ACTA MATHEMATICA VIETNAMICA
Vol. 13 No 1 (1988)

ON THE PARABOLIC PSEUDODIFFERENTIAL OPERATORS OF VARIABLE ORDER IN SOBOLEV SPACES WITH WEIGHTED NORMS

NGUYEN MINH CHUONG

Elliptic pseudodifferential operators (ψ.d.o.) of variable order have been first studied by A. Unterberger and J. Bokobza [6], and then by many others, for instance by M.I. Visik and G.E. Eskin [8—10], R. Beals [1], L.R. Volevich [12], etc... In our recent papers [2—5] we have investigated parabolic ψ.d.o of variable order. Most of our results, however, were announced without proofs. The aim of the present paper is to give the detailed proofs of the main results presented in [4].

The paper consists of 5 sections. In Section 1 we introduce Sobolev spaces of variable order with weighted norms. In Section 2 we define the class of symbols of the operators to be considered. In Section 3 we present some auxiliary results. Section 4 deals with boundary value problems of parabolic ψ.d.o. in a bounded cylindrical domain of R^{n+1} in Sobolev spaces with weighted norms. Finally, in Section 5 we consider boundary value problems in a noncylindrical domain.

I. SOBOLEV SPACES OF VARIABLE ORDER WITH WEIGHTED NORMS

Denote by $x=(x_0, x_1, \ldots, x_n)=(x_0, x') \in R^{n+1}$, a point of the $(n+1)$-dimensional Euclidean space R^{n+1} and $\xi=(\xi_0, \xi_1, \ldots, \xi_n)=(\xi_0, \xi') \in R_{n+1}$ its dual variables. Let $\varphi(x) \in C^\infty(R^{n+1})$, $\varphi(x)=\text{Const}$ for $|x|>R$, where R is a real number and $|\,.\,|$ is the Euclidean norm, and the oscillation of the function $\varphi(x)$ is sufficiently small, $\varphi_+=\max\limits_x \varphi(x)$, $\varphi_-=\min\limits_x \varphi(x)$. Let

$\omega_k(x_n)=x_n^k+0(x^k)$, when $|x_n|\leqslant 1$, $\omega_k(x_n)\neq 0$ when $x_n\neq 0$ and $\omega_k(x_n)=(\operatorname{Sgn} x_n)^k$ when $|x_n|\geqslant 1$.

Denote by $H_{\varphi(x),\gamma,M}(R^{n+1}) \equiv H_{\varphi(x),\gamma,M}$ the space of functions $u(x)$ equal to zero when $x_0 < 0$ and having the following finite norm

$$\|u\|_{\varphi(x),\gamma,M} = \sum_{k=0}^{M} \left(\| \overset{2}{F}{}^{-1} [\langle\xi\rangle_\gamma^{\varphi(x)+k} \widetilde{\omega_k u}] \|_0 + \|\omega_k u\|_{\varphi-\varepsilon+k,\gamma} \right)$$

where $\|\cdot\|_0$ is the norm in $L_2(R_{n+1})$, $\langle\xi\rangle_\gamma = 1 + |\xi|^{\frac{1}{\gamma}} + |\xi'|$ $\gamma > 1$, $\|\cdot\|_{S\gamma}$, is the usual Sobolev-Slobodeskii norm, and $Fu = \tilde{u}$ is the Fourier transform of u, $\overset{2}{F}{}^{-1}C(x,\xi)$ is the inverse Fourier transform of $C(x,\xi)$ with respect to the second variable ξ.

Let $R^{n+1}_{\pm}$ be the half space $x_n > 0$ $(x_n < 0)$. Denote by $H_{\varphi(x),\gamma,M}(R^{n+1}_{\pm})$ the space of functions $u(x)$ admitting a continuation lu onto R^{n+1} and having the following finite norm

$$\|u\|^{\pm}_{\varphi(x),\gamma,M} = \inf_{l} \|lu\|_{\varphi(x),\gamma,M}$$

We say that $u(x) \in \overset{0}{H}{}^{\pm}_{\varphi(x),\gamma,M}$ if $u(x) \in H_{\varphi(x),\gamma,M}(R^{n+1})$ and vanishes when $x_n < 0$ $(x_n > 0)$ and when $x_0 < 0$.

Later, in Section 4, we shall define other kinds of Sobolev spaces of variable order.

2. CLASS OF SYMBOLS

We say that $A(x,\xi) \in O_{\alpha(x),\gamma,M}$ if

a/ $A(x,\xi) \in C^\infty$ in $x \in R^{n+1}$, is continuous for $|\xi_0|^{2/\gamma} + |\xi'|^2 = 1$ together with its derivatives of all orders with respect to ξ_n, and is analytic in ξ_0 for $\operatorname{Im} \xi_0 > 0$

b/ $A(x, t^\gamma\xi_0, t\xi') = t^{\alpha(x)} A(x,\xi)$, $t > 0$, $\operatorname{Im}\xi_0 \geqslant 0$.

Denote by $E_{\alpha(x),\gamma,M}$ the class of functions $A(x,\xi) \in O_{\varphi(x),\gamma,M}$ which satisfy the following conditions:

c/ $|A(x,\xi) - A(x,\eta)| \leqslant C(|\xi_0 - \eta_0|)^{\frac{1}{\gamma}} + |\xi' - \eta'| \times$

$\times [(1 + |\xi_0|^{\frac{1}{\gamma}} + |\xi'|)^{\alpha(n)-1} + (1 + |\eta_0|^{\frac{1}{\gamma}} + |\eta'|)^{\alpha(x)-1}]$

d/ $A(x,\xi) \neq 0$ $\forall x, \forall\xi \neq 0$, $\operatorname{Im}\xi_0 \geqslant 0$, $\operatorname{Im}\xi' \leqslant 0$.

We say that $A(x,\xi) \in Y^1_{\alpha(x),\gamma,M}$ if $A(x,\xi)$ satisfies condition a/ and, additionally

i) $A(x,\xi) \in C^\infty$ in (x,ξ) when $\xi \neq 0$.

ii) $A(x,\xi) = 0$ when $|x| > R - \delta_0$, where R and δ_0 are positive numbers,

iii) $|D^p_x D^{q''}_{\xi''} D^{q_n}_{\xi_n} A(x,\xi)| \leqslant C_{pM}\langle\xi\rangle_\delta^{\alpha(x)+\varepsilon-|q''|-q_n}$

where $0 \leqslant |q''| \leqslant 1$, $0 \leqslant q_n \leqslant M+1$, $1 \leqslant |p| < \infty$, $|q''| + q_n > 0$, $q'' = (q_1, \dots, q_{n-1})$, $q = (q_o, q'', q_n)$, $p = (p_o, p_1, \dots, p_n)$, $|q''| = q_1 + \dots + q_{n-1}$, $\xi'' = (\xi_1, \dots, \xi_{n-1})$, and ε is an arbitrary positive number, $\varepsilon = 0$ when $p = 0$.

Finally, we denote by $D^o_{\alpha(x), \gamma, M}$ the class of functions $G(x, \xi)$ satisfying conditions a/ b/ i) ii) and, in addition,

$$D^{p_n}_{x_n} D^{q^{(n)}}_{\xi^{(n)}} G(x^n, 0, 0, -1) = (-1)^{|q^n|} e^{-i\pi\alpha(x)} D^{p_n}_{x_n} D^{q^{(n)}}_{\xi^{(n)}} G(x^{(n)}, 0, 0, 1),$$

where $\xi^{(n)} = (\xi_o, \xi_1, \dots, \xi_{n-1})$, $x^{(n)} = (x_o, x_1, \dots, x_{n-1})$, $q^{(n)} = (q_o, q_1, \dots, q_{n-1})$

3. SOME AUXILIARY RESULTS

In this section we present some lemmas and propositions that we shall need for proving our main results.

LEMMA 1. *Let* $A(x, \xi) = \langle \xi \rangle_\gamma^{\alpha(x)} A_\gamma(x, \xi)$

and $|D^p_x D^{q''}_{\xi''} D^{q_n}_{\xi_n} A_\gamma(x, \xi)| \leqslant C_{pq''q_n} \langle \xi \rangle_\gamma^{-|q''| - q_n}$, (1)

Then

$$|D^p_x D^{q''}_{\xi''} D^{q_n}_{\xi_n} A(x, \xi)| \leqslant C'_{p_n q_n} \langle \xi \rangle_\gamma^{\alpha(x) + \varepsilon - |q''| - q_n} \qquad (2)$$

Proof. This follows from Leibnitz formula and the inequality

$$|D^p_x \langle \xi \rangle_\gamma^{\alpha(x)}| \leqslant C \langle \xi \rangle_\gamma^{\alpha(x) + \varepsilon} \qquad (3)$$

LEMMA 2. *Under the same conditions as in Lemma 1 we have*

$$|D^p_x A(x, \xi) - D^p_x A(x, \eta)| \leqslant C |\xi - \eta| \left[\langle \xi \rangle_\gamma^{\alpha(x) - 1 + \varepsilon} + \langle \eta \rangle_\gamma^{\alpha(x) - 1 + \varepsilon}\right] \qquad (4)$$

Proof. We give only a sketch of the proof since the argument is analogous to the one in [9].

First, when $p = 0$, using the Lagrange formula we can write $|A(x, \xi) - A(x, \eta)| \leqslant |D_{\xi_n} A(x, \zeta)| |\xi - \eta|$, $\zeta = \xi + \theta(\xi - \eta)$, $0 < \theta < 1$.

If $\alpha(x) - 1 \geqslant 0$, then in view of the obvious inequality

$$\langle \zeta \rangle_\gamma \leqslant C_1 (\langle \xi \rangle_\gamma + \langle \eta \rangle_\gamma)$$

the desired inequality follows from the hypotheses of the lemma.

Suppose now that $\alpha(x) - 1 < 0$. If $|\zeta - \eta| \frac{1}{\gamma} < \frac{\langle \xi \rangle_\gamma + \langle \eta \rangle_\gamma}{4}$ then:

$$\langle \zeta \rangle_\gamma > \frac{\langle \xi \rangle_\gamma + \langle \eta \rangle_\gamma}{4}.$$

Consequently,

$$| D_{\xi_n} A(x, \zeta) | \leqslant C_2 \langle\zeta\rangle_\gamma^{\alpha(x)-1} \leqslant C_3 \left(\langle\zeta\rangle_\gamma^{\alpha(x)-1} + \langle\eta\rangle_\gamma^{\alpha(x)-1} \right)$$

Whence (4).

On the other hand, if $| \xi - \eta |^{\frac{1}{\gamma}} \geqslant \frac{\langle\xi\rangle_\gamma + \langle\eta\rangle_\gamma}{4}$,

then

$$| A(x, \xi) - A(x, \eta) | \leqslant | A(x, \xi) | + | A(x, \eta) | \leqslant$$
$$\leqslant C_4 \left(\langle\xi\rangle_\gamma^{\alpha(x)} + \langle\eta\rangle_\gamma^{\alpha(x)} \right) \leqslant$$
$$\leqslant C_5 \left(\langle\xi\rangle_\gamma^{\alpha(x)-1} + \langle\xi\rangle_\gamma^{\alpha(x)-1} \right) | \xi - \eta |.$$

Thus the estimate (4) holds for $p = 0$

For $| p | \geqslant 1$ the proof is similar.

PROPOSITION 1. *Let* $A(x, \xi) \in Y^1_{\alpha(x), \gamma, M}$ *and* $B(x, \xi) \in Y^1_{\beta(x), \gamma, M}$. *Then for the* ψ. d. o. *A and B with symbols* $A(x, \xi)$ *and* $B(x, \xi)$ *respectively, we have*

$$AB = (AB) + J \tag{5}$$

where (AB) *is a* ψ. d. o. *with symbol* $A(x, \zeta) \cdot B(x, \zeta)$ *and J is an operator satisfying*

$$\| Ju \|_{S, \gamma, M} \leqslant C \| u \|_{S + \alpha_+ + \beta_+ - 1 + \varepsilon_0, \gamma, M} \tag{6}$$

where $C = Const$, $\varepsilon_0 > 0$ *and* $\| \cdot \|_{r, \gamma, M}$ *is the Sobolev-Slobodeskii norm with weights (see [10]).*

PROPOSITION 2. *Let* $a(x) \in C_0^\infty (R^{n+1})$. *Then*

$$\| au \|_{\alpha(x), \gamma, M} \leqslant \max_x | a(x) | \; \| u \|_{\alpha(x), \gamma, M} + C(a) \| u \|_{\alpha(x) - \frac{1}{3}, \gamma, M} \tag{7}$$

where

$$C(a) \leqslant C \max | D^k a(x) |, \quad | k | \leqslant | \varphi_+ | + M + 1$$

The proofs of Propositions 1 and 2 can be found in [3] (p. 11 — 12).

PROPOSITION 3. *Suppose that* $A(x, \xi) \in Y^1_{\alpha(x), \gamma, M}$ *Then the* ψ. d. o. *A with symbol* $A(x, \xi)$ *acts boundedly from* $H_{\varphi(x), \gamma, M}$ *into* $H_{\varphi(x) - \alpha(x), \gamma, M}$.

Proof. In view of Proposition 1 and the relation

$$D^m_{x_n} \omega_k (x_n) = C_{mk} (x_n) \; \omega_{k-n} (x_n), \quad C_{mk}(x_n) \in C_0^\infty (R^1)$$

the proof is similar to that of Lemma 5. 2 in [3] (p. 15).

LEMMA 3. *Let* $| \varphi(x) | \leqslant \frac{1}{2}$ *and* $\varphi_+ - \varphi_- < \frac{1}{2}$. *Then for the operator* θ_+ *(the operator of multiplication with a Heaviside function* $\theta (x_n)$*), we have*

$$\| \theta l u \|_{\varphi(x),\gamma,M} \leqslant C \| u \|^{+}_{\varphi(x),\gamma,M}$$

Proof. Set $B = \Lambda_{+}^{\varphi(x)}\, \theta_{+}\, \Lambda_{-}^{-\varphi(x)}$, where

$\Lambda_{\pm}^{\pm\varphi(x)}$ is the Ψ.d.o. with symbol $(\xi_n \pm i(1 + |\xi''|) \pm \xi_0^{\frac{1}{\gamma}})^{\pm\varphi(x)}$

Then as in [11], it is not difficult to show that the operator B acts boundedly from $H_{0,M}(R_{+}^{n+1})$ into $\overset{\circ}{H}{}^{+}_{0,M}(R^{n+1})$. The desired inequality follows by noting that

$$(\xi_n - i(1 + |\xi''|) - \xi^{\frac{1}{\gamma}})^{\varphi(x)} \in \Upsilon^{1}_{\varphi(x),\gamma,M}$$

LEMMA 4. *The inequality*

$$\left| D_x^p\, D_{\xi}^{q'}\, A(x,\xi) \right| \leqslant C_{pq'} \langle \xi \rangle_{\gamma}^{\alpha(x)+\varepsilon - |q'|},\ |q'| \geqslant 1 \tag{9}$$

implies

$$\left| D_x^q\, D_{\xi}^{q'}\, A(x,\xi) \right| \leqslant C_{pq'}\, \langle \xi \rangle_{\gamma}^{\alpha(x)+\varepsilon-1} \left| \xi' \right|^{1-|q'|} \tag{10}$$

Proof. Obvious, since

$$\langle \xi \rangle_{\gamma} > |\xi'|$$

4. BOUNDARY VALUE PROBLEMS IN CYLINDRICAL DOMAINS

Let $\Omega_T = (0,T) \times \Omega \subset R^{n+1}$, where Ω is a bounded domain with a smooth boundary $\partial\Omega$, and let $S = [0,T] \times \partial\Omega$

Let $A(x,\xi) = A_{+}(x,\xi)$. $A_{-}(x,\xi)$, where ord $A_{+}(x,\xi) = \chi(x)$ with $\chi(x)$ being a smooth function on S, and $A_{-}(x,\xi) = \alpha(x) - \chi(x)$. Denote by $\{\psi_j\, \Omega_j\}$ a partition of unity on $\overline{\Omega}_T$. Let $\lambda_j(x)$ be a smooth function such that

$$\lambda_j(x) = \chi(x) + \delta_j(x),\ |\ \delta_j(x)| < \frac{1}{2},\ \forall x \in \Omega_j$$

Denote by $H_{(\chi(x))+\beta(x),\gamma,M}(\Omega_{+})$ the space of functions $u\ (x)$, $x \in \Omega_T$ with the finite norm

$$\| u \|^{\Omega_T}_{(\chi(x))+\beta(x),\,\gamma,\,M} = \inf \sum_j \sum_{k=0}^{M} \| \Lambda^{\lambda_j(x)+\beta(x)+k}\, \omega_k\, \psi_j\, lu \|_0 + $$
$$+ \| lu \|_{\lambda-\varepsilon,\,\gamma},$$

where lu is the continuation of u onto $\mathbf{R}^{n+1}$, $lu = 0$ when $x_0 < 0$, $\Lambda^{\varphi(x)}$ is the Ψ. d. o. with symbol $< \xi >_{\gamma}^{\varphi(x)}\, \omega_k^{(x)} \in C^{\infty}\ (\overline{\Omega}_T)$, $\omega_k(x) = a_k(x)\, r^k + 0\,(r^k)$,

$0 < C_1 \leqslant a_k(x) \leqslant C_2$, r is the distance from x to S, $\omega_0(x) \equiv 1$ in $\overline{\Omega}_T$, $\lambda_- = \min_j \min_x (\lambda_j(x) + \beta(x))$ and $\varepsilon > 0$.

The space $\overset{o}{H}_{(\chi(x) + \beta(x),\, \gamma,\, M}(\Omega_T)$ is defined as the closure of the set $C_0^\infty(\Omega)$ in the norm of $H_{(\chi(x)) + \beta(x),\, \gamma,\, M}(\Omega_T)$. The space $H'_{(\chi(x)) + \beta(x), \gamma, M}$ is defined as usual.

Let $d_{jk}(x, \xi^{(n)})$ be the functions defined by the following formulas :

For $Q \geqslant L$:

$$d_{jk}\left(x, \xi^{(n)}\right) = i\pi' \xi^{j-1} + G_k(x, \xi)\ A_-^{-1}(x, \xi),\ 1 \leqslant k \leqslant Q,\ i \leqslant j \leqslant Q - L, \tag{11}$$

$$d_{jk}\left(x, \xi^{(n)}\right) = E_{jk}\left(x, \xi^{(n)}\right) - \pi' \pi^+ B_j(x, \xi)\, \xi_+^{L-Q}\ A_+^{-1}(x, \xi) \times$$
$$\times\ \pi^+ \xi_+^{Q-L} \times G_k(x, \xi)\, A_-^{-1}(x, \xi),\ 1 \leqslant k \leqslant Q,\ Q - L < j \leqslant Q \tag{12}$$

For $L > Q$:

$$d_{jk}\left(x, \xi^{(n)}\right) = E_{jk}\left(x, \xi^{(n)}\right) - \pi' \pi^+ B_j(x, \xi)\, \xi^{L-Q} A_+(x, \xi) \times$$
$$\times\ \pi^+ G_k(x, \xi) \times \xi^{Q-L} A^{-1}(x, \xi),\ 1 \leqslant k \leqslant Q,\ 1 \leqslant j \leqslant L \tag{13}$$

$$d_{jk}(x, \xi^{(n)} = \pi' \pi^+ B_j(x, \xi)\xi_n^{k-1-Q} A_+^{-1}(x, \xi),\ Q + 1 < k \leqslant L,\ i < j \leqslant L, \tag{14}$$

where π' is the operator

$$\pi' f(x, \xi^{(n)}, \xi_n) = \lim_{y_n \to +0} \frac{1}{2n} \int_{-\infty}^{\infty} f(x, \xi^{(n)}, \xi_n)\, e^{-i y_n \xi_n}\, d\xi_n, \tag{15}$$

π^+ is the operator defined by $\pi^+ \widetilde{f} = \widetilde{\theta_+ f}$

$$\xi_\pm = \xi_n \pm i \mid \xi'' \mid \pm \xi_0^{\frac{1}{\gamma}}$$

Denote by $[d_{jk}(x, \xi^{(n)}]$ the matrix of functions $d_{jk}(x, \xi^{(n)})$.

Let us now consider the following problem in the domain Ω_T:

$$p^+ A_n + \sum_{k=1}^{Q} G_k(k) = f(x),\ x \in \Omega_T \tag{16}$$

$$\gamma_1 p^+ B_{jn} + \sum_{k=1}^{Q} E_{jk}\rho_k = g_j(x^{(n)}),\ x^{(n)} \in S,\ 1 \leqslant j \leqslant L. \tag{17}$$

where p^+ and γ_1 are the operators of restriction to Ω and S respectively, and A, B_j, G_k and E_{jk} are ψ. d. o. with symbols $A(x, \xi)$, $B_j(x, \xi)$, $G_k(x, \xi)$ and $E_{jk}(x, \xi)$ respectively.

THEOREM 1 *Assume that*

1) $A(x,\xi) \in \gamma^{1}_{\alpha(x),\gamma,M} \cap E_{\alpha(x),\gamma,M}$, $G_k(x,\xi) \in D_{\alpha_k(x),\gamma,M}$

$E_{jk}(x,\xi)\ D^{0}_{\beta_j(x)+\alpha_k(x)-\alpha(x)+1,\gamma,M}$, $B_j(x,\xi) \in O_{\beta_j(x),\gamma,M}$

and $\beta_j(x) < \lambda_-^{*} + Q - L - \frac{1}{2}$, $\lambda_-^{*} = \min\limits_{j,\,x} \lambda_j(x)$, *and* $B_j(x,\xi)$ *satisfies, in addition, the conditions i) — iii) in Section 2.*

2) At each point $x \in S$ *the condition det* $[d_{jk}(x, \xi^{(n)})] \neq 0$ *is satisfied in the local coordinate system corresponding to* x.

Then the operator defined by (16) — (17) is a topological isomorphism from the space

$$\left(H^{0}_{(\chi(x))+Q-L,\gamma,M}(\Omega_T),\ H'_{(\chi(x))-\alpha(x)+Q-L+\alpha_k+\frac{1}{2},\gamma,0}(S)\right)$$

to the space

$$H\left(\chi(x))-\alpha(x)+Q-L,\gamma,M(\Omega_T),\ H'_{(\chi(x))-\beta_j(x)+Q-L-\frac{1}{2}\ \gamma,0}(S))\right)$$

Proof. First, in the half space R^{n+1}_+, we consider the problem

$$A(u_+,\rho_k) = p'+(Aru_+ + \sum_{k=1}^{Q} G_K r' \rho_K) = f, \tag{18}$$

$$B_j(u_+,\rho_k) = \gamma'_1\ \rho'_+ B_j r' u_+ + \sum_{k=1}^{Q} E_{jK} r'\rho_k = g_j\ ,\ 1 \leqslant j \leqslant L \tag{19}$$

where Ar is the ψ. d. o. with symbol $A(x,\xi)r(\xi') \in Y^{1}_{\alpha(x),\gamma,M}$, ord $A(x,\xi) = \alpha(x)$, $A(x,\xi)\ r(\xi') \neq 0\ \forall x,\ \forall \xi \neq 0$, Im $\xi_0 \geqslant 0$, Im $\xi' = 0$ and $G_k r'$, $B_j r'$, $E_{jk} r'$ are the ψ. d. o. with symbols

$$G_k(x,\xi)\ r'(\xi'') \in D^{0}_{\alpha_k(x),\gamma,M},\ \text{ord}\ G_k(x,\xi) = \alpha_k(x),$$

$$B_j(x,\xi)\ r'(\xi'') \in Y^{1}_{\beta_j(x),\gamma,M},\ \text{ord}\ B_j(x,\xi) = \beta_j(x) <$$

$$< \chi(x) + \delta + Q - L - \frac{1}{2},\ E_{jk}\left(x^{(n)},\xi^{(n)}\right) r'(\xi'') \in D^{0}_{\beta_j}(x) + \alpha_k(x) - \alpha(x) + 1,\gamma,M$$

and $P'+$, γ'_1 are the operators of restriction to the half space R^{n+1}_+ and to the plane $x_n = 0$ respectively. The functions $r(\xi')$ and $r'(\xi'')$ are defined by the relations

$$r(\xi) = (\xi_n + i\,|\,\xi''\,|)^M / (\xi_n + i(1 + |\,\xi''\,|))^M \tag{20}$$

$$r'(\xi'') = |\,\xi''\,|^M / 1 + |\,\xi''\,|^M \tag{21}$$

Set $\mathcal{U} = (\mathcal{A}, \mathcal{B}_j)$, $\Phi = (f, g_j)$ (22)

For $Q \geqslant L$, the parametric V of $\mathcal{U}$ is constructed as follows

$$V_o \Phi = A_+^{-1} r_+ \Lambda_+^{L-Q} \theta_+ \Lambda_+^{Q-L} A_-^{-1} r_- lf -$$

$$\sum_{j,k=1}^{Q} A_+^{-1} r_+ \Lambda_+^{L-Q} \theta_+ \Lambda_+^{Q-L} A_-^{-1} r' G_k r' V_k \phi, \tag{23}$$

$$V_K \Phi = \sum_{j=1}^{L} h_{kj}^{(o)} r' c_j, \quad 1 \leqslant k \leqslant Q \tag{24}$$

$$V = (V_o, V_k),$$

where $h_{kj}^{(o)}, c_j$ are the operators with symbols

$h_{kj}^{(o)}(x^{(n)}, \xi^{(n)})$, $c_j(x, \xi^{(n)})$, $h_{kj}^{(o)}(x, \xi^{(n)}) = h_{k_j}(x^{(n)}, 0, \xi^{(n)})$ and $[h_{k_j}(x, \xi^{(n)})]$ is the inverse of the matrix $[d_{jk}(x, \xi^{(n)})]$. The functions $c_j(x, \xi^{(n)})$ and $r_\pm(\xi')$ are defined by

$$c_j(x, \xi^{(n)}) = i\pi' \xi_+^{j-1} \tilde{l}f(\xi) A^{-1}(x, \xi), 1 \leqslant j \leqslant Q - L \tag{25}$$

$$c_j(x, \xi^{(n)}) = \tilde{g}_j(\xi^{(n)}) + \pi' \pi^+ B_j(x, \xi) A_+^{-1}(x, \xi) \xi_+^{Q-L} \times$$

$$\times \pi^+ \xi_+^{Q-L} A_-^{-1}(x, \xi) \tilde{l}f(\xi), \quad 1 \leqslant k \leqslant Q, Q - L < j \leqslant Q, \tag{26}$$

$$r_\pm(\xi') = (\xi_n \pm i \mid \xi' \mid)^M / (\xi_n \pm i(1 +) \mid \xi'' \mid))^M$$

For $Q < L$ the parametric V of $\mathcal{U}$ is defined by the formulas

$$\check{V}_0 \Phi = A_+^{-1} r_+ \Lambda_-^{L-Q} \theta_+ \Lambda_-^{Q-L} A_-^{-1} rlf -$$

$$\sum_{k=1}^{Q} A_+ r_+ \Lambda_-^{L-Q} \theta_+ \Lambda_-^{Q-L} A_-^{-1} r' G_k r' \check{V}_k \Phi +$$

$$\sum_{k=Q+1}^{L} A_{+k} r' h_{kj}^{(n)}(g_j - \gamma_1 \rho^+ B_j r' A_+^{-1} \Lambda_-^{L-Q} r_+ \theta_+ A_-^{-1} \Lambda_-^{Q-L} r l f), \tag{27}$$

$$\check{V}_k \Phi = \sum_{j=1}^{L} h_{kj}^{o} r' (g_j - \gamma_1 \rho^+ B_j r' A_+^{-1} r_+ \Lambda_-^{L-Q} \theta_+ A_-^{-1} \Lambda_-^{Q-L} r lf), \tag{28}$$

$$\check{V} = (\check{V}_o, \check{V}_k) \qquad 1 \leqslant k \leqslant Q$$

where A_{+k} is the operator with symbol $A_+^{-1}(x, \xi) \xi_n^{k-1-Q}$ and $h_{kj}^{(0)}$ is the operator with symbol $h_{kj}^{(o)}(x, \xi^{(n)})$ defined similarly to the case $Q \geqslant L$ by the functions $d_{jk}(x, \xi^{(n)})$ mentioned in the formulas (13), (14).

Now with the help of Cauchy formula for integral decomposition (see [8], [10]), and using Proposition 1 on the commutators in Section 3, it is not difficult to show that the above defined operator V ($\check{V}$) is just the parametric of $\mathcal{U}$. Further, note that the lemma of Visik—Eskin on Volterra operators (see [9], p.179—180) is still valid in Sobolev spaces with weighted norms. Using then the partition of unity $\{\psi_j, \Omega_j\}$ on $\bar{\Omega}_T$, and reasoning as in [8], we obtain the desired conclusion. This completes the proof of Theorem 1.

5. THE CASE OF NONCYLINDRICAL DOMAINS

Let Ω_T^* be a «curvilinear» cylinder with base Ω and lateral surface S^*, nowhere tangent to the plane x_0 = Const. Assume that the section Ω_t of Ω_t^* by the plane $x_0 = t$ is an n-dimensional domain with a smooth boundary. In this case the condition iii) is replaced by

iii') $| D_x^p D_{\xi_0}^{q_0} A(x, \xi) | \leqslant C_{pq_0} < \xi >_\gamma^{\alpha(x)+\varepsilon-\gamma q_0}$

$| D_x^p D_{\xi_0}^{q_0} D_{\xi''}^{q''} D_{\xi_n}^{q_n} A(x, \xi) | \leqslant C_{pM} < \xi >_\gamma^{\alpha(x)+\varepsilon-\gamma q_0 - | q'' | - q_n}$

where $0 \leqslant q_0 < \infty$, $0 \leqslant | q'' | \leqslant 1$, $0 \leqslant q_n \leqslant M_{+1}$, $1 \leqslant | p | < \infty$
$| q'' | + q_n > 0$ and $\varepsilon = 0$ when $p = 0$.

By the change of variables $\bar{x}_0 = x_0$, $\bar{x}' = S_j(x_0, x')$, the equation of $\Omega_j \cap S^*$ takes the form $\bar{x}_n = 0$.

In the same way as in [5], [9], it is easy to derive the following

THEOREM 2. *Assume that the hypotheses of Theorem 1, with iii') replacing iii), are fulfilled in the domain Ω^*. Then the operator defined by (16) – (17) is a one-to-one continuous mapping of the space*

$$\left(\overset{o}{H}_{(\chi(x))+Q-L, \gamma, M} (\Omega_T^*), H'_{(\chi(x))-\alpha(x)+Q-L+\alpha_k(x)+\frac{1}{2}, \gamma, 0} (S)^* \right)$$

onto the space

$$\left(H_{(\chi(x))-\alpha(x)+Q-L, \gamma, M} (\Omega_T^*), H'_{(\chi(x))-\beta_j(x)+Q-L-\frac{1}{2}, \gamma, 0} (S)^* \right)$$

REFERENCES

[1] R. Beals, *A general calculus of pseudo-differential operators*, Duke Math. J., 42 (1975), 1 — 42.

[2] Nguyễn Minh Chương, *Parabolic pseudo-differential operators of variable order*, Soviet Math. Dokl. 23 (1981), 675 — 678.

[3] Nguyễn Minh Chương, *On the isomorphism of Sobolev spaces of variable order*, Math. Sboruik, 121 (1983), 3—17 (Russian).

[4] Nguyễn Minh Chương, *Parabolic pseudo-differential operators of variable order in Sobolev spaces with weighted norms*, Soviet Math. Dokl., 25 (1982), 132–165.

[5] Nguyễn Minh Chương, *Parabolic systems of pseudo-differential equations of variable order*, Soviet Math. Dokl., 55(1982), 636–639.

[6] A. Unterberger. j. Bokobka, *Les operateur pseudodifferentiels d'ordre variable*, Comptes Rendus Acad. Sci. (Paris), 261 (1965), 2271–2273.

[7] M.I. Visik-G.I. Eskin, *Singular elliptic equations and systems of vaviable order*, Dokl. Akad. Nauk USSR 156 (1964), 243–246 (*Russian*).

[8] M.I. Visik–G.I Eskin, *Equations in convolution of variable order*, Trudy Mosk. Ob. 16(1967), 26–49 (*Russian*).

[9] M.I. Visik-G.I. Eskin, *Parabolic convolution equations in a bounded domain*. Math. Sbornik 71 (1976), 162–190 (*Russian*)

[10] M.I. Visik-G.I. Eskin, *Convolution equations in a bounded domain in the spaces with weighted norms*, Math. Sbornik 69 (1966). 65–110 (*Russian*).

[11] M.I. Visik-G.I. Eskin, *Sobolev-Sotobodeskii spaces of variable orber with weighted norms and its applications to mixed boundary value problems*, Sibirs. Math. J. 9 (1968), 973–997 (*Russian*).

[12] L.R. Volevich, *Local properties of non homogenous pseudodifferential operators* Trudy Mosk. Ob. 16(1967), 51–98 (*Russian*).

Received September 10, 1986

INSTITUTE OF MATHEMATICS, P.O.BOX 631, BO HO, HANOI, VIETNAM