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# ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A CLASS OF EVOLUTION EQUATIONS

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## I. INTRODUCTION

Let $V$ and $H$ be real separable Hilbert spaces such that $V$ is densely continuously and compactly imbedded into $H$. Identifying $H$ and its dual space we obtain $V \subset H \subset V^*$ where $V^*$ is the dual space of $V$. We denote by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$ the norm in $V$, $H$ and $V^*$, respectively, and by $(.,.)$ the pairing between $V^*$ and $V$ as well as the scalar product in $H$. Let $S = [0, T]$ be a finite interval of the real axis. For an arbitrary Hilbert space $X$ we denote by $L^2(S, X)$ and $C(S, X)$ the usual spaces of the quadratically integrable and the continuous functions on $S$ with values in $X$, respectively. Let us define

$$\mathcal{V} = L^2(S, V), \ \mathcal{H} = L^2(S, H), \ \mathcal{V}^* = L^2(S, \mathcal{V}^*),$$

$$\langle f, u\rangle = \int_S (f(t), u(t))dt, \ f \in \mathcal{V}^*, \ u \in \mathcal{V},$$

$$Y = \mathcal{V} \cap C(S, H), \ \|u\|_Y = \|u\|^2_{\mathcal{V}} + \|u\|_{C(S,H)}, \ u \in \mathcal{Y},$$

$$W = \{u \in \mathcal{V} : u' \in \mathcal{V}^*\}, \ \|u\|^2_w = \|u\|^2_{\mathcal{V}} + \|u'\|^2_{\mathcal{V}^*}, \ u \in W.$$

where $u'$ is the derivative of $u \in \mathcal{V}$ in the sense of distributions on $S$ with values in $V$.

We consider the following initial value problem

$$\begin{cases} u' + A(u, u) + B(u, u) = 0 \\ \qquad u(0) = a \in H, \ u \in W \end{cases} \tag{1.1}$$

where A and B are operators with the following properties

(I) $A \in (H \times V \to V^*)$,

(II) $A(.,v) \in (H \to V^*)$ is continuous for each $v \in V$,

(III) $\forall u \in H \ \|A(u, o)\|_* \leqq M(|u| + 1)$, $M =$ const $> 0$,

(IV) $\forall v_1, v_2 \in V, \ u \in H, \ (A(u, v_1) - A(u, v_2), v_1 - v_2) \geqq m\|v_1 - v_2\|^2$, $m =$ const $> 0$,

(V) $\forall v_1, v_2 \in V, u \in H, \ \|A(u, v_1) - A(u, v_2)\|_* \leqq M\|v_1 - v_2\|$

I₁) $B \in (\mathcal{Y} \times \mathcal{Y} \to \mathcal{V}^*)$, $\langle B(u, v), w\rangle = \int_S b((u, t), v(t), w(t))dt$,

e

II₁) $b \in (V \times V \times V \to R)$ 3 — linear, $b(x,y,y) = 0\ \forall x, \forall y \in V$,

$$| b(x, y, z) | \leqq M_1( | x | \| x \| \ | y | \ \| y \| )^{\frac{1}{2}} \ \| z \|,$$

$$| b(x, y, z) | \leqq K(z) | x | \ | y | \qquad \forall x,y \in V,\ z \in V,$$

re $M_1$ is a positive constant and K(z) is bounded.

The purpose of the present paper is to study the existence and uniqueness olutions for Problem (1. 1). In the case where the operator $A$ depends only ne variable ($A(u,v) = Cv$) Problem (1.1) was proved by Galerkin's method 2]. In this paper the solution existence will be shown by using the fixed it theorem of Schauder.

It is to be noticed that the space $W$ defined above is continuously imbed- into $C(S, H)$ (see e.g. [4]) so that the initial condition $u(o) = a \in H$ in (1.1) es sense. For the investigation in the next sections we also note that $W$ is pactly imbedded into $\mathcal{H}$ (see[5]). Furthermore, the function $A(u, v)$ defined $A(u, v))(t) = A(u(t)), v(t))$, $u \in \mathcal{H}$, $v \in \mathcal{V}$, belongs to $\mathcal{V}^*$ (see [1]).

The paper consists of three sections. After the introduction (Section I) we ve the existence and the uniqueness of solutions for Problem (1.1) in Section nd Section III, respectively.

Problem (1.1) has many applications in the hydrodynamics (see also [2]). application to the Marangoni equation will be given in a subsequent paper.

## II. THE EXISTENCE THEOREM

THEOREM 2.1. *Let (I)-(V) and $(I_1)$, $(II_1)$ be satisfied. Then the initial value oblem (1.1) has a solution.*

*Proof*. 1) By virtue of the Conditions (IV) and (V) we have for all $x, y \in V$, $) \in H$ and $t \in S$

$$(A(u(t), x) - A(u(t), y), x - y) \geqq m \| x - y \|^2,$$

d

$$\| A(u(t), x) - A(u(t), y) \|_* \leqq M \| x - y \|^2.$$

Fro.n Theorem 1 in [2] it follows that for each $u \in \mathcal{H}$ (arbitrary, but fixed) e initial value problem

$$\begin{cases} v' + A(u,v) + B(v, v) = 0 \\ v(0) = a \in H,\ v \in W \end{cases} \tag{2.1}$$

is a unique solution. We denote by Su this solution. It is clear that $S \in (\mathcal{H} \to \mathcal{V} \subset \mathcal{H})$. We shall prove by the fixed point theorem of Schauder that the perator $S$ has a fixed point and, consequently, Problem (1.1) has a solution.

2) We first show that the operator $S \in (\mathcal{H} \to \mathcal{H})$ is continuous. It is easy to ee that for any fixed element $v \in \mathcal{V}$ the mapping $[s, u] \to A(u, v(s))$ satisfies he Cara theodory condition and

$$\| A(u, v(s)) \|_* \leqq \| A(u, v(s)) - A(u, o) \|_* + \| A(u, o) \|_*$$
$$\leqq M \| v(s) \| + \| A(u, o) \|_*$$
$$\leqq M ( \| v(s) \| + | u | + 1).$$

herefore, it follows from Krasnoselski's Theorem on the continuity of the emyzki operator that the operator $A(.,v) \in (\mathcal{H} \to \mathcal{V}^*)$ is continuous [6].

Let $\{u_n\} \subset \mathcal{H}$ be a sequence converging to $u$ in $\mathcal{H}$. Let $v_n = Su_n$, $v = Su$. hen, for all $t \in S$ we have

$$O = \int_0^t ((v' + A(u, v) + B(v, v) - v'_n - A(u_n, v_n) - B(v_n, v_n)(s), v(s) - v_n(s))ds$$

$$\geqq \frac{1}{2} | (v - v_n)(t) |^2 + \int_0^t \{(A(u(s), v(s)) - A(u_n(s), v(s)), v(s) - v_n(s)) + (A(u_n(s), v(s)) - A(u_n(s), v_n(s)), v(s) - v_n(s)) + ((B(v - v_n, v)(s), v(s) - v_n(s))\} ds$$

$$\geqq \frac{1}{2} | v(t) - v_n(t) |^2 + m \int_0^t \| v(s) - v_n(s) \|^2 ds - \int_0^t \| A(u(s), v(s)) - A(u_n(s), v(s)) \|_* . \| v(s) - v_n(s) \| ds - M \int_0^t | v(s) - v_n(s) |^{\frac{1}{2}} | v(s) |^{\frac{1}{2}} \| v(s) \|^{\frac{1}{2}} \| v(s) - v_n(s) \|^{\frac{3}{2}} ds$$

$$\geqq \frac{1}{2} | v(t) - v_n(t) |^2 + \frac{m}{2} \int_0^t \| v(s) - v_n(s) \|^2 ds - \frac{1}{2m} \| A(u(s), v(s)) - a(u_n, v \|_{\mathcal{V}^*}^2 - C \int_0^t \| v(s) \|^{\frac{1}{2}} | v(s) - v_n(s) |^{\frac{3}{2}} ds$$

$$\geqq \frac{1}{2} | v(t) - v_n(t) |^2 + \left(\frac{m}{2} - \delta\right) \int_0^t \| v(s) - v_n(s) \|^2 ds - \frac{1}{2m} \| A(u, v) - A(u_n, v) \|_{\mathcal{V}^*}^2 - C_0(\delta) \int_0^t \| v(s) \|^2 | v(s) - v_n(s) |^2 ds, \qquad (2.2)$$

ere $\delta = \text{const} > 0$ is arbitrary such that $\frac{m}{2} - \delta > 0$ and $C_0(\delta)$ is a posi-
constant depending only on $\delta$.
From the inequality (2.2) it followsws that for all $t \in S$ we have

$$|v(t) - v_n(t)|^2 \leqq C(n) + \int_0^t k(s)\,|v(s) - v_n(s)|^2\,ds,$$

ere $C(n) := \frac{1}{m} \|A(u,v) - A(u_n, v)\|^2_{\mathcal{V}^*} \to 0$ as $n \to \infty$ and $k(s) :=$ $(\delta)\,\|v(s)\|^2 \in L^1(s)$. Using Gronwall's lemma we obtain from the last quality

$$\|v - v_n\|_{C(S,H)} \to 0 \text{ if } n \to \infty. \tag{2,3}$$

view of (2.2) we also have

$$\left(\frac{m}{2} - \delta\right)\int_0^T \|v(s) - v_n(s)\|^2\,ds \leqq \frac{1}{2}\,C(n) + \frac{1}{2}\int_0^T k(s)\,\|v(s) - v_n(s)|^2\,ds.$$

This inequality shows that

$$\|v - v_n\|^2_{\mathcal{V}} \leqq C\,C(n) + C\,\|v - v_n\|_{C(S,H)}.$$

Hence $\|v - v_n\|_{\mathcal{V}} \to 0$ $(n \to \infty)$, and consequently;

$$\|v - v_n\|_{\mathcal{H}} \to 0 \text{ if } n \to \infty. \tag{2.4}$$

3) Now we prove the compactness of the operator $S$.
Let $v$ be the solution of Problem (2.1) for $u \in \mathcal{H}$ (i.e. $v = Su$). Then we ve

$$0 = \int_0^t ((v' + A(u,v) + B(v,v))(s), v(s))\,ds$$

$$= \int_0^t ((v + A(u(s), v(s)) - A(u(s), o) + A(u(s), o), v(s))\,ds$$

$$\geqq \frac{1}{2}|v(t)|^2 - \frac{1}{2}|a|^2 + m\int_0^t \|v(s)\|^2\,ds - \int_0^t \|A(u(s), o)\|_*\,\|v(s)\|\,ds$$

$$\geqq \frac{1}{2}|v(t)|^2 - \frac{1}{2}|a|^2 + \frac{m}{2}\int_0^t \|v(s)\|^2 ds - \frac{1}{2m}\int_0^t \|A(u(s), o)\|^2_* ds$$

$$\geqq \frac{1}{2}|v(t)|^2 - \frac{1}{2}|a|^2 + \frac{m}{2}\int_0^t \|v(s)\|^2 ds - \frac{1}{2m}\int_0^t M^2\,|u(s)| + 1)^2 ds$$

$$\geqq \frac{1}{2}|v(t)|^2 + \frac{m}{2}\int_0^t \|v(s)\|^2 ds - C\left(1 + \int_0^t |u(s)|^2\,ds\right),\ C = \text{const} > 0. \tag{2.5}$$

From this inequality it follows that

$$|v(t)|^2 \leqq C\,(1 + \|u\|^2_{\mathcal{H}}) \qquad C = \text{const} > 0. \tag{2.6}$$

his implies

$$\| v \|^2_{C(S,H)} \leqq (1 + \| u \|^2_{\mathscr{H}}). \tag{2.7}$$

sing the inequality (2. 5) we obtain

$$\frac{m}{2} \| v \|^2_{\mathscr{V}} \leqq C(1 + \| u \|^2_{\mathscr{H}}).$$

i. e

$$\| v \|^2_{\mathscr{V}} \leqq C(1 + \| u \|^2_{\mathscr{H}}). \tag{2.8}$$

n the other hand,

$$\| v' \|_{\mathscr{V}^*} = \| -A(u, v) - B(v, v) \|_{\mathscr{V}^*}$$
$$\leqq \| A(u, v) - A(u, o) \|_{\mathscr{V}^*} + \| A(u, o) \|_{\mathscr{V}^*} + \| B(v, v) \|_{\mathscr{V}^*}$$
$$\leqq M \| v \|_{\mathscr{V}} + C(1 + \| u \|_{\mathscr{H}}) + \| B(v, v) \|_{\mathscr{V}^*}.$$

or all $w \in \mathscr{V}$ we have

$$< B(v, v), w > = \int_S b(v(s), v(s), w(s))\, ds$$
$$\leqq M \int_S | v(s) | \, \| v(s) \| \, \| w(s) \| \, ds$$
$$\leqq M_1 \left( \int_S | v(s) |^2 \, \| v(s) \|^2 \, ds \right)^{\frac{1}{2}} \left( \int_S \| w(s) \|^2 \, ds \right)^{\frac{1}{2}}.$$

om this it follows that

$$\| B(v, v) \|_* = M_1 \| v \|_{C(S,H)} \| v \|_V.$$

ing this inequality and (2. 7), (2. 8) we have

$$\| B(v, v) \|_{\mathscr{V}^*} \leqq C(1 + \| u \|^2_{\mathscr{H}}) \quad C = \text{const} > 0. \tag{2.9}$$

ice $v$ is a solution of the equation (2. 1) we see that

$$\| v' \|_{\mathscr{V}^*} \leqq \| B(v, v) \|_{\mathscr{V}^*} + \| A(u, u) - A(u, o) \|_{\mathscr{V}^*} + \| A(u, o) \|_{\mathscr{V}^*}.$$

om the last inequality and (III), (V), (2.8) and (2.9) we obtain

$$\| v' \|_{\mathscr{V}^*} \leqq C(1 + \| u \|_{\mathscr{H}} + \| u \|^2_{\mathscr{H}}), \; C = \text{const} > 0 \tag{2.10}$$

e inequalities (2. 8) and (2. 10) give the following estimate

$$\| v \|_W \leqq C(1 + \| u \|_{\mathscr{H}} + \| u \|^2_{\mathscr{H}}), \; C = \text{const} > 0.$$

nce, the operator $S$ maps a bounded subset of the space $\mathscr{H}$ into a bounded set of the space $W$. Since $W$ is compactly imbedded into $\mathscr{H}$ it follows that operator $S$ is compact.

We now prove that $S(E) \subset E$ for a suitable set $E \subset \mathscr{H}$.

From (2.5) we have for all $t \in S$

$$| v(t) |^2 \leqq \lambda \left(1 + \int_0^t \| u(s) \|^2 \, ds\right), \; \lambda = \text{const} > 0.$$

This implies

$$e^{-2\lambda t}|v(t)|^2\,dt \leqq \int_s e^{-2\lambda t}\left(1+\int_0^t |u(s)|^2\,ds\right)dt$$

$$\leqq -\frac{1}{2}e^{-2\lambda t}\left(1+\int_0^t |u(s)|^2\,ds\right)\Big|_0^T + \frac{1}{2}\int_s e^{-2\lambda t}|u(t)|^2\,dt$$

$$\leqq \frac{1}{2}+\frac{1}{2}\int_0 e^{-2\lambda t}|u(t)|^2\,dt$$

Let $E$ be the set defined by

$$E = \{ u \in \mathscr{H} : \int_s e^{-2\lambda t}|u(t)|^2\,dt \leqq 1 \}.$$

is clear that E is a closed bounded convex subset of the space $\mathscr{H}$ and rthemore $S(E) \subset E$. By the fixed point theorem of Schauder it follows then at the operator $S$ has a fixed point. This completes the proof of **Theorem 2.1.**

## III. UNIQUENESS THEOREM

In the previous Section, under assumptions **(I) — (V)** and $(I_1)$, $(II_1)$ we ıve proved the existence of a solution of the initial value Problem (1.1). In this ection we shall show the uniqueness of this solution if the operator $A$ satisfies certain regularity condition.

We make the following assumption:

'I) For each solution $u$ of Problem (1. 1), the function $\varphi_\delta(u)$ defined by

$$\varphi_\delta(u) = \max\left\{0,\ \sup_{\substack{z=0\\ z\in V}} \frac{1}{|z|}\left(\|A(u+z,u) - A(u,u)\|_* - \delta\|z\|\right)\right\}$$

$$\forall\ u \in v,\ \delta \in R,$$

ıtisfies the condition

$$\varphi_{mo}(u(.)) \in L^2(S) \text{ for } m_0 < m,$$

here $m$ is the constant in the condition (IV).

In the sequel we need the following

LEMMA 3.1. *Under the assumption (IV) we have for all $m_0 \in (0, m)$:*

$$(A(u+z,\ u+z) - A(u,u),\ z) \geqq m_1\ \|z\|^2 - \rho(u)\,|z|^2\ \forall\ u, z \in V,$$

*here*

$$m_1 := \frac{m - m_0}{2},\quad \rho(u) := \frac{1}{2(m-m_0)}(\varphi_{mo}(u))^2$$

*Proof*. See [1], Lemma 1. 2

**Remark 3.1.** The assumption (VI) is equivalent to the following

$$\rho(u(.)) \in L^1(S) \tag{3.1}$$

THEOREM 3.1. *Let the conditions (I) — (VI) and ($II_1$), ($II_2$) be satisfied. hen the initial value problem (1. 1) has a unique solution.*

*Proof.* Let $u$ be a solution of the Problem (1.1), which satisfies the Contion (3. 1) and let $\bar{u}$ be another solution of the Problem (1. 1). We set $z = \bar{u} - u$.

Using Lemma 3.1 and Remark 3.1 we have for all $t \in S$

$$0 = \int_0^t (\bar{u}' + A(\bar{u}, \bar{u}) + B(\bar{u}, \bar{u}) - u' - A(u, u) - B(u, u))(s), z(s))ds$$

$$= \int_0^t ([z' + A(u + z, u + z) - A(u, u) + B(z, \bar{u})](s), z(s))ds$$

$$\geq \frac{1}{2} | z(t) |^2 + \int_0^t \{m_1 \| z(s) \|^2 - \rho(u(s)) | Z(s) |^2 -$$

$$- M( | z(s) |^{\frac{1}{2}} | \bar{u}(s) |^{\frac{1}{2}} \cdot \| \bar{u}(s) \|^{\frac{1}{2}} \| z(s) \|^{\frac{3}{2}} \} ds$$

d consequently

$$| z(t) |^2 + C_1 \int_0^t \| z(s) \|^2 ds \leqq \int_0^t k(s) | z(s) |^2 ds, \tag{3.2}$$

ıere $C_1 = 2(m_1 - \delta)$ and the constant $\delta > 0$ is chosen so that $m_1 - \delta > 0$, d

$$k(s) = 2 (C(\delta) | \bar{u}(s) |^2 \| \bar{u}(s) \|^2 + \rho(u(s))).$$

ıce $\rho(u(.)) \in L^1(S)$ and $\int_0 | \bar{u}(s) |^2 \| \bar{u}(s) \|^2 ds \leqq \| \bar{u} \|^2_{C(S,H)} + \| \bar{u} \|^2_v < \infty$,

have then $k(s) \in L^1(S)$.

By Gronwall's lemma the inequality (3.2) implies that $z = 0$. This completes proof of Theorem 3.1.

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