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APPROXIMATING AN OPTIMAL CONTROL PROBLEM OF QUANTUM PROCESSES BY THE FINITE ELEMENT METHOD

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1. INTRODUCTION

In recent years, much attention has been paid to control problems of objects governed by equations of quantum mechanics, equations of electrodynamics and equations of quantum fields of more general nature. Such problems are encountered in nuclear energetics, automatics and computer engineering techniques (see [1], [2—6] and references therein). While approximate methods have been extensively developed for solving various classes of control problems (see [7] and references therein), approximate methods for solving control problems of quantum processes have been little studied.

The aim of the present paper is to suggest a scheme based on the finite element method for approximating a nonlinear optimal control problem of quantum processes governed by nonstationary Schrödinger equations.

In Section 2 we describe the optimal control problem to be studied. Then in Section 3 we develop the approximation scheme for this problem. The main results are formulated in Section 4 and established in Section 5.

Throughout the paper we shall use the notations of the book [8].

2. OPTIMAL CONTROL PROBLEM OF QUANTUM PROCESSES

Let Ω be a bounded domain of R^n, T be a given finite, positive number. Let $Q = \Omega \times [o, T]$, $S = \Gamma \times [0, T]$ where Γ is the boundary of Ω. Consider the following system

$$\frac{\partial \psi}{\partial t} - i \sum_{k,j=1}^{n} \frac{\partial}{\partial x_k}\left(a_{kj}(x) \frac{\partial \psi}{\partial x_j}\right) + ia(x)\,\psi + iu(x)\psi = 0 \qquad (2.1)$$

$$(x, t) \in Q,\ i = \sqrt{-1},$$

$$\psi\,|_S = 0, \qquad (2.2)$$

$$\psi\,|_{t=0} = \varphi(x),\ x \in \Omega, \tag{2.3}$$

$$\|\varphi\| = 1,\ \|\cdot\| = \|\cdot\|_{L_2(\Omega)}\ , \tag{2.4}$$

where

(i) $a_{k_j}(x)$, $a(x)$ belong to the real functional space $L_\infty(\Omega)$,

(ii) $u(x) \in L_q(\Omega)$, $n = \begin{cases} \alpha > 1 & n = 1 \\ n + \varepsilon & \forall \varepsilon > 0,\ n = 2, \\ n & n > 2 \end{cases}$

(iii) $a(x)$, $u(x) \geqslant 0$ almost everywhere in Q,

(iv) $a_{kj}(x) = a_{jk}(x)$ $\quad k, j \in [1, 2, .., n]$,

(v) $\mu_1 |\xi|^2 \leqslant \sum_{k,j=1}^{n} a_{kj}(x)\, \xi_k\, \xi_j \leqslant \mu_2 |\xi|^2$,

(vi) $|\partial a_{kj} / \partial x_l| \leqslant \mu$, $k, j, l \in [1,2, .., n)$,

(vii) Ω is a ball , or a ball layer, or a parallelepiped or Ω can be transformed into one of these domains with the aid of a regular transformation $y = y(x) \in C^2(\bar{\Omega})$.

μ_1, μ_2, μ_3, are fixed and positive constants, $\xi = (\xi_1, ..., \xi_n)$ is an arbitrary vector of R^n, $|\xi|^2 = \xi_1^2 + \ldots + \xi_n^2$.

DEFINITION 1. *A function* Ψ *is said to be a generalized solution in* $W_2^{1,1}(Q)$ *of the problem (2.1) – (2.3), if* ψ *belongs to* $\overset{\circ}{W}{}_2^{1,1}(Q)$ *and satisfies*

$$\int_Q \Big(-\psi\bar{\eta}_t + i \sum_{k,j=1}^{n} a_{k_j}\, \psi_{x_k}\, \bar{\eta}_{x_j} + ia\,\psi\bar{\eta}\Big)\, dx\, dt + \\ + i \int_Q u\, \psi\, \bar{\eta}\, dx\, dt = \int_\Omega \varphi\, \overline{\eta(x,0)}\, dx \tag{2.5}$$

for all η *from* $\widehat{W}{}^1_{2,o}(Q)$.

Let

$$\varphi \in W^2_{2,o}(\Omega). \tag{2.6}$$

Then the problem (2.1) – (2.3) has a unique generalized solution in $W_2^{1,1}(Q)$ [2]). Furthermore, this solution belongs to $\overset{\circ}{W}{}_2^{2,1}(Q)$. Consequently, $\psi(x,T;u)$ makes sense and $\psi(x, T; u) \in \overset{\circ}{W}{}_2^2(Q)$. From [2] we also have

$$\|\psi(x,t,u)\| = \|\varphi\| = 1 \qquad \forall t \in [o, T] \tag{2.7}$$

Let $z(x)$ be a given function from $W^2_{2,0}(\Omega)$. Suppose the control $u(x)$ must belong to a bounded, closed and convex subset $\mathcal{U}$ of $L^+_q(\Omega)$ $(L^+_q(\Omega) = \{u(x) : u(x) \in L_q(\Omega),\ u(x) \geqslant 0$ almost everywhere in $\Omega\})$. The control problem we are concerned with is to minimize the functional

$$J(u) = \int_\Omega |\psi(x, T; u) - z(x)|^2 dx = \| \psi(x, T; u) - z(x) \|^2 \tag{2.8}$$

subject to the above constraints.

This problem arises from the control of quantum processes ([1] – [6]). It is known ([6]) that under the above mentioned conditions it always has a solution.

3. APPROXIMATING THE PROBLEM BY THE FINITE ELEMENT METHOD

Co sider the set of functions $\eta(x, t)$ of the form $\eta(x, t) = w(x)\ \Phi(t)$, where $w(x) \in \overset{\circ}{W}{}^1_2(\Omega)$, $\Phi(t) \in C(0, T)$. As is well known, this set is dense in $\overset{\circ}{W}{}^{1,1}_{2,0}(Q)$ ([8]). Putting $\eta(x, t) = w(x)\ \Phi(t)$ in (2.5) and integrating by parts yields

$$\int_0^T \left[\left(\frac{\partial \psi}{\partial t}, w \right) + i \int_\Omega \left(\sum_{k, j=1}^n a_{kj}(x)\, \psi_{x_k} \overline{\eta}_{x_j} + \right.\right.$$

$$\left.\left. + a(x) + u(x))\, \psi\, \overline{w} \right) dx \right] \Phi(t)\, dt = -\Phi(0)\, (\psi(x, 0) - \varphi, w), \text{ where}$$

$$(\cdot, \cdot) = (\cdot, \cdot)_{L_2(\Omega)}.$$

Let us denote

$$[u, \xi] = \int_\Omega \left(\sum_{k, j=1}^n a_{kj}\, \psi_{x_k} \overline{\xi}_{x_j} + (a + u)\, \psi \overline{\xi} \right) dx,\ \{u, \xi\} = \int_\Omega \mathcal{L}\psi\, \overline{\mathcal{L}\xi}\, dx,$$

where $\mathcal{L}\psi = -\sum_{k, j=1}^n \frac{\partial}{\partial x_k} \left(a_{kj}(x) \frac{\partial \psi}{\partial x_j} \right) + (a + u)\, \psi.$

Then under the assumptions (i) through (v) $[\psi, \xi]$ generates a new scalar product in $\overset{\circ}{W}{}^1_2(\Omega)$ and the norm $[\cdot] = [\cdot, \cdot]^{1/2}$ is equivalent to the previous norm $\| \cdot \|^{(1)}_{2, \Omega}$ of $\overset{\circ}{W}{}^1_2(\Omega)$; under the assumptions (i) through (vii) $\{\psi, \xi\}$ generate a new scalar product in $\overset{\circ}{W}{}_2^2(\Omega)$ and the norm $\{\cdot\} = \{\cdot, \cdot\}^{1/2}$ is equivalent to the previous norm $\|\cdot\|_{2,\Omega}^{(2)}$ of $\overset{\circ}{W}{}_2^2(\Omega)$ ([8]). Hence, we can write

$$\int_0^T \left(\left(\frac{\partial \psi}{\partial t} \right) w \right) + i\,[\psi, w] \Big) \Phi(t)\, dt = -\Phi(o)\, (\psi(x, 0) - \varphi, w).$$

Since $\phi(t)$ is arbitrary, we get

$$\left(\frac{\partial \psi}{\partial t}, w \right) + i\,[\psi, w] = 0 \text{ almost everywhere on } [o, T] \tag{3.1}$$

$$(\psi(x, 0), w) = (\varphi, w) \tag{3.2}$$

This may be accepted as a definition of the generalized **solution.**

DEFIN TION 2. *A function ψ is said to be a generalized solution in $\overset{\circ}{W}{}_2^1(Q)$ of the problem (2 . 1) — (2 . 3), if for almost every $t \in (O, T)$ $\psi(x, t)$ belongs to $\overset{\circ}{W}{}_2^1(\Omega)$ and for almost every $t \in (O, T]$ the equalities (3.1), (3.2), hold for any $w(x) \in \overset{0}{W}{}_2^1(\Omega)$ and if $\psi(x, t)$ prossesses the derivative with respect to t $\partial\psi / \partial t \in L_2(Q)$.*

To approximate the control problem (2.8) let us first approximate the problem (2 . 1) — (2 . 3). For that, we take the equalities (3 . 1) — (3 . 2) and approximate (3 . 1) and (3 . 2) with respect to x, then with respect to t.

Let $w_1, w_2, \ldots, w_N$ be a basis system in $W^2_{2,0}(\Omega)$ [(9)]. The set of linear combinations of the kind $\sum_{k=1}^{N} a_k w_k$ generates a subspace $H^{(N)}$ of $W^2_{2,0}(\Omega)$ (and of $\overset{0}{W}{}_2^1(\Omega)$).

We shall seek the approximate solution of the problem (2 . 1) — (2 . 3) in the form

$$\psi_N(x, t) = \sum_{k=1}^{N} a_k^N(t)\, w_k(x), \tag{3.3}$$

where the coefficients a_k^N are functions of $t \in [O, T]$ determined by the following system of ordinary differential equations

$$\left(\frac{\partial \psi_N}{\partial t}, w_k \right)(t) + i\,[\psi_N, w_k]\,(t) = 0 \tag{3.4}$$

with inital conditions

$$(\psi_N(x, O) - \varphi, w_k) = 0,\ k \in [1, 2, \ldots, N], \tag{3.5}$$

which are obtained from (3 . 1) — (3 . 2) by the Galerkin's method. The problem (3 . 4) — (3 . 5) can be rewritten in the form

$$\widehat{B}\,\frac{da}{dt} + i\,\widehat{A}\,a = 0 \tag{3.6}$$

$$\widehat{B}\,a\,(0) = a_{(o)}, \tag{3.7}$$

where $a\,(t) = (a_1^N\,(t), \ldots, a_N^N\,(t))^T$, $a_{(o)} = (a_{(o),1} \ldots, a_{(o),N})^T$, $a_{0,k}^N = (\varphi,\ w_k)$, $\widehat{B} = (B_{kj})$, $\widehat{A} = (A_{k_j})$, $B_{kj} = B_{jk} = (w_k,\ w_j)$, $A_{kj} = A_{jk} = [w_k, w_j]$, $k, j \in [1, 2, \ldots, N]$.

Clearly, the matrix $\widehat{B}$ is non degenerate and the matrix $\widehat{A}$ is positively defined ([9]). Hence,

$$\frac{da}{dt} + i\,\widehat{B}^{-1}\,\widehat{A}\,a = 0,\ a\,(0) = \widehat{B}^{-1}\,a_{(o)} \tag{3.8}$$

From the theory of ordinary differential equations we know that the solution of the latter problem exists, is unique and is of the form

$$a = \exp\,\{i\,\widehat{B}^{-1}\,\widehat{A}\,t\}\,a_{(o)}. \tag{3.9}$$

It follows that the approximate solution $\psi_N\,(x,\ t) = \sum_{k=1}^{N} a_k^N\,(t)\ W_k\,(x)$ exists and is unique.

Of course, the system (3. 6) is not convenient for practical use and the formula (3. 9) is enormous. To improve upon these shortcomings we discretize the system (3. 6) with respect to t.

We cover the interval $[0, T]$ by a grid $t_j = j\,\tau,\ \tau = T/M,\ j \in [0,1,\ldots, M]$ and consider the system

$$\widehat{B}\,\frac{a(j+1) - a(j)}{\tau} + i\,\widehat{A}\,\frac{a(j+1) + a(j)}{2} = 0 \tag{3.10}$$

$$\widehat{B}\,a(0) = a_{(0)}.$$

To this system we associate the minimization problem

$$I_{N,\,M}(u^N) = \left\| \sum_{k=1}^{N} a_k^N\,(M)\ W_k - \sum_{k=1}^{N} z_k^N\ w_k \right\| \to \inf \tag{3.11}$$

$$u^N = \sum_{k=1}^{n} u_k^N\ w_k \in \mathcal{U} \cap H^N\,(w_1, \ldots, w_N) \equiv \mathcal{U}^N \tag{3.12}$$

We shall assume that for every system $\{w_1, \ldots, w_N\}$ and every grid ω_τ one can compute by any available minimization method the approximate value $I^*_{N,M} + \varepsilon_{N,M}$ of the infimum $I^*_{N,M}$ of the function $I_{N,M}(u^N)$ in $\mathcal{U}^N$ subject to conditions (3. 10) and the approximate control $u^N_{M,\varepsilon} \in \mathcal{U}^N$ Such that

$$I^*_{N,M} \leqslant I_{N,M}\left(u^N_{M,\varepsilon}\right) \leqslant I^*_{N,M} + \varepsilon_{N,M}, \tag{3.13}$$

where $\varepsilon_{N,M}$ converges to zero as N and M tend simultaneously to infinity.

It is worth pointing out that approximate methods for solving optimal control problems in formulations analogous to (3.10) — (3.13) are also studied by several authors (see [7], [10] and references therein).

4. FORMULATION OF THE MAIN RESULTS

THEOREM 1. *For every N the system (3.8) has a unique solution and*

$$\| \psi(x, t) - \psi_N(x, t) \| \leqslant c\, \theta_1(N) \to 0 \text{ as } N \to \infty, \tag{4.1}$$

where $\theta_1(N)$ is the rate of convergence of $\varphi_N(x)$ to $\varphi(x)$ in $L_2(\Omega)$,

$$\| \psi(x, t) - \psi_N(x, t) \|_{W_2^1(\Omega)} \leqslant c\, \theta_2(N) \to 0 \text{ as } N \to \infty, \tag{4.2}$$

where $\theta_2(N)$ is the rate of convergence of $\varphi_N(x)$ to $\psi(x)$ in $W_2^1(\Omega)$,

$$\| \psi(x, t) - \psi_N(x, t) \|_{W_2^2(\Omega)} \to 0 \text{ as } N \to \infty. \tag{4.3}$$

Furthermore, for every N and M the system (3. 10) *has a unique solution and*

$$\| \psi(x, t_j) - \psi_N(x, t_j) \|^2 \leqslant c\left((\theta_1(N))^2 + \tau \right). \tag{4.4}$$

Throughout the sequel $c, c_1, c_2, \ldots,$ denote a generic positive constants independent of N, M.

COROLLARY. *Let $\Omega = (c, d)$. Consider a grid of $[c, d]$ $c = x_0 < x_1 < \ldots < x_{N-1} < x_N = d$, $h_j = x_j - x_{j-1}$, $h = max\, h_j$, $j \in [1, 2, \ldots, N]$ and the system of functions*

$$w_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h_j}, & x \in (x_{j-1}, x_j) \\[2mm] \dfrac{x_{j+1} - x}{h_{j-1}}, & x \in (x_j, x_{j+1}),\ j \in [1, \ldots, N-1] \\[2mm] 0, & x \in (x_j, x_{j+1}) \end{cases}$$

$$w_0(x)=\begin{cases} \dfrac{x_1 - x}{h_1}, & x \in (x_0, x_1) \\ 0, & x \bar{\in} (x_0, x_1) \end{cases} \tag{4.5}$$

$$w_N(x) = \begin{cases} \dfrac{x - x_{N-1}}{h_N}, & x \in (x_{N-1}, x_N) \\ 0, & x \bar{\in} (x_{N-1}, x_N) \end{cases}$$

Then $\theta_1(N) = h^2$, $\theta_2(N)=h$, *consequently*

$$\begin{aligned} &\| \psi(x, t_j) - \psi_N(x, t_j) \|^2 \leqslant c(h^4 + \tau), \\ &\| \psi(x, t_j) - \psi_N(x, t_j) \|^2_{W_2^1(\Omega)} \leqslant c(h^2 + \tau). \end{aligned} \tag{4.6}$$

THEOREM 2. *Let* u_* *be a solution of the problem (2.8),* $u_*^{N,M}$ *be a solution of the problem (3. 11) — (3. 12).*

Then

$$\lim_{\substack{N\to\infty \\ M\to\infty}} I^*_{N,M} = J^*, \tag{4.7}$$

$$\begin{aligned} &- c(\| u_* - u_*^{N,M} \|_q + \sqrt{\theta_1(N) + \tau}) \leqslant \\ &\leqslant I^*_{N,M} - J^* \leqslant c\sqrt{\theta_1(N) + \tau}. \end{aligned} \tag{4.8}$$

For the onedimensional case (4. 5) we have

$$\begin{aligned} &- c(\| u_* - u_*^{N,M} \|_q + \sqrt{h^4 + \tau}) \leqslant \\ &\leqslant I^*_{N,M} - J^* \leqslant c\sqrt{h^4 + \tau}. \end{aligned} \tag{4.8'}$$

If the sequence $\{ u^N_{M,\varepsilon} \}$ *is determined from (3.13), then*

$$0 \leqslant J(u^N_{M,\varepsilon}) - J^* \leqslant c(\| u_* - u^N_{m,\varepsilon} \|_q + \sqrt{\theta_1(N) + \tau}) + \varepsilon_{N,M}$$

5. PROOFS OF THE CONVERGENCE THEOREMS

Proof of Theorem 1: First we prove a priori estimates of ψ_N. Multiplying both sides of the equations (3. 4) by $a_k^N(t)$ and adding from 1 to N then integrating with respect to $t' \in (0, t)$ we obtain

$$\int_o^t \left(\left(\frac{\partial \psi_N}{\partial_t}, \psi_N \right) + i\, [\psi_N, \psi_N] \right) dt' = 0, \tag{5.1}$$

$$\| \psi_N \|^2 (t) = \| \psi_N \|^2 (0),\ 0 \leqslant t \leqslant T$$

Now. multiplying both sides of the equations (3.4) by $\frac{d}{dt} a_k^N (t)$ and adding from 1 to N yields

$$\left(\frac{\partial \psi_N}{\partial t}, \frac{\partial \psi_N}{\partial t} \right) (t) + i \left[\psi_N, \frac{\partial \psi_N}{\partial t} \right] (t) = 0 \tag{5.2}$$

Taking the imaginary part, we get

$$\left[\psi_N, \frac{\partial \psi_N}{\partial t} \right] (t) + \left[\frac{\partial \psi_N}{\partial t}, \psi_N \right] (t) = 0,$$

$$\frac{d}{dt} [\psi_N, \psi_N] (t) = 0,$$

hence

$$[\psi_N] (t) = [\psi_N] (0) \tag{5.3}$$

Taking the real part, we obtain

$$\left\| \frac{\partial \psi_N}{\partial t} \right\|^2 = - \operatorname{Im} \left(L\psi_N, \frac{\partial \psi_N}{\partial t} \right) \leqslant \left\| L\psi_N \right\|^2 / 4\varepsilon + \varepsilon \left\| \frac{\partial \psi_N}{\partial t} \right\|^2,$$

hence

$$\left\| \frac{\partial \psi_N}{\partial t} \right\|^2 \leqslant c \left\| L\psi_N \right\|^2 = c \left\{ \psi_N \right\}^2. \tag{5.4}$$

By an analogous argument, we have

$$\left(\frac{\partial \psi_N}{\partial t}, \frac{\partial}{\partial t} L\psi_N \right) - i \left(L\psi_N, \frac{\partial}{\partial t} L\psi_N \right) = 0,$$

$$\left[\frac{\partial \psi_N}{\partial t} \right]^2 - i \left(L\psi_N, \frac{\partial}{\partial t} L\psi_N \right) = 0,$$

$$\operatorname{Re} \left(L\psi_N, \frac{\partial}{\partial t} L\psi_N \right) = 0,\ \frac{d}{dt} \left\{ L\psi_N \right\}^2 = 0.$$

Thus,

$$\{ \psi_N \} (t) = \{ \psi_N \} (0) \tag{5.5}$$

From (5. 4), (5. 5) we conclude

$$\left\| \frac{\partial \psi_N}{\partial t} \right\|^2 + \left\{ \psi_N \right\}^2 (t) \leqslant c \left\{ \psi_N(x, 0) \right\}^2 \tag{5.6}$$

Now, to estimate the rate of convergence of ψ_N to ψ as N tends to infinity, let $\xi_N = \psi - \psi_N$. Then

$$\left(\frac{\partial \xi_N}{\partial t}, w_k\right) + i\,[\xi_N, w_k] = 0, \quad (\xi_N, w_k) = 0, \; k \in [1, 2, \ldots, N],$$

From (5.3) — (5.6), (3.1), (3.2) we can write

$$[\xi_N(t)] = [\varphi(x) - \psi_N(x, 0)], \tag{5.7}$$

$$\{\xi_N(t)\} = \{\varphi(x) - \psi_N(x, 0)\}. \tag{5.8}$$

$$\left\|\frac{\partial \psi_N}{\partial t}\right\|^2 (0) + \{\psi_N\}^2 (t) \leqslant c\,\{\varphi(x) - \psi_N(x, 0)\}^2. \tag{5.9}$$

Since $\psi_N(x, 0)$ is the orthogonal projection of $\varphi(x)$ *on* H^N, the strong convergence of ψ_N to ψ in the norms of W_2^1 and W_2^2 respectively follows from (5.7) — (5. 9). We have thus proved (4.1) — (4.3).

If $\Omega = (c, d)$ an w_j are defined by (4. 5) then from [9] (th. 1, p. 102) we obtain

$$\| \psi(x, t) - \psi_N(x, t) \| \leqslant c\; h^2 \tag{5.10}$$

$$\| \psi(x, t) - \psi_N(x, t) \|_{W_2^1(\Omega)} \leqslant c\, h. \tag{5.11}$$

Turning to the system (3.10), let $\tilde{a}(k) = Ba(k)$. Then

$$\tilde{a}(j+1) + i\frac{\tau}{2}\hat{A}\hat{B}^{-1}\tilde{a}(j+1) =$$

$$= \tilde{a}(j) - i\frac{\tau}{2}\hat{A}\hat{B}^{-1}\tilde{a}(j).$$

Since all the eigenvalues of the matrix $\hat{A}\hat{B}^{-1}$ are positive ([9], p. 217), it follows that $\tilde{a}(j+1) = \left(E + i\frac{\tau}{2}\hat{A}\hat{B}^{-1}\right)^{-1}\left(E - i\frac{\tau}{2}\hat{A}\hat{B}^{-1}\right)\tilde{a}(j)$,

and

$$\|\tilde{a}(j+1)\| \leqslant \|\tilde{a}(j)\| \leqslant \ldots \leqslant \|\tilde{a}(0)\| = \|a_{(0)}\|.$$

Thus,

$$\|\Psi_N(x, t_{j+1})\| \leqslant \|\Psi_N(x, t_j)\| \leqslant \ldots \leqslant \|\varphi_N(x)\|. \tag{5.13}$$

which means that the system (3.10) is absolutety stable.

Let $z(x, t_{j+1}) = \psi_N(x, t_j) - \psi(x, t_j)$. Then it is easy to prove that

$$\| z(x, t_j + 1) \|^2 \leqslant \| z(x, 0) \|^2 + k\, c\, \tau^2 \leqslant$$

$$\leqslant \| z(x, 0) \|^2 + c\tau \leqslant c\,(\theta(N) + \tau).$$

If $\Omega \equiv (c,d)$ then we have from (5.10), (5.11)

$$\| z(x,tj) \|^2 \leqslant c(h^4 + \tau).$$

The proof of Theorem 1 is complete.

Proof of Theorem 2. We shall need some lemmas.

LEMMA 1. *There exists a number $N^* \in N^+$ such that for all numbers $N > N^*$ $U_N = U \cap H_N \neq \phi$ and U_N is a convex, closed set of H_N.*

The proof of this Lemma is straightforward.

LEMMA 2. *Let all the conditions of Theorem 1 hold. Then for every $u \in U$, $N > N^*$ we have*

$$\left| J(u) - I_{N,M}\left(u_M^N\right) \right| \leqslant c\, \sqrt{\theta_1(N) + \tau}\,.$$

For the onedimensional case (4.5) the following more accurate estimate holds:

$$\left| J(u) - I_{N,M}\left(u_M^N\right) \right| \leqslant c\, \sqrt{h^4 + \tau}\,.$$

Proof. Clearly,

$$\left| J(u) - I_{N,M}\left(u_M^N\right) \right| \leqslant \Big| \| \psi(x, T; u) - z(x) \|^2 -$$

$$- \left\| \psi_N\left(x, T;\, u_M^N\right) - z_N(x) \right\|^2 \Big| \leqslant \Big(\| \psi(x, T; u) \| + \| z(x) \| +$$

$$+ \left\| \psi_N\left(x, T;\, u_M^N\right) \right\| + \| z_N(x) \| \Big) \Big(\left\| \psi(x, T; u) - \psi_N\left(x, T;\, u_M^N\right) \right\| +$$

$$+ \| z(x) - z_N(x) \| \Big)$$

Taking account of (4.4), we get

$$\left| J(u) - I_{N,M}\left(u_M^N\right) \right| \leqslant c\left(\sqrt{\theta_1(N) + \tau} + \sqrt{\theta_1(N)}\right) \leqslant c\,\sqrt{\theta_1(N) + \tau}.$$

LEMMA 3. *Let all the conditions of Theorem 1 hold and u be any control from U. Let u^N is the orthogonal projection of u on U_N. Then*

$$\left| J(u) - I_{N,M}(u^N) \right| \leqslant c\left(\| u^N - u \|_q + \sqrt{\theta_1(N) + \tau} \right).$$

For the onedimensional case (4.5) we have

$$\left| J(u) - I_{N,M}(u^N) \right| \leqslant c\left(\| u^N - u \|_q + \sqrt{h^4 + \tau} \right).$$

Proof. Using the inequality for coefficients of the equations (2.1) ([2] and [6], p.18) we can write

$$|J(u) - I_{N,M}(u^N)| = \big| \| \psi(x,T;u) - z(x) \|^2 - \| \psi(x,T;u^N) - z_N(x) \|^2 \big| \leqslant$$

$$\leqslant (\| \psi(x,T;u^N) \| + \| z(x) \| + \| \psi_N(x,T;u^N) \| + \| z_N(x) \|) \times$$

$$\times (\| \psi(x,T;u^N) - \psi_N(x,T;u^N) \| + \| z(x) - z_N(x) \|) \leqslant$$

$$\leqslant C (\| \psi(x,T;u^N) - \psi_{N,M}(x,T;u^N) \| + \| z(x) - z_N(x) \|) \leqslant$$

$$\leqslant c(\| \psi(x,T;u^N) - \psi(x,T;u) \| + \| \psi(x,T;u) - \psi_{N,M}(x,T;u^N) \| +$$

$$+ \| z(x) - z_N(x) \|) \leqslant c(\| u^N - u \|_q + \sqrt{\theta_1(N) + \tau}).$$

Let us now prove Theorem 2. As we have seen above, the set of all optimal controls of the problem (2.8) U_* is non empty. Let us peak $u_* \in U_*$. According to Lemma 1, for $N > N^*$, $U_N \neq \phi$. For $u_*^N \in U_N$ it follows from Lemma 2 that

$$I^*_{N,M} \leqslant I_{N,M}(u_*^N) \leqslant J(u_*) + c\sqrt{\theta_1(N) + \tau}.$$

Further, the function (3.11) attains its infimum on the conpact set U_N, i. e. $I^*_{N,M} > -\infty$ and $(U_N)^* \neq \phi$. Taking any control $u_*^{N,M} \in U_N^*$, we get from Lemma 3

$$J^* \leqslant J(u_*^{N,M}) \leqslant I_{N,M}(u_*^{N,M}) + c(\| u_* - u_*^{N,M} \|_q + \sqrt{\theta_1(N) + \tau})$$

$$\leqslant I^*_{N,M} + c(\| u_* - u_*^{N,M} \|_q + \sqrt{\theta_1(N) + \tau}).$$

Hence,

$$-c(\| u_* - u_*^{N,M} \|_q + \sqrt{\theta_1(N) + \tau}) \leqslant I^*_{N,M} - J^* \leqslant c\sqrt{\theta_1(N) + \tau}.$$

On the other hand

$$\| u_* - u_*^{N,M} \|_q \to 0 \quad \text{as } N, M \to \infty.$$

Then

$$\lim_{\substack{N\to\infty \\ M\to\infty}} I^*_{N,M} = J^*.$$

Consider a sequence $\{ u^N_{M,\varepsilon} \}$ determined by (4.13). Clearly

$$0 \leqslant J\left(u^N_{M/\varepsilon}\right) - J^* = \left[J\left(u^N_{M/\varepsilon}\right) - I_{N,M}\left(u^N_{M,\varepsilon}\right)\right] +$$
$$+ \left[I_{N,M}\left(u^N_{M,\varepsilon}\right) - I^*_{N,M}\right] + \left[I^*_{N,M} - J^* \right].$$

From (3.13), (4.1), (4.2) we then obtain

$$J\left(\mu^N_{M,\varepsilon}\right) - J^* \leqslant C\left(\left\| u_* - u^N_{M/\varepsilon} \right\|_q + \sqrt{\theta_1(N) + \tau}\right) + \varepsilon_{N,M}.$$

This means that the sequence $\left\{u^N_{M,\varepsilon}\right\}$ is a minimizing sequence for the problem (2.8).

The proof of Theorem 2 is complete.

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