper, let $(\Omega, \mathcal{A}, P)$ be a complete probability space, $(\mathcal{A}_n)$ an increasing sequence of sub-σ-fields of $\mathcal{A}$, F a real separable Banach space and F^* the topological dual of F. As usual, L^1_F denotes the Banach space of all (equivalence classes of) $\mathcal{A}$ — measurable functions $X : \Omega \to F$ such that

$$E(\| X \|) = \int_{\Omega} \| X \| \, dP < \infty.$$

All sequences (X_n), considered in this paper are assumed to be adapted to $(\mathcal{A}_n)$, i. e., X_n is $\mathcal{A}_n$ — measurable for any n.

A sequence (X_n) is said to be a martingale, if $X_n(m) = X_n(m \geqq n \in \in N)$, where $X_n(m)$ denotes the $\mathcal{A}_n$ — conditional expectation of X_m(cf. [7]).

It is clear that every martingale (X_n) is a mil, i. e.

$$\forall_{\varepsilon > 0} \; \exists_p \forall_n \, (p \leq n) \; P\left(\sup_{p \leq q \leq n} \| X_q(n) - X_q \| > \varepsilon \right) < \varepsilon. \tag{1.1}$$

The above-mentioned concept has been recently introduced by Talagrand [9], in which the following results are proved.

THEOREM 1.1. ([9], *Theorem 4). Every real-valued mil* (X_n) *with* $\lim_n \inf E(|X_n|) < \infty$ *converges a. e.*

THEOREM 1.2. ([9], *Theorem 6). Let* (X_n) *be an E-valued mil with* $\lim_n \inf E(\| X_n \|) < \infty$. *Suppose that* (X_n) *converges scalarly to zero a.e. that is for each* $x^* \in F^*$ *the sequence* $(x^*(X_n))$ *converges to zero a.e. Then* (X_n) *converges (strongly) to zero a.e.*

THEOREM 1.3 ([9], *Theorem 8). Let* (X_n) *be an E-valued mil with* $\lim_n \inf E(\| X_n \|) < \infty$. *Then there exists a unique decomposition:* $X_n = M_n + P_n$, *where* (M_n) *is an* L^1*- bounded martingale and* (P_n) *is a mil that goes to zero a. e.*

Let us remark that the proof of the above Theorem 8 in [9] could be improved as follows

a) The function $g = E^q h$, defined in the proof of Theorem 8 [9] is not suitable. It should be replaced by the same function h. This replacement is essential, since the inequality « $f_i \leq g$ a. e. », used in the proof is not true even in the real-valued case. This inequality should be thus replaced by « $f_i \leq h$ a. e. ».

b) The martingale (M_n) is indeed uniformly integrable since Talagrand proved that for each $q \in N$ we have $\| M_q \| \leq E^q(h)$ a. e. This observation is essential for the proof of the uniqueness of the decomposition for games which become fairer with time in the next section.

DEFINITION 2. 1 (see, [1, 2,3]). A sequence (X_n) in L^1_F is said to be a *game which becomes fairer with time,* if

$$(X_n(m)-X_n) \xrightarrow{P} 0 \text{ as } m \geqq n \to \infty, \text{ i.e.,}$$
$$\forall_{\varepsilon>0} \ \exists_p \ \forall_{m\geqq n\geqq p} \ P(\| X_n(m) - X_n \| > \varepsilon) < \varepsilon. \tag{2.1}$$

Clearly, by (1.1) and (2.1), every mil is a game which becomes fairer with time.

Now, let (X_n) be a real—valued sequence such that

a) (X_n) converges in L^1,

b) (X_n) does not converge a. e.

Then by (a), (X_n) is a game which becomes fairer with time. On the other hand, by $(a - b)$ it follows from Theorem 1. 1 that (X_n) can not be a mil.

The main purpose of this note is to establish some decomposition and limit theorems for F—valued games which become fairer with time, using the structure results of Talagrand [9] mentioned in the previous section. The following lemma is the main tool which will be frequently needed in the sequel.

LEMMA 2. 1. *If (X_n) is an F-valued game which becomes fairer with time, then (X_n) contains a subsequence (X_{n_k}) which is a mil.*

Proof. Let (X_n) be as in the lemma Then by Definition 2. 1, we have

$$\forall_k \cdot \exists_{n_k \nearrow} \ \forall_{n\geqq n_k} \ P(\| X_{n_k}(n) - X_{n_k} \|) > 2^{-k}) < 2^{-k}. \tag{2. 2}$$

We shall show that the subsequence (X_{n_k}) is a mil. Indeed, let $\varepsilon > 0$ be given.

Choose $p \in N$ such that $2^{-p+1} < \varepsilon$. Then for all $k \geqq p$, by (2. 2) it follows that

$$P(\sup_{p\leq q\leq k} \| X_{n_q}(n_k) - X_{n_q} \| > \varepsilon)$$
$$\leq \sum_{q=p}^{k} P(\| X_{n_q}(n_k) - X_{n_q} \| > \varepsilon)$$
$$\leq \sum_{q=p}^{k} P(\| X_{n_q}(n_k) - X_{n_q} \| > 2^{-q})$$
$$\leq \sum_{q=p}^{k} 2^{-q} \leq \sum_{q=p}^{\infty} 2^{-q} = 2^{-p+1} < \varepsilon.$$

Thus by Definition 1. 1, (X_{n_k}) is a mil which completes the proof of the lemma.

THEOREM 2. 3. *Let (X_n) be an L^1-bounded real-valued game which becomes fairer with time. Then there is an $X \in L^1_R$ such that $X_n \xrightarrow{P} X$ as $n \to \infty$.*

Proof. Let (X_n) be an L^1-bounded game which becomes fairer with time. To prove the theorem, it suffices to show that every subsequence of (X_n) contains a subsequence which converges a.e. For this purpose, let (X_{m_k}) be a subsequence of (X_n). It is clear that (X_{m_k}) is itself a game which becomes fairer with time. Thus by Lemma 2.2, it follows that there exists a subsequence (n_k) of (m_k) such that (X_{n_k}) is a mil. This fact together with Theorem 1.1 shows that (X_{n_k}) must converge a. e. The proof of the theorem is thus complete.

LEMMA 2. 4. *Let (P_n) be an L^1-bounded F-valued game which becomes fairer with time. Suppose that for each $x^* \in F^*$, $x^*(P_n) \xrightarrow{P} 0$ as $n \to \infty$. Then $P_n \xrightarrow{P} 0$ as $n \to \infty$.*

Proof Let (P_n) be as in the lemma. Again, it suffices to show that every subsequence of (P_n) contains a subsequence which converges to zero a. e. For this purpose, let (P_{m_k}) be a subsequence of (P_n). Clearly (P_{m_k}) is itself a game which becomes fairer with time. From this, it follows by Lemma 2.2 that (P_{m_k}) contains a subsequence (P_{n_k}) which is a mil. Consequently, by the hypothesis of the lemma and Theorem 1. 1 we have $x^*(P_{n_k}) \to 0$ a. e. as $k \to \infty$. Hence, by Theorem 1. 2, $P_{n_k} \to 0$ a. e. as $k \to \infty$. Hence, $P_n \xrightarrow{P} 0$ as $n \to \infty$ and the lemma is proved.

We are now in a position to prove a decomposition theorem for games fairer with time which is the main result of the paper.

THEOREM 2.5. *Let (X_n) be an L^1-bounded F-valued game which becomes fairer with time. Then there exists a unique decomposition: $X_n = M_n + P_n$, where (M_n) is a uniformly integrable martingale and (P_n) is a game fairer with time that goes to zero in probability.*

Proof. Let (X_n) be as in the theorem. By Lemma 2.2, it follows that there exists a subsequence (X_{n_k}) of (X_n) which is a mil. Thus by Theorem 1.3 and the remark about this theorem, (X_{n_k}) can be written in a unique form: $X_{n_k} = M_{n_k} + P_{n_k}$, where (M_{n_k}) is a uniformly integrable martingale and (P_{n_k}) is a mil that goes to zero a. e. Hence, (X_n) can be also written in a form

$X_n = M_n + P_n$, where (M_n) is the uniformly integrable martingale which is defined in the natural way by the uniformly integrable martingale (M_{n_k}) and (P_n) contains the subsequence (P_{n_k}) which has been just proved to be a mil converging to zero *a.e.* The main part of the proof is devoted to show that (P_n) is a probability potential, i.e. (P_n) goes to zero in probability. To do this let $x^* \in F^*$. Then by Theorem 2.3 it follows that$(x^*(X_n))$ converges in probability to some function $X^{x^*} \in L^1_R$. Hence the uniformly integrable martingale $(x^* (M_{n_k}))$ converges to X^{x^*} a.e. and in L^1 since the mil $(x^* (P_{n_k}))$ converges to zero a.e. Further, since (M_n) is a uniformly integrable martingale, we infer that $(x^* (M_n))$ must converge itself to X^{x^*} a.e. and in L^1. This implies that $(x^* (P_n))$ converges to zero in probability since $X_n = M_n + P_n$ $(n \in N)$ and $(x^* (X_n))$ converges to X^{x^*} in probability. On the other hand, (P_n) is also a game which becomes fairer with time such that $\sup E(\| P_n \|) < \infty$, so by Lemma 2.4, (P_n) converges to zero in probability. This proves the existence of the decomposition given in the theorem. Thus it remains to prove only that the decomposition is unique. This fact depends on our remark about Theorem 1.3. Namely, since the martingale (M_n) is uniformly integrable and (P_n) is a probability potential so such a decomposition is always unique. Thus the theorem is completely proved.

COROLLARY 2. 6. *Let (X_n) be a uniformly integrable sequence in L^1_E. Then the following conditions are equivalent:*
a) (X_n) is a game which becomes fairer with time.
b) (X_n) has a Riesz decomposition: $X_n = M_n + P_n$, where (M_n) is a martingale and (P_n) is an L^1 — potential, i.e. $\lim_n E(\| P_n \|) = 0$.

c) (X_n) is an L^1 — amart, i. e.
$E(\| X_n(m) - X \|) \to 0$ as $m \geqq n \to \infty$.

Proof. $(c \to a)$ follows from the same definitions. $(a \to b)$ is an easy consequence of the theorem. Finally, $(b \to c)$ has been recently proved by the author [5] even for the multivalued case.

COROLLARY 2. 7. *Let F be a Banach space with the Radon—Nikodym property and (X_n) be an F — valued L^1 — bounded game which becomes fairer with time. Then (X_n) converges in probability to some $X \in L^1_F$.*

Clearly, Proposition 2 [6] and Theorem 2. 1 [8] are easy consequences of the real—valued version of the latter Corollary.

Remark 2. 8. The Talagrand structure method used in [9] suggests a new simple proof of the Chatterji's martingale limit theorem [4]. Indeed, let F be a Banach space with the Radon-Nikodym property and (X_n) be an L^1—bounded F—valued martingale. Then it follows from Theorem 1. 3 and the remark about this theorem that (X_n) can be written in a unique form: $X_n = M_n + P_n$, where (M_n) is a uniformly integrable martingale and (P_n) goes to zero a. e. Further, a much more simpler proof given in [4, 7] shows that as a uniformly integrable martingale in a Banach space with the Radon—Nikodym property, (M_n) must be a regular martingale which converges in particular, a.e. hence so does the martingale (X_n). This completes the proof of the Chatterji's martingale limit theorem given in [4].

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