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MULTIDIMENSIONAL QUANTIZATION. IV
THE GENERIC REPRESENTATIONS

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1. INTRODUCTION

Today physicists observe that many particles admit also internal symmetry, for example under classical Lie groups like SU(2), SU(3),... rather than the ordinary external symmetry under Lorentz, Poincaré, conformal, ..., groups. The total internal-external symmetry picture suggests some generalization of the Orbit Method in the multidimensional quantization context by using quantized vector fiber bundles of finite or infinite dimensions [1—10] rather than the line ones [13], [14].

At about the same time and independently, M. Duflo [11] extended the Orbit Method to arbitrary Lie groups. His construction proposes also holomorphic induction from unitary representations of finite or infinite dimensions. Our multidimensional quantization procedure could be viewed as a geometric version of this construction. We shall prove in this paper (Theorem 1) that every Duflo's generic representation can actually be obtained from our multidimensional quantization procedure.

Recently, R. L. Lipsman [15] has made a detailed analysis of the structure of generic representations: By induction in steps, every generic representation can be obtained from some square-integrable representation. Using this result, we shall prove (Theorem 2) that the numerous equivalent compactness criteria [3, 4] are multiply applicable to these generic representations of connected and simply connected Lie groups. This fact enables us in many cases to provide a method of topological invariants [4] using the KK-theory [12] in the study of the structure of group C*-algebras.

2. THE GENERIC REPRESENTATIONS

Let $G = G(\mathbb{R})$ be the Lie group of real points of a complex algebraic R-group G, $\mathcal{G} = \text{Lie}\, G$ its Lie algebra and $\mathcal{G}^* = \text{Hom}_R(\mathcal{G}, R)$ the dual vector space. Denote by $\mathcal{AP}(G)$ the set of all ϕ in $\mathcal{G}^*$ which are admissible and well-polarizable (the definitions of these concepts will be recalled in the proof of

Theorem 1). To each $\phi \in \mathcal{AP}(G)$ one associates a canonical finite set $\mathcal{X}_G^i(\phi)$ of irreducible unitary representations of a two-fold cover of the stability group G_ϕ. Set

$$\mathcal{B}(G) = \{(\phi, \tau);\ \phi \in \mathcal{AP}(G),\ \tau \in \mathcal{X}_G^i(\phi)\}.$$

It is easy to see that $\mathcal{B}(G)$ is a G-space with the natural action of G.

M. Duflo [11] has constructed a map $(\phi, \tau) \mapsto \pi(\phi, \tau)$ from $\mathcal{B}(G)$ to the dual $\widehat{G}$ of equivalence classes of irreducible unitary representations of G, and has shown that this map factors to an injection $\mathcal{B}(G)/G \hookrightarrow G$, the image of which consists of *generic classes* in the sense that its complement in $\widehat{G}$ is of null Plancherel measure. As in the nilpotent case the Duflo's construction of classes $\pi(\phi, \tau)$ is based on induction on dim G. Lastly, R. L. Lipsman [15] proved that there is a Plancherel co-null set U in G, contained in $\mathcal{B}(G)/G \subset \widehat{G}$, all elements of which are ordinarily induced from representations which are square-integrable modulo their projective kernel. A few years ago J. Rosenberg [16, Th. 4.8] showed that for extensions of nilpotent groups having square-integrable representations by reductive groups the L^2-cohomology spaces of homogeneous holomorphic line bundles associated to some special polarization vanish except in one degree.

Our multidimensional quantization procedure [1—10] introduced a new concept of polarization where not only is the stabilizer extended to a larger group and a complex subordinate Lie subalgebra, but also its character can be extended to an irreducible unitary representation (of any dimension) of the polarization Lie subgroup and subordinate complex Lie algebra. The foundation of the same construction of induced representations in L^2-cohomology spaces $(L^2 - \mathrm{Coh}_k)$ Ind $(G; \mathcal{P}, H, \rho, \sigma_0)$ has been given in [1—10].

THEOREM 1. *Every generic representation* $\pi(\phi, \tau)$ *can be obtained from the multidimensional quantization procedure.*

An equivalent form of this theorem is

LEMMA. *For every Duflo's data* (ϕ, τ) *we can choose a polarization* $(\mathcal{P}, H, \rho, \sigma_0)$ *such that*

$$\pi(\phi, \tau) = (L^2 - Coh_k)\ Ind\ (G; \mathcal{P}, H, \rho, \sigma_0).$$

Proof. I. Following Duflo [11], $\phi \in \mathcal{G}^*$ is *admissible* iff the character $\chi_\phi(\exp X) = \exp(\frac{i}{h} X, \phi)$ of the connected component of identity $e \in G$ can be extended to the two-fold cover $G_\phi^{\mathcal{G}}$ of G_ϕ such that $\chi_\phi(e, \varepsilon) = -1$ where G is the fibered product

$$\begin{array}{ccccccc} 0 \to Z/2Z & \longrightarrow & G_\phi^{\mathcal{G}} & \longrightarrow & G_\phi & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \to Z/2Z & \longrightarrow & M_p(\mathcal{G}/\mathcal{G}_\phi) & \to & S_p(\mathcal{G}/\mathcal{G}_\phi) & \to & 1 \end{array}$$

and ε is the generator of $Z/2Z$.

Recall that $\Phi \in \mathcal{G}^*$ is *polarizable* iff there exists a complex subalgebra $\mathcal{P} \subset \mathcal{G}_{\mathbf{C}}$ which is a maximal isotropic for Kirillov form B_{Φ}, and is *well-polarizable* iff the complex polarization $\mathcal{P}$ is solvable and the Pukanszky condition is fulfilled, $H \cdot \Phi = \Phi + \mathcal{P}^{\perp}$ where H is the corresponding analytic subgroup of G with Lie algebra $\mathcal{P}$.

Let (Φ, τ) be a Duflo's data. Then Φ is admissible and well-polarizable and τ is a unitary continuous representation of G_{Φ} in a separable Hilbert space, whose restriction to $(G_{\Phi})^{\mathcal{G}}_{0}$ is a multiple of χ_{Φ}.

II. It is now easy to see that there exists at least a (τ, Φ) — polarization [1, Def. 1.1.] $(\mathcal{P}, \rho, б_0)$ such that: (a) $\mathcal{P}$ is a Lie subalgebra of $\mathcal{G}_{\mathbf{C}}$ containing $\mathcal{G}_{\Phi}$, (b) The subalgebra $\mathcal{P}$ is invariant under all $\mathrm{Ad}_{\mathcal{G}_{\mathbf{C}}} x$, $x \in G_{\Phi}$. (c) The vector space $\mathcal{P} + \overline{\mathcal{P}}$ is a complexification of a real Lie subalgebra $\mathcal{M}$, i. e. $\mathcal{M} = (\mathcal{P} + \overline{\mathcal{P}}) \cap \mathcal{G}$. (d) All the subgroups M_o, H_o, M, H are closed in G, where M_o (resp, H_o) is the connected subgroup of G with Lie algebra $\mathcal{M}$ (resp., $\mathcal{H} = \mathcal{P} \cap \mathcal{G}$) and $M = M_o \cdot G_{\Phi}$, $H_o = H_o \cdot G_{\Phi}$.
(e) $б_0$ is an irreducible unitary representation of H_o in a Hilbert space V such that:

1) The restriction $б_o \big|_{G_{\Phi} \cap H_o}$ is a multiple of the restriction of τ to $G_{\Phi} \cap H_o$

2) The point $б_0$ is fixed under the natural action of G_{Φ} in the dual of H_o.
(f) ρ is a representation of the complex Lie algebra $\mathcal{P}$ in V satisfying all the Nelson conditions for H_o, and $\rho \mid \mathcal{H} = dб_o$.

Following [1, Th. 1], the Duflo's generic representation $\pi(\Phi, \tau)$ is equivalent to one of the representations $(L^2 - \mathrm{Coh}_k)$ Ind $(G; \mathcal{P}, \rho, б_0)$ of G in the L^2-cohomology with coefficients in $C^\infty(G; \mathcal{P}, \rho, б_o)$.

III. In accordance with [2, Th. 2], we can construct from each (τ, Φ)-polarization $(\mathcal{P}, \rho, б_0)$ a (χ_{Φ}, τ)-polarization $(L, \rho, б_0)$ and conversely, where the triple $(L, \rho, б_0)$ satisfies the following conditions:

(a) L and $L + \overline{L}$ are integrable complexified tangent distributions.

(b) L is invariant under all operators Adx, $x \in G$.

(c) $L \cap \overline{L}$ and $L + \overline{L}$ are the complexifications of some integrable real distributions $L_{\mathcal{H}}$ and $L_{\mathcal{M}}$ respectively.

(d) The polarization subgroups M_o, H_o, M, H are closed (see [2] for definitions).

(e) L is a weakly Lagrangian distribution, i.e.

1) The restriction $\sigma_o|_{G_\Phi \cap H_o}$ is a multiple of the restriction of the restriction τ to $G_\Phi \cap H$

2) The point σ_o is fixed under the natural action of G_Φ on the dual $\widehat{H}_o$ of H_o.

(f) ρ is a representation of the complex Lie subalgebra $\mathcal{P}$ in V satisfying all the Nelson conditions for H_o and $\rho|_{\mathcal{H}} = d\sigma_o$

IV. By Theorem 3 of [2], we have a Hilbert fiber bundle with affine connection $(\mathcal{E}_V, \nabla)$ and then the Duflo's generic representation $\pi(\Phi, \tau)$ is equivalent to the natural representation of G on the space of partially invariant partially holomorphic sections of this bundle, and the Lie derivative of this partially invariant holomorphically induced representation $(L^2\text{-Coh}_k)\,\mathrm{Ind}\,(G\,;\,\mathcal{P}, \rho, \sigma_o)$ is equivalent to the representation of our Lie algebra

$$X \longrightarrow \frac{i}{\hbar} f_x$$

where $\widehat{f} = f + \frac{\hbar}{i} \Delta_{\xi} f$. The proof of Theorem 1 and of the Lemma is complete.

3. THE COMPACTNESS CRITERIA FOR THE GENERIC REPRESENTATIONS

Let us recall Lipsman's analysis of the structure of generic representations. Let $(\Phi, \tau) \in \mathfrak{B}(G)$ be a Duflo's data, $\pi = \pi(\Phi, \tau)$, N be the unipotent radical of G, $\mathcal{N} =$ Lie N its Lie algebra and $\theta = \Phi|_{\mathcal{N}}$ the restriction of Φ to $\mathcal{N}$. Set $G^1 = G_o N$, $\mathcal{G}^1 =$ Lie G^1 and $\Phi^1 = \Phi|_{\mathcal{G}^1}$ It was shown that $\Phi^1 \in \mathcal{A}\mathcal{P}(G^1)$ and there exists $\tau^1 \in \mathcal{X}^i_{G^1}(\Phi^1)$ canonically defined by τ such that $\pi(\Phi, \tau) =$
$= \mathrm{Ind}^G_{G^1}\, \pi_{G^1}\, (\Phi^1, \tau^1)$.

Repeat the procedure with $(\Phi^1, \tau^1) \in \mathcal{B}(G^1)$. Let N^1 be the unipotent radical of G^1,... $\theta^1 = \Phi^1|_{n^1}$,... The sequence stabilizes for a finite number of steps, say r,

$$G^r = (G^r)_{\theta^r} \cdot N^r, (\Phi^r, \tau^r) \in \mathcal{B}(G^r).$$

By induction in steps one has

$$\pi(\Phi, \tau) = \mathrm{Ind}^G_{G^1}\, \mathrm{Ind}^{G^1}_{G^2} \dots \mathrm{Ind}^{G^{r-1}}_{G^r}\, \pi_{G^r}(\Phi^r, \tau^r) =$$

$$= \mathrm{Ind}^G_{G^r}\, \pi_{G^r}(\Phi^r, \tau^r).$$

Let $\gamma = \pi_{N^r}(\theta^r)$ be the Kirillov representation of N^r determined by θ^r. Then selecting a Levi factor S of G^r which lies in $(G^r)_{\theta^r}$, we have $G^r = S.N^r$, a semidirect product. Since S fixes θ^r, γ extends canonically to a (perhaps projective) representation $\tilde{\gamma}$ of S on the space of γ. It is well known that there is a canonically determined (perhaps projective) irreducible unitary representation ω of S such that $\pi_{G^r}(\Phi^r, \tau^r) = (\omega \otimes \tilde{\gamma}) \times \gamma$. By passing to two-fold covering if necessary, we can view ω as an ordinary representation of S. If $\zeta = \Phi|_{Lie\, S}$ there exists $\nu \in \mathcal{X}^i_S(\zeta)$, canonically determined by τ such that ω $\omega = \pi_S(\xi, \nu)$.

THEOREM 2. *For every generic representation π and every function $\varphi \in L^1(G)$, the operator $\pi(\varphi)$ is compact if and only if in every induction step* Ind G^{i-1}_{Gi} *the compactness criteria (see* [3]) *hold.*

Proof. Observe that the compactness criteria proved by the author in [3] can be applied to the irreducible unitary representations induced by CCR-representations of a closed invariant subgroup. We must only check these conditions for every induction step of generic representations.

From the construction, it is easy to see that G^i is an invarian closed subgroup of G^{i-1}.

Lipsman shows that one can always restrict oneself to the square-integrable $\pi_S(\xi, \nu)$, see [15]. Then $\pi_S(\phi^r, \tau^r)$ is CCR by the Gelfand-Piateskij-Shapiro's theorem on CCR-property of square-integrable representations.

In every induction step one now applies the concrete analysis of the induced representations in the space of functions of two variables as was done in [3]. This completes the proof of Theorem 2.

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