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ON THE CONVERGENCE OF WEIGHTED SUMS OF MARTINGALE DIFFERENCES

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1. INTRODUCTION

Let (Ω, F, P) be a probability space, $(F_n, n \geqq 1)$ an increasing sequence of sub $-\sigma-$ *fields* of F and $(X_n, n \geqq 1)$ a sequence of real — valued random variables adapted to $(F_n, n \geqq 1)$, i. e., each X_n is F_n — measurable. Throughout this paper we will use the following definitions and notations :

A sequence $(X_n, n \geqq 1)$ is said to be uniformly integrable, if

$$\sup_{n \in N} \{ \int_{|X_n| > a} |X_n| \, dP \} \to 0 \text{ as } a \to \infty \tag{1.1}$$

Note that (1. 1) implies

$$\sup_{n \in N} \{ a P (| X_n | > a) \} \to 0 \text{ as } a \to \infty. \tag{1.2}$$

A sequence $(X_n, n \geqq 1)$ is said to be a martingale difference, if $E(X_n | F_{n-1}) = 0$ for all $n \geqq 1$.

An array (a_{nk}) of real numbers is said to be a Toeplitz matrix, if for some $M < \infty$ the following conditions are satisfied

$$\begin{cases} (i) & \lim_{n \to \infty} a_{nk} = 0, k \geqq 1, \\ (ii) & \Sigma_k |a_{nk}| \leqq M, n \geqq 2. \end{cases} \tag{1.3}$$

The stochastic convergence of the weighted sums $s_n = A_n^{-1} \Sigma_{k=1}^n a_k X_k$ or $s_n = \Sigma_{k=1}^n a_{nk} X_k$, where $(X_n, n \geqq 1)$ is a sequence of independent random variables, (a_k) is a sequence of positive real numbers and $A_n = (\Sigma_{k=1}^n a_k) \uparrow \infty$, was systematically studied by B. Jamison, S. Orey and W. Pruitt [4], A. Stout [6] and many others. The purpose of this paper is to extend some of the above results to martingale differences $(X_n, n \geqq 1)$.

In Section 2, we shall show that if (a_{nk}) is a Toeplitz matrix then $\Sigma_{k=1}^{n} a_{nk} X_k \to 0$ in probability for any uniformly integrable martingale difference $(X_n, n \geq 1)$ if and only if $\max_{k \leq n} | a_{nk} | \to 0$ as $n \to \infty$. The same result is also obtained if tail probabilities of (X_n) are uniformly bounded by tail probabilities of a random varialbe $X \in L^1$. In Section 3, we shall study the convergence of $(\Sigma_{k=1}^{n} a_{nk} X_k, n \geqq 1)$ in L^P $(1 \leq p < 2)$ wtih (a_{nk}) being a Toeplitz matrix such that $\max_{k \leqslant n} | a_{nk} | \to 0$ as $n \to \infty$. In Section 4, we shall study the almost sure convergence of $(A_n^{-1} \Sigma_{k=1}^{\infty} a_k X_k, n \geqq 1)$ where $(X_n, n \geqq 1)$ is a martingale difference, (a_k) is a sequence of positive real numbers and $A_n > 0$, $A_n \uparrow \infty$.

Recall that a sequence $(X_n, n \geqq 1)$ of random variables is said to have uniformly bounded tail probabilities by tail probabilities of a random variable $X \in L^P$ $p > 0)$ in symbols $(X_n) \prec X \in L^P$, if there exists a positive constant C such that

$$P(|X_n| > x) \leq C \; P(|X| > x)$$

for all $x > 0$ and $n = 1, 2 ...$

Other definitions and notations related to the problem can be found in [7].

2. CONVERGENCE IN PROBABILITY

Throughout this section (a_{nk}) is assumed to be a Toeplitz matrix. Let $S_n = \Sigma_{k=1}^{n} a_{nk} X_k (n \geqq 1)$.

Let us begin with the following

LEMMA 1. *Let (a_{nk}) be a Toeplitz matrix such that* $\max_{k \leq n} | a_{nk} | \to 0$ *as* $n \to \infty$, *$(X_n, n \geqq 1)$ a uniformly integrable martingale difference. Then* $S_n \to 0$ *in* L^1.

Proof. We first establish the following fact:

If $f_n : \mathbf{R} \to \mathbf{R}^+$ where $0 \leq f_n \leq 1$ for all $n \geqq 1$ and $\sup_{n \in N} (x \; f_n(x)) \to 0$ as $x \to \infty$, then

$$\operatorname*{Sup}_{n \ni N} \left(\frac{1}{y} \int_0^y x \, f_n(x) \; dx \right) \to 0 \tag{2.1}$$

as $y \to \infty$.

To see this, put $f^*(x) = \sup_{n \in N} (x \, f_n(x))$.

Clearly,

$$\operatorname*{Sup}_{n\in N}\left(\frac{1}{y}\int_0^y x f_n(x)\, dx\right) \leq \frac{1}{y}\int_0^y f^*(x)\, dx \text{ for all } y > 0.$$

Thus, it suffices to show that

$$\frac{1}{y}\int_0^y f^*(x)\, dx \to 0 \quad \text{as } y \to \infty.$$

Since $f^*(x) \to 0$ as $x \to \infty$, for any fixed $\varepsilon > 0$, there exists an $x_0(\varepsilon) > 0$ such that if $x > x_0(\varepsilon)$, $y > x_0(\varepsilon)$ then

$$0 \leq f^*(x) < \varepsilon,$$

$$\frac{1}{y}\int_0^{x_0(\varepsilon)} f^*(x)\, dx \leq \frac{x_0^2(\varepsilon)}{2y} \to 0 \quad as\ y \to \infty,$$

$$\frac{1}{y}\int_{x_0(\varepsilon)}^{y} f^*(x)\, dx \leq \frac{\varepsilon}{y}\int_{x_0(\varepsilon)}^{y} dy = \frac{\varepsilon}{y}(y - x_0(\varepsilon)) < \varepsilon.$$

The result follows, since

$$\frac{1}{y}\int_0^y f^*(x)\, dx = \frac{1}{y}\left\{\int_0^{x_0(\varepsilon)} f^*(x)\, dx + \int_{x_0(\varepsilon)}^{y} f^*(x)\, dx\right\}.$$

Combining (2, 1) and (1, 2) yields

$$\sup_{n\in N}\left(\frac{1}{y}\int_0^y x\, P(|X_n| > x)\, dx\right) \to 0 \tag{2. 2}$$

as $y \to \infty$.

Now, put

$$X_{nk} = a_{nk} X_k I(|X_k| \leq |a_{nk}|^{-1})$$

where $I(A)$ denotes the indicator function of the set A, and

$$Z_n = \Sigma_{k=1}^n [X_{nk} - E(X_{nk} \mid F_{k-1})].$$

We can suppose $a_{nk} \neq 0$ for all n and k. From the assumption we have for n large enough

$$\begin{aligned} E|Z_n|^2 &\leq \Sigma_{k=1}^n E|X_{nk} - E(X_{nk} \mid F_{k-1})|^2 \\ &= \Sigma_{k=1}^n [E|X_{nk}|^2 - (E(X_{nk} \mid F_{k-1}))^2] \\ &\leq \Sigma_{k=1}^n E|X_{nk}|^2 \end{aligned}$$

$$\leq 2\,\Sigma_{k=1}^{n} |a_{nk}|^2 \int_{\{0 < x \leq |a_{nk}|^{-1}\}} x\; P(|X_k| > x)\, dx$$

$$\leq 2C\,\Sigma_{K=1}^{n} |a_{nk}| \left\{ \frac{1}{|a_{nk}|^{-1}} \int_{\{0 < x \leqq |a_{nk}|^{-1}\}} x\, P(|X| > x)\, dx \right\}$$

$$\leq 2C\,\Sigma_{k=1}^{n} |a_{nk}| \cdot \varepsilon \leq 2\,CM\,\varepsilon.$$

Thus,

$$F\,|Z_n| \to 0. \tag{2.3}$$

On the other hand, since $E(X_n \mid F_{n-1}) - 0$ for all $n \geqslant 1$, we obtain

$$E(X_{nk} \mid F_{k-1}) = -E(a_{kn} X_k\, I(|X_k| > |a_{nk}|^{-1}) \mid F_{k-1}).$$

Consequently, for n large enough,

$$E\,|\,\Sigma_{k=1}^{n} E(X_{kn} \mid F_{k-1})|$$

$$\leq \Sigma_{k=1}^{n} |a_{nk}|\, E(|X_k|\, I(|X_k| > |a_{nk}|^{-1}))$$

$$\leqq \Sigma_{k=1}^{n} |a_{nk}| \left\{ \int_{\{x > |a_{nk}|^{-1}\}} P(|X_k| > x)\, dx \right\}$$

$$\leq \Sigma_{k=1}^{n} |a_{nk}| \cdot \varepsilon \leqslant M\varepsilon$$ (by the uniform integrability of $(X_n, n \geqq 1)$).

Hence $\Sigma_{k=1}^{n} E(X_{nk} \mid F_{k-1}) \to 0$ in L^1. This together with (2.3) completes the proof of the lemma.

THEOREM 1. *Suppose that (a_{nk}) is a Toeplitz matrix. The three following statements are equivalent:*

(i) $\max_{k \leq n} |a_{nk}| \longrightarrow 0$ *as* $n \to \infty$;

(ii) for any uniformly integrable martingale difference $(X_n, n \geqq 1)$ $S_n \to 0$ *in* L^1;

(iii) for any uniformly integrable martingale difference $(X_n, n \geqq 1)$ $S_n \to 0$ *in probability.*

Proof. (i) → (ii) by Lemma 1.

(ii) → (iii): trivial.

(iii) → (i).

Supposing that $S_n \to 0$ in probability for any uniformly integrable martingale difference $(X_n, n \geqq 1)$, we must show that $\max_{k \leqq n} |a_{nk}| \to 0$ as $n \to \infty$. To do this, it suffices to take

$(X_n, n \geqq 1)$ as a sequence of independent random variables with $E X_n = 0$, $EX_n^2 < \infty$ and $P(X_n \neq 0) > 0$ $n = 1, 2, \ldots$ The rest of the proof follows by using Theorem 3.4.5 [7].

COROLLARY 1. *Suppose that (a_{nk}) is a Toeplitz matrix. $S_n \to 0$ in probability for any martingale difference $(X_n) \prec X \in L^1$ if and only if $\max_{k \leq u} |a_{nk}| \to 0$ as $n \to \infty$.*

Proof. It is clear that $E|X| < \infty$ implies the uniform integrability of $(X_n, n \geq 1)$. The corollary follows from Theorem 1.

Remark 1. We see from Theorem 1 that if $(M_n, \geqslant 1)$ is a martingale with increments $D_n = M_n - M_{n-1}$, $D_0 = 0$ such that $(D_n) \prec X \in L^1$, then $M_n = 0$ (n^{-1}) in probability and in L^1. Recently J. Elton (see [1]) has proved that $M_n = 0$ (n^{-1}) almost surely if $D_1 D_2, \ldots$ are identically distributed with $D_1 \in L \operatorname{Log}^+ L$. He has also constructed a very interesting example which shows that if $X \in L^1$, $EX = 0$ and $X \notin L \operatorname{Log}^+ L$ then there exists a martingale difference $(D_n, n \geq 1)$ with the same distribution as X but $n^{-1} \Sigma_{k=1}^n D_k$ diverges almost surely.

3. CONVERGENCE IN L^p $(1 \leqq p < 2)$

Throughout this section (a_{nk}) denotes a Toeplitz matrix with $\max_{k \leqq n} |a_{nk}| \to 0$ as $n \to \infty$. Let $s_n = \Sigma_{k=1}^n a_{nk} X_k$.

THEOREM 2. *Let $(X_n, n \geqq 1)$ be a martingale difference such that $(X_n) \prec X \in L^p (1 \leqslant p < 2)$. Then $E(|S_n|^p) \to 0$ as $n \to \infty$.*

Proof. Suppose first that $1 < p < 2$. Applying the Burkholder inequality (see [2], p.23) to the martingale array $(S_{nj} = \Sigma_{k=1}^j a_{nk} X_k, 1 \leqq j \leq n)$, we have

$$E|S_n|^p = E|\Sigma_{k=1}^n a_{nk} X_k|^p \leqq B(p) E\{(\Sigma_{k=1}^n a_{nk}^2 X_k^2)^{\frac{p}{2}}\} \tag{3.1}$$

where $B(p)$ is a positive constant depending only on p.

Now, put

$Y_{nk} = a_{nk} X_k I(|X_k| \leqq |a_{nk}|^{-1})$, where I(A) denotes the indicator function of the set A, and

$Z_{nk} = a_{nk} X_k - Y_{nk}$.

By the C_r — inequality $E \mid X + Y \mid^r \leq C_r(E \mid X \mid^r + E \mid Y \mid^r)$,

where $C_r = 1$ *if* $0 < r \leq 1$ and $C_r = 2^{r-1}$ if $r \geqq 1$, and (3.1) we have

$$E \mid S_n \mid^p \leq B(p)\, E \{ [\Sigma_{k=1}^n (Y_{nk} + Z_{nk})^2 \ ]^{\frac{p}{2}} \} \tag{3.2}$$

$$\leq B(p)\, E \{[2\, \Sigma_{k=1}^n (Y_{nk}^2 + Z_{nk}^2)]^{\frac{p}{2}}$$

$$= 2^{\frac{p}{2}} B(p)\, E \{[\Sigma_{k=1}^n (Y_{nk}^2 + Z_{nk}^2)]^{\frac{p}{2}} \}$$

$$\leq 2^{\frac{p}{2}} B(p) \ \{ E\, (\Sigma_{k=1}^n (Y_{nk}^2)^{\frac{p}{2}} +$$

$$+ E\, (\Sigma_{k=1}^n Z_{nk}^2)^{\frac{p}{2}} \},$$

since $(a + b)^2 \leq 2(a^2 + b^2)$ for any two real numbers a and b.

Next, using again the C_r — inequality with $0 < r = P/2 \leq 1$, and the assumption that $(X_n) \prec X \in L^p$, we have

$$E \left(\Sigma_{k=1}^n \Big| Y_{nk}^2\right)^{p/2} \leq \Sigma_{k=1}^n \ E \mid Y_{nk} \mid^P$$

$$= \Sigma_{k=1}^n \mid a_{nk} \mid^p \int_{\{0 < x \leq \mid a_{nk} \mid^{-1}\}} x^p \, d\, P\, (\mid X_k \mid \leq x)$$

$$= p\, \Sigma_{k=1}^n \mid a_{nk} \mid^p \int_{\{0 < x \leq \mid a_{nk} \mid^{-1}\}} x^{p-1}\, P\, (\mid X_k \mid > x)\, dx$$

$$\leq C_p \Sigma_{k=1}^n \mid a_{nk} \mid \Big\{ \frac{1}{\mid a_{nk} \mid^{1-p}} \int_{\{x < x \leq \mid a_{nk} \mid^{-1}\}} x^{p-1}\, P\, (\mid X \mid > x)\, dx. \tag{3.3}$$

$\leq C_p M\varepsilon$, for n large enough, because

$$\sup_{k \leq n} \mid a_{nk} \mid^{p-1} \int_{\{0 < x \leq \mid ank \mid^{-1}\}} x^{p-1}\, P\, (\mid X \mid < x)\, dx \to 0$$

as $\sup_{k \leq n} \mid a_{nk} \mid \to 0$ as $n \to \infty$.

Likewise, we obtain

$$E\left(\Sigma_{k=1}^{n} Z_{nk}^{2}\right)^{p/2} \to 0 \text{ as } n \to \infty, \tag{3.4}$$

which, together with (3.3) and (3.2), completes the proof for the case $1 < p < 2$.

When $p = 1$, the uniform integrability of $(X_n, n \geqslant 1)$ follows from the assumption $X_n \prec X \in L^1$. Indeed

$$\sup_{n \in N} E[\,|X_n|\, I(|X_n| > a)] = \sup_{n \in N} \int_{x > a} P(|X_n| > x)\, dx$$
$$\leqslant C \int_{x > a} P(|X| > a)\, dx \to 0 \text{ as } a \to \infty.$$

This together with Lemma 1 yields the assertion.

4. ALMOST SURE CONVERGENCE

Throughout this section the following assumptions are made: $(X_n, n > 1)$ is a martingale difference, $a_k > 0$ $k = 1,2, \ldots, A_n > 0$ and $A_n \uparrow \infty$, $a_{n}/A_n \to 0$ as $n \to \infty$. $S_n = \Sigma_{k=1}^{n} a_k X_k$ denotes the partial weighted sums.

Our next purpose is to study the almost sure convergence of $(S_n/A_n, n \geqslant 1)$, which will imply that $n^{-1/p} \Sigma_{k=1}^{n} X_k \to 0$ a.s. $1 \leqslant p < 2$ if $(X_n) \prec X \ni L^p$.

For this we shall need the following well known fact.

LEMMA 2 (Kronecker Lemma). *Let $(x_n, n \geqslant 1)$ be a sequence of real numbers such that $\Sigma_n x_n$ converges, and let $(b_n, n \geqslant 1)$ be a monotone sequence of positive constants with $b_n \uparrow \infty$. Then*

$$b_n^{-1} \Sigma_{k=1}^{n} b_k x_k \to 0.$$

THEOREM 3. *Let $a_n > 0$, $A_n > 0$, $A_n \uparrow \infty$, $a_n/A_n \to 0$ and $(X_n, n \geqslant 1)$ a martingale difference such that $(X_n) \prec X$ with $EN(|X|) < \infty$. Suppose that*

$$\int_0^{\infty} x\, P(|X| > x) \int_{y \geqq x} \frac{N(y)}{y^3}\, dy\, dx < \infty, \tag{4.1}$$

$$\int_1^{\infty} P(|X| > x) \int_1^{x} \frac{N(y)}{y^2}\, dy\, dx < \infty. \tag{4.2}$$

Then $S_n / A_n \to 0$ a.s.

Proof. Put $Y_n = X_n\, I(|X_n| \leq A_n / a_n)$,

$$T_n = \Sigma_{k=1}^{n} a_k Y_k.$$

Clearly,

$$\Sigma_{k=1}^{\infty} P(X_k \neq Y_k) = \Sigma_{k=1}^{\infty} P(|X_k| > A_k / a_k)$$

$$\leq C\,\Sigma_{k=1}^{\infty} P(|X| > A_k / a_k) = C\,\Sigma_{k=1}^{\infty} \int_{\{x > A_k / a_k\}} d\,P(|X| \leqslant x)$$

$$= C\int_0^{\infty} N(x)\, dP(|X| \leqslant x) = CEN(|X|) < \infty.$$

Thus, the sequences (T_n / A_n) and (S_n / A_n) converge on the same set and to the same limit. We shall show that the series T_n / A_n converges a.s. to zero.

Now, the same method as that used in the proof of Theorem 2.1 [3] gives :

$$\Sigma_{k=1}^{n} (a_k / A_k)^2\, E\,[Y_k - E\,(Y_k \mid F_{k-1})]^2$$

$$= \Sigma_{k=1}^{n} (a_k / A_k)^2 \{ EY_k^2 - (E\,(Y_k \mid F_{k-1}))^2 \}$$

$$\leq \Sigma_{k=1}^{n} (a_k / A_k)^2\, E\,|Y_k|^2$$

$$= 2\,\Sigma_{k=1}^{n} (a_k / A_k)^2 \int_{\{0 < x \leqslant A_k / a_k\}} xP(|X_k| > x)\, dx$$

$$\leq 2C\,\Sigma_{k=1}^{n} (a_k / A_k)^2 \int_{\{0 < x \leqslant A_k / a_k\}} xP(|X| > x)\, dx$$

$$= 2C\int_0^{\infty} xP(|X| > x) \sum_{\{k\,:\,A_k / a_k \geq x\}} (a_k / A_k)^2\, dx$$

$$\leq 4C \int_0^{\infty} xP(|X| > x) \int_x^{\infty} \frac{N(y)}{y^3}\, dy\, dx < \infty,$$

where, for the last inequality we have used the fact that

$$\sum_{\{k : A_k / a_k \geqq x\}} (a_k / A_k)^2 = \lim_{u \to \infty} \sum_{\{k\,:\,x \leq A_k / a_k \leq u\}} (a_k / A_k)^2$$

$$= \lim_{u \to \infty} \int_x^u \frac{dN(y)}{y^2} = \lim_{u \to \infty} \left(\frac{N(u)}{u^2} - \frac{N(x)}{x^2} + 2\int_x^u \frac{N(y)}{y^3}\, dy \right)$$

and also

$$\frac{N(u)}{u^2} \leq 2\int_u^{\infty} \frac{N(y)}{y^3}\, dy \to 0 \text{ as } u \to \infty.$$

Hence, in view of the martingale convergence theorem and the Kronecker Lemma, we get

$$A_n^{-1} \ \Sigma_{k=1}^{n} \ a_k[Y_k - E(Y_k \mid F_{k-1})] \to 0 \ \ a.s. \qquad (4.3)$$

Note that

$$0 = E(X_n \mid F_{n-1}) = E(Y_n \mid F_{n-1}) + E(X_n I(\mid X_n \mid > A_n / a_n) \mid F_{n-1})$$

and

$$\Sigma_{k=1}^{\infty} (a_k / A_k) \, E(\mid X_k \mid I(\mid X_k \mid > A_k / a_k))$$

$$= \Sigma_{k=1}^{\infty} (a_k / A_k) \int_{\{x > A_k / a_k\}} P(\mid X_k \mid > x) dx$$

$$\leq C \, \Sigma_{k=1}^{\infty} (a_k / A_k) \int_{\{x > A_k \setminus a_k\}} P(\mid X \mid > x) dx$$

$$= C \int_1^{\infty} P(\mid X \mid > x) \sum_{\{k : 1 \leqslant A_k / a_k \leqslant x\}} (a_k / A_k) \, dx$$

$$\leq C \int_1^{\infty} P(\mid X \mid > x) \int_1^{x} \frac{N(y)}{y^2} \, dy \, dx < \infty.$$

Hence, by the Kroneckcr Lemma,

$$A_n^{-1} \ \Sigma_{k=1}^{n} \ a_k \ E(Y_k \mid F_{k-1}) \to 0 \qquad a.\ s. \qquad (4.\ 4)$$

which together with (4. 3) completes the proof.

Remark 2. (i) If $(X_n, n \geq 1)$ is a sequence of independent random variables with $(X_n) \prec X \in L^1$ and if $p = 1$ then we can see that $E(X_n I(X_n I(\mid X_n \mid > > A_n / a_n) = E(X_n I(\mid X_n \mid > A_n / a_n)) = c_n \to 0$ as $n \to \infty$. By the Toeplitz Lemma, we have $A_n^{-1} \Sigma_{k=1}^{n} a_k c_k \to 0$, i. e. (3. 4) holds without the assumption (4. 2). In this case, we obtain Theorem 2 of [1].

(ii) If the independence of $(X_n, \ n \geqq 1)$ is omitted, one must use the assumption (4.2). For example, we consider $a_{nk} = 1/n$ for $k \leq n$, $a_{nk} = 0$ for $k > n$, i. e. $S_n = \Sigma_{k=1}^{n} X_k$. In this case, if $E \mid X \mid < \infty$ then $EN(\mid X \mid) < \infty$ and (3.1) holds. On the other hand.

$$\int_1^{\infty} P(\mid X \mid > x) \int_1^{x} \frac{N(y)}{y^2} \, dy dx = \int_1^{\infty} \text{Log}\, x P(\mid x \mid > x) \, dx$$

$$E \mid X \mid \text{Log}^+ \mid X \mid .$$

Thus, $S_{n/n} \to 0$ a. s. only if $X \in L \ \text{Log}^+ L$, as seen in Remark 1.

(iii) If $1 < p < 2$, $a_n = 1$, $n = 1, 2, \ldots$, $A_n = n^{1/p}$, we have.

COROLLARY 2. *If* $1 < p < 2$, $a_n = 1$, $A_n = n^{1/p}$ *(*X_n, $n \geq 1$*) is a martingale difference such that* $(X_n) < X \in L^p$, *then* $n^{-1/p} \Sigma_{k=1}^{n} X_k \to 0$ *a.s as* $n \to \infty$.

Proof. Note that (1.3) is satisfied by $a_{nk} = n^{-1/p}$ for $n = 1, 2, \ldots$, $k = 1, 2, \ldots, n$. It is easy to check that $N(x) - x^p$ so (4. 1) and (4. 2) hold if $X \in L^p$. It remains to use Theorem 2 to complete the proof.

The next result deals with the case whe the weights (a_n) are bounded and (A_n) are p-norms of $a_1, a_2, \ldots, a_n$. Recall from [3]:

LEMMA 3. (see [3], Lemma 2.1). *Let* $A_n = (\Sigma_{k=1}^{n} a_k^p)^{1/p}$, $n = 1, 2, \ldots, 0 < p \leq 2$, $(a_n) \in l_\infty$, $a_n > 0$ *and* $A_n \uparrow \infty$. *There exists a positive constant C such that for* $x \in R^+$ *large enough*

$$N(x) \leq C x^p \log x.$$

PROPOSITION 1. *Let* $1 < p < 2$, $a_n > 0$, $(a_n) \in l_\infty$, $A_n = (\Sigma_{k=1}^{n} a_k^p)^{1/p}$, $A_n \uparrow \infty$. *If* $(X_n, n \geq 1)$ *is a martingale difference such that* $(X_n) \prec X \in L^p$ Log$^+$ *L then* $S_{n/A_n} \to 0$ *a. s. as* $n \to \infty$.

Proof. Using Theorem 2 and Lemma 3, it suffices to check the assumptions (4. 1) and (4.2). Indeed, we have

$$\int_0^\infty x P(|X| > x) \int_x^\infty \frac{N(y)}{y^3} dy dx \leq C \int_0^\infty x P(|X| > x) \int_x^\infty \frac{y^p \log y}{y^3} dy dx$$

$$\leq C_1 \int_0^\infty x^{p-1} \log x P(|X| > dx = C_1 E |X|^p \log |X| < \infty.$$

In the same way, we obtain (4.2) when $X \in L^p \log L$

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