PARTIALLY ORDERED SETS OF SEQUENCES AND THE RELATED BOOLEAN ALGEGRAS

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In this paper we consider partially ordered sets such as $(S, \leq)$ where S is a set of sequences (i. e., functions defined on finite or infinite ordinals) and where for every element x and y of S we define $x \leq y$ if and only if x is a (physical) extension of y (including the trivial extension of remaining indentical). We prove that of every such $(S, \leq)$ there exists a suprema preserving ismorphism from $(S, \leq)$ into a complete Boolean algehra $(C, \subseteq)$ which also preserves the infima of finite subsets of S. In many instances this embedding enables passing from Forcing to Boolean-valued techniques.

In what follows by a *sequence* we mean a function defined on a (finite or infinite) ordinal which also determines the type of the sequence. Thus, a sequence defined on an ordinal k is of type k and can be visualized as a k-tuple of symbols. For instance, (a, b, b) is a sequence of type 3, whereas $(b, c, c, \ldots, t, t)$ is a sequence of type $\omega + 2$.

We say that a sequence x is an *extension* of the sequence y if and only if y is an initial segment of x or $y = x$. Thus, (a, b, a) is an extension of itself as well as of (a, b). On the other hand, neither of (a, b) and (a, m) is an extension of the other.

For every sequence x and y we define $x \leq y$ as:

(1) *$x \leq y$ if and only if x is an extension of y*

If (1) holds and $x \neq y$ then we write $x < y$ and we say that x is a *proper extension* of y.

Let S be any set of sequences (not necessarily all of the same type). Then it can be readily verified that $(S, \leq)$ is a partially ordered set. Thus, we may use the usual terminology. Accordingly, since $(a, b, a) \leq (a, b, a)$ we say that (a, b, a) is less than or equal (in fact, equal) to (a, b, a). Again, since $(a, c, c, d, \ldots) < (a, c)$ we say that $(a, c, c, d, \ldots)$ is strictly less than (a, c). Also, since $(b, b, m, \ldots) \not\subseteq (b, a, m, \ldots)$ and $(b, a, m, \ldots) \not\subseteq (b, b, m, \ldots)$ we say that $(b, b, m, \ldots)$ and $(b, a, m, \ldots)$ are uncomparable (i. e., *not comparable*).

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In what follows we always let $(S, \leq)$ be a partially ordered set of sequences and we let S_1 be defined as:

(2) $$S_1 = S - \{\max(S, \leq)\}$$

Clearly, if $(S, \leq)$ has no maximum then $S_1 = S$.

We define a mapping h from S into the powerset (i.e., the set of all subsets) $P(S_1)$ of S_1 as follows:

(3) $$h(x) = \{y \mid y < x \text{ or } y \text{ and } x \text{ are uncomparable}\}$$

Thus, if $S = \{(\beta, \gamma, \beta), (\beta, \gamma, \gamma, \alpha, \ldots), (\beta, \gamma), (\beta, \gamma, \gamma), (\beta, \gamma, \gamma, \gamma, \ldots)\}$ and if we let $a = (\beta, \gamma, \beta)$, $b = (\beta, \gamma, \gamma, \alpha, \ldots)$, $c = (\beta, \gamma)$, $d = (\beta, \gamma, \gamma)$, $e = (\beta, \gamma, \gamma, \gamma, \ldots)$ then according to (2) we have:

$$S = \{a, b, c, d, e,\} \text{ and } S_1 = \{a, b, d, e\}$$

and according to (3) we have:

$h(a) = \{b, d, e\}$, $h(b) = \{a, e\}$, $h(c) = \{a, b, d, e\}$, $h(d) = \{a, b, e\}$, $h(e) = \{a, b\}$

which, as expected, is a mapping from S into $P(S_1)$.

Clearly, from (3) it follows readily that for every element x and y of S we have:

(4) $$y \notin h(x) \text{ if and only if } y \geq x$$

Also, we remark that in $(S, \leq)$ two elements have a lower bound if and only if they are comparable (and therefore, in $(S, \leq)$ simply ordered subsets only may have lower bounds) from which it can be shown easily that $(S, \leq)$ is a distributive partially ordered set.

It is well known that $(P(S_1), \subseteq)$ is a complete Boolean algebra (i.e., a complemented distributive complete partially ordered set).

THEOREM. *Let $(S, \leq)$ be a partially ordered set of sequences and $(P(S_1), \subseteq)$ be the complete Boolean algebra of all subsets of S_1. Then the mapping h as given by (3) is one-to-one from S into $P(S_1)$ such that h preserves comparability and the existing least upper bounds of subsets of S and the existing greatest lower bounds of finite subsets of S, i.e., for every element (sequence) x and y of S,*

(5) $h(x) = h(y)$ *if and only if* $x = y$

6) $x \leq y$ *if and only if* $h(x) \subseteq h(y)$

and for every subset E of S

(7) $h(\text{lub } E) = \bigcup (h[E])$ *whenever the left side of the equality exists*

(8) $h(\text{glb } E) \subseteq \bigcap (h[E])$ *whenever the left side of the inclusion exists*

(9) $h(\text{glb } E) = \bigcap (h[E])$ *whenever the left side of the equality exists and E is a finite subset of S.*

Proof. To prove (5), let us observe that $h(x) = h(y)$ if and only if for every $z \in S$ it is the case that $z \notin h(x)$ if and only if $z \notin h(y)$, and by (4), it is the case that $z \geq x$ if and only if $z \geq y$, and therefore if and only if $x = y$.

To prove (6), let us observe that $x \leq y$ if and only if for every $z \in S$ it is the case that $z \geq y$ implies $z \geq x$, and by (4), if and only if $z \notin h(y)$ implies $z \notin h(x)$, and therefore if and only if $h(x) \subseteq h(y)$.

To prove (7), we show that $x \notin h(\text{lub } E)$ if and only if $x \in h(y)$ for every $y \in E$. But $x \notin h$ (lub E) by (4), if and only if $x \geqq$ lub E, and therefore if and only if $x \geqq y$ for every $y \in E$, and again by (4), if and only if $x \notin h(y)$ for every $y \in E$.

To prove (8), we show that $x \notin h(\text{glb } E)$ if $x \notin h(y)$ for some $y \in E$. But if $x \notin h(y)$ for some $y \in E$ then by (4) we have $x \geqq y$ for some $y \in E$ and hence $x \geq$ glb E and therefore $x \notin h(\text{glb } E)$.

To prove (9), as mentioned earlier, we observe that $(S, \leqq)$ is such that if E is a finite subset of S and glb E exists then E is a (finite) simply ordered subset of S and (glb E) $\in E$. But then (9) follows readily.

Remark. In the existing literature (e. g., [2, p. 50]), a partially ordered set such as $(S, \leq)$ is usually embedded (since $(S, \leq)$ is separative [2, p. 49]) in the complete Boolean algebra $(RO(S), \subseteq)$, where $RO(S)$ is the set of all the regular open sets of the topology defined on S by taking the sets $[x] = \{z \mid z \in S \text{ and } z \leq x\}$ for basic open sets (cf. [1], [3]). However, this embedding although ordered preserving, does not preserve suprema and, in general, infima. For instance, for every x and y of S it can be readily seen (cf. [3, p. 175]) that $[x] \cup [y]$ is a both open and closed set and yet it is not equal to $[x \vee y]$. Thus, the embedding resulting from our Theorem, in the case of partially ordered sets such as $(S, \leq)$, is far superior to the embeddings found in the exisling literature.

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