AN OSCILLATION CRITERION FOR AN N^{th} ORDER DIFFERENTIAL EQUATION WITH DAMPED TERM

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The aim of this note is to give a new oscillation criterion for the equation $^{(n)}(t) + p(t)\,x^{(n-1)}(t) + q(t)\,x(t) = 0$, $t \in [t_0, \infty)$, where n is even, $p(t)$ and $q(t)$ 'e non-negative continuous functions on $[t_0, \infty)$.

Consider the nth order equation with damped term

$$x^{(n)}(t) + p(t)\,x^{(n-1)}(t) + q(t)x(t) = 0, \qquad n \text{ even}, \tag{1}$$

here $p, q : [t_0, \infty) \to [0, \infty)$ are continous and $q(t)$ is not identically zero on ıy ray of the form $[t^*, \infty)$ for some $t^* \geq t_0$.

We shall restrict our attention to solutions of (1) which exist on some ray $, \infty)$. A solution of (1) is called oscillatory if it has no largest zero ; otherwise is nonoscillatory. Equation (1) is said to be oscillatory if every solution is cillatory.

For second order equation

$$\ddot{x}(t) + p(t)\dot{x}(t) + q(t)x(t) = 0, \qquad \left(\cdot = \frac{d}{dt}\right), \tag{2}$$

ıere $p, q : [t_0, \infty) \to R = (-\infty, \infty)$ are continuous, Yan [7] proved that the nditions

$$\limsup_{t\to\infty} t^{-\alpha} \int_{t_0}^{t} (t-s)^{\alpha} S^{\beta} q(s)ds = \infty,$$

d

$$\limsup_{t\to\infty} t^{-\alpha} \int_{t_0}^{t} [(t-s)\,p(s)s + \alpha s - \beta(t-s)]^2 (t-s)^{\alpha-2} s^{\beta-2} ds < \infty$$

: some $\alpha \in (1, \infty)$ and $\beta \in [0, 1)$, are sufficient so that all solutions of (2) are :illatory. His result improved those obtained by Kamenev [3] and Yeh [8, 9].

In this note we proceed further in this direction and present a new oscilla-n theorem which extends and improves Yan's criterion.

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The following three lemmas will be needed in the proofs of our results. The t two can be found in [5] and the third appeared in [4].

LEMMA 1. *Let u be a positive and n-times differentiable function on an interval* $, \infty]$. *If* $u^{(n)}$ *is of constant sign and not identically zero on any interval of th m* $[t^*, \infty)$, *then there exist a* $t_u \geq t_0$ *and an integer l,* $0 \leq l \leq n$ *with* $n + l$ *ı for* $u^{(n)}$ *nonnegative or* $n + l$ *odd for* $u^{(n)}$ *nonpositive and such that*

$$l > 0 \text{ implies } u^{(k)}(t) > 0 \text{ for } t \geqslant t_u \ (k = 0, 1, \ldots, l-1)$$

$$l \leq n-1 \text{ implies } (-1)^{e+k} u^{(k)}(t) > 0 \text{ for } t \geqq t_u (k = l, l+1, \ldots, n-1).$$

LEMMA 2. *If the function u is as in Lemma 1 and*

$$u^{(n-1)}(t)\, u^{(n)}(t) \leq 0 \text{ for every } t \geqslant t_u .$$

ı for every λ, $0 < \lambda < 1$, *we have*

$$u(\lambda t) \geqq \frac{2^{1-n}}{(n-1)!}[1/2 - |\lambda - 1/2|]^{n-1} t^{n-1} |u^{(n-1)}(t)| \tag{3}$$

all large t.

LEMMA 3. *Let*

$$\lim_{t \to \infty} \int_{\bar t}^{t} \exp\left(-\int_{\bar t}^{s} p(\tau) d\tau\right) ds = \infty \text{ for every } \bar t \geq to. \tag{4}$$

en if $x(t)$ *is a nonoscillatory solution of (1), we have* $x(t)\, x^{(n-1)}(t) > 0$ *for large* $t \geqq t_0$.

Our result is as follows:

THEOREM 1. *Let condition (4) hold. Suppose for some* $\alpha \in (1, \infty)$ *and* $\beta \in [0, n-1)$.

$$\limsup_{t \to \infty} t^{-\alpha} \int_{t_0}^{t} (t-s)^{\alpha} s^{\beta} q(s)\, ds = \infty, \tag{5}$$

$$\limsup_{t \to \infty} t^{-\alpha} \int_{t_0}^{t} [(t-s)\, p(s)\, s + \alpha s - \beta(t-s)]^2 (t-s)^{\alpha-2} s^{\beta-n} ds < \infty \tag{6}$$

en every solution of equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0$ for $\geqq t_1 \geqq t_0$. By Lemma 3, there exists $t_2 \geqq t_1$ such that $x^{(n-1)}(t) > 0$ for $t \geqslant t_2$. om (1) we obtain $x^{(n)}(t) \leq 0$ for $t \leq t_2$. The hypotheses of Lemma 1 are satisfied $[t_2, \infty)$, which implies that there exists $t_3 \geqq t_2$ such that $\dot x(t) > (0)$ for $t \geqq t_3$. is easy to check that we can apply Lemma 2 for $u = \dot x$, $\lambda = 1/2$ and conclude it there is a $t_4 \geqq t_3$ so that $\dot x[t/2] \geq \dfrac{2^{4-2n}}{(n-2)!} t^{n-2} x^{(n-1)}[t/2]$ for $t \geqq t_4$. ing the fact that $x^{(n-1)}(t)$ is a positive non-increasing function we obtain

$$\dot{x}\left[\frac{t}{2}\right] \geqq \frac{2^{4-2n}}{(n-2)!} t^{n-2} x^{(n-1)}(t) \text{ for } t \geqq t_4. \tag{7}$$

Define $w(t) = \dfrac{x^{(n-1)}(t)}{x[t/2]}$. Then it follows from equation (1) that

$$\dot{w}(t) = -q(t)\frac{x(t)}{x[t/2]} - p(t)\,w(t) - 1/2\,\frac{\dot{x}[t/2]}{x[t/2]}\,w(t)$$

Using (7) and the fact that x is positive nondecreasing function on $[t_4, \infty)$ we get

$$\dot{w}(t) + \frac{2^{3-2n}}{(n-2)!} t^{n-2} w^2(t) + p(t)\,w(t) + q(t) \leq 0, \; t \geqq t_4.$$

Hence

$$\int_{t_4}^{t} (t-s)^{\alpha} s^{\beta} \dot{w}(s)\,ds + \int_{t_4}^{t} \frac{2^{3-2n}}{(n-2)!}(t-s)^{\alpha} s^{\beta+n-2} w^2(s)\,ds +$$

$$+ \int_{t_4}^{t} (t-s)^{\alpha} s^{\beta} p(s)\,w(s)\,ds + \int_{t_4}^{t} (t-s)^{\alpha} s)^{\alpha} s^{\beta} q(s)\,ds \leq 0.$$

Noting that

$$\int_{t_4}^{t} (t-s)^{\alpha} s^{\beta} \dot{w}(s)\,ds = \alpha \int_{t_4}^{t} (t-s)^{\alpha-1} s^{\beta} w(s)\,ds$$

$$-\beta \int_{t_4}^{t} (t-s)^{\alpha} s^{\beta-1} w(s)\,ds - w(t_4)(t-t_4)^{\alpha} t_4^{\beta},$$

we obtain

$$\int_{t_4}^{t} (t-s)^{\alpha} s^{\beta} q(s)\,ds \leq w(t_4)(t-t_4)^{\alpha} t_4^{\beta} - 2\int_{t_4}^{t} f(t,s)\,g(t,s)\,ds -$$

$$-\int_{t_4}^{t} g^2(t,s)\,ds,$$

where

$$f(t,s) = \frac{1}{2}\sqrt{\frac{(n-2)!}{2^{3-2n}}}\,(t-s)^{\frac{\alpha}{2}-1} s^{\frac{\beta-n}{2}} \left[(t-s)\,p(s)\,s + \alpha s - \beta(t-s)\right],$$

$$g(t,s) = \sqrt{\frac{2^{3-2n}}{(n-2)!}}\,(t-s)^{\frac{\alpha}{2}} s^{\frac{\beta+n-2}{2}} w(s).$$

use the fact that $-2f(t,s)\,g(t,s) \leq f^2(t,s) + g^2(t,s)$, then we divide by .
l take limit superior as $t \to \infty$ to obtain

$$\lim_{t\to\infty} \sup t^{-\alpha} \int_{t_4}^{t} (t-s)^{\alpha} s^{\beta} q(s)\, ds \leqslant w(t_4)\, t_4^{\beta}$$

$$+ \frac{(n-2)!}{2^{5-2n}} \lim_{t\to\infty} \sup t^{-\alpha} \int_{t_4}^{t} (t-s)^{\alpha-2} \beta^{n-2} [(t-s)\; p(s)s +$$

$$+ \alpha s - \beta(t-s)]^2 \; ds < \infty$$

:ch contradicts condition (5). This completes the proof.

Let $p(t) = 0$. The conditions (4) and (6) are automatically satisfied and thus have:

COROLLARY 1. *Suppose for some $\alpha \in (1, \infty)$ and $\beta \in [0, n-1)$, condition (5) is sfied. Then all solutions of equation (1) are oscillatory.*

Remark 1. Corollary 1 improves and generalizes our Theorem 1 in [1].

One can extend the above result to the following equation.

$$x^{(n)}(t) + p(t)\, x^{(n-1)}(t) + q(t)\, f(x(t)] = 0, \tag{8}$$

ven, where p and q are as given above, $f: R \to R$ is coctinuous and $xf(x) > 0$ $x \neq 0$. We state:

COROLLARY 2. *Suppose*

$$f'(x) \textit{ exists and } f'(x) \geqq k > 0, \left(' = \frac{d}{dx}\right), \tag{9}$$

some constant k and all $x \neq 0$. If conditions (4) — (6) hold, then equation (8) scillatory.

If the function f in (8) is not monotonic (i.e. if condition (9) fails), we have following result.

COROLLARY 3. *Suppose*

$$\frac{f(x)}{x} \geqslant \alpha_1 > 0 \qquad \textit{for } x \neq 0. \tag{10}$$

onditions (4) — (6) hold, then equation (8) is oscillatory.

Remark 2. *The result of this paper holds for $\alpha = 0$ and $\beta \in [0, n-1]$. For ils we refer the reader to our Theorem 2 and 3 in* [2].

Remark 3. *If $n = 2$, the functions p and q need not to be of fixed sign and ce our theorem and Yan's Theorem in* [7] *are the same.*

We enlarge the domain of applicability of Theorem 1 by combining conons (5) and (6), and obtain the following results:

THEOREM 2. *Let conditions (5) and (6) in Theorem 1 be replaced by*

$$\lim_{t\to\infty} \sup t^{-\alpha} \int_{t_0}^{t} (t-s)^{\alpha-2} s^{\beta} [(t-s)^2 q(s) - \tag{11}$$

$$2^{2n-5}(n-2)!\, \{(t-s)p(s)s + \alpha s - \beta(t-s)\}^2]ds = \infty,$$

some $\alpha \in (1, \infty)$ and $\beta \in [0, n-1]$. Then the conclusion of Theorem 1 holds.

Proof. The proof is similar to that of Theorem 1 and hence is omitted.

COROLLARY 4. *Let conditions (5) and (6) in corollary 2 be replaced by condition (11). Then the conclusion of corollary 2 holds.*

COROLLARY 5. *Let conditions (5) and (6) in corollary 3 be replaced by conditions (11). Then the conclusion of corollary 3 holds.*

The following examples are illustrative:

Example 1. Consider the differential equation

$$x^{(n)}(t) + ct^{-n}x(t) = 0, \tag{12}$$

where n is even, $t \geqslant 1$ and $c > 2^{2n-5}(n-2)!\,(n-1)^2$. Condition (11) with $\alpha=2$ and $\beta = n-1$ takes the form

$$\limsup_{t\to\infty} t^{-2}\int_1^t \{[c-2)^{2n-5}(n-2)!\,(n-1)^2\,]\,[t^2 s^{-1} - 2t + s] -$$
$$- 2^{2n-3}(n-2)!\,[s-(n-1)\,(t-s)]\}\,ds = \infty$$

Thus codition (11) is satisfied and Theorem 2 ensures the oscillation of the solutions of equation (12). On the other hand, it is easy to verify that conditions (5) and (6) fail fo rall $\alpha \in (1, \infty)$ and $\beta \in [0, n-1)$, and hence Theorem 1 cannot be applied here. We note that our results in [1] hold for $\beta = 0$, and hence fail to apply to equation (12).

Example 2. Consider the differential equation

$$x^{(n)}(t) + t^{-1}x^{(n-1)}(t) + ct^{-n}x(t) = 0, \tag{13}$$

where n is even, $t \geqq 1$ and $c > 2^{2n-5}(n-2)!\,(n-2)^2$. As in example 1, we let $\alpha = 2$ and $\beta = n-1$ and note that

$$\limsup_{t\to\infty} t^{-2}\int_1^t \{[c - 2^{2n-5}(n-2)!\,(n-2)^2\,]\,[t^2 s^{-1} - 2t + s -$$
$$- 2^{2n-3}(n-2)!\,[s-(n-2)\,(t-s)]\,\}\,ds = \infty,$$

which shows that condition (11) is verified, and hence all solutions of equation (13) are osillatory by Theorem 2. Once again Theorem 1 fails to apply to equation (13), since $\beta < n-1$ and hence condition (5) is violated, also Theorem 2 in [2] cannot be applied to equation *(13)*, since $p(t) \neq 0$. One can easily check that Yan's Theorem in [7] and Yeh's Theorems in [8, 9], cannot be applied to equations (12) and (13).

Example 3. Consider the differential equation

$$x^{(n)}(t) + t^{-1}x^{(n-1)}(t) + c\,t^{-n}x(t)\ \exp\ (\sin\ x(t)) = 0,\ t \geqq 1, \tag{14}$$

where n is even and $c > \mathrm{e}\,2^{2n-5}(n-2)!\,(n-2)^2$. Here

$$\frac{f(x)}{x}\exp(\sin x) \geqq \frac{1}{\mathrm{e}} \text{ for all } x,$$

and condition (11) is satisfied for $\alpha = 2$ and $\beta = n-1$. The hypotheses of Corollary 5 are satisfied and hence all solutions of equation (14) are oscillatory. It is easy to check that Theorem 2 in [1], Theorem 3 in [2] and Corollary 3 are not applicable to equation (14).

xample 4. The differectial equation

$$x^{(n)}(t) + t^{-1} x^{(n-1)}(t) + ct^{-n} \sinh x(t) = 0, \ t > 0 \qquad (15)$$

: n is even and $c > 2^{2n-5} (n-2)! \, (n-2)^2$, is oscillatory by Corollary $\alpha = 2$ and $\beta = n - 1$. As we mentioned in the above examples the results – 3] and [7 – 9] fail to apply to equations (13) – (15). We believe that scillatory behavior of the equations (13) – (15) are not deducible from known oscillation criteria.

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