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CAUCHY'S PROBLEM FOR SECOND ORDER PARABOLIC EQUATIONS WITH RANDOM PARAMETER(*)

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Ordinary differential equations with random parameter have been the subject of study of many papers (see [2], [4], [5] and References therein). This paper is devoted to studying the existence of solutions for a class of linear equations in partial derivatives which contain a random parameter. We consider the following Cauchy's problem:

$$Lu \equiv \frac{\partial u}{\partial t} - \sum_{K,j=1}^{n} a_{kj}(\omega, t, x) \frac{\partial^2 u}{\partial x_k \, \partial x_j}$$

$$- \sum_{k=1}^{n} b_k(\omega, t, x) \frac{\partial u}{\partial x_k} - c(\omega, t, x)\, u = f(\omega, t, x) \qquad (1)$$

with the initial condition:

$$u(\omega, o, x) = \varphi(\omega, x) \qquad (2)$$

where ω is a parameter, taken from a complete measurable space $(\Omega, \mathcal{A})$. $a_{kj}(\omega, ., .)$, $b_k(\omega, ., .)$ $(k,j = 1, 2, ..., n)$, $c(\omega, ., .)$. $f(\omega, ., .)$ and $\varphi(\omega, .)$ are complex-valued functions possessing the following properties:

a) for each $\omega \in \Omega$, $a_{kj}(\omega, ., .)$, $b_k(\omega, ., .)$ $(k,j = 1, 2, ..., n)$, $c(\omega, ., .)$ and $f(\omega, ., .)$ are defined and continuous on $[0, T(\omega)] \times \mathbf{R}^n$, where $T(.)$ is a strictly positive measurable function:

b) for each $(t,x) \in \mathbf{R}_+ \times \mathbf{R}^n$ these functions considered or functions of ω are measurable on the set $\Omega_t = \{\omega \in \Omega : t \in [0, T(\omega)]\}$;

c) φ is defined on $\Omega \times \mathbf{R}^n$, continuous in x and measurable in ω.

The parabolicity of (1) means that for each $\omega \in \Omega$, there exists $\delta(\omega) > 0$ such that

$$\mathrm{Re} \sum_{k,j=1}^{n} a_{kj}(\omega, t, x)\, \sigma_k\, \sigma_j \geqslant \delta(\omega) |\sigma|^2 \qquad (3)$$

(*) This work was performed while the author stayed at Hanoi Institute of Mathematics, on leave of absence from Pedagogical Institute of VINH.

for all $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbf{R}^n$, $t \in [0, T(\omega)]$ and $x \in \mathbf{R}^n$, where $|\sigma| = (\sum_{i=1}^{n} \sigma_i^2)^{\frac{1}{2}}$ (see [1]).

For (1) to have solutions, just as in the deterministic case, we have to require that all the above functions satisfy Hölder's condition(☆). Our main result is the following:

THEOREM. *In addition to the above stated hypotheses, we suppose that:*

a) for each $\omega \in \Omega$, the functions a_{kj}, b_k ($k, j = 1, 2, \ldots, n$), c and f satisfy the Hölder's condition with index $\alpha(\omega) > 0$ and constants $B(\omega) > 0$.

b) for each $\omega \in \Omega$

$$|\varphi(\omega, x)| \leqslant K(\omega)\, exp\, \{h(\omega)\, |x|^2\} \tag{4}$$

$$|f(\omega, t, x)| \leqslant K(\omega)\, exp\, \{h(\omega)\, |x|^2\} \tag{5}$$

where $K(\omega)$ and $h(\omega)$ are constants, possibly depending on ω.

Then there exist measurable functions g, T_0, K_0 on Ω and a function u defined on Graph $T_0 \times \mathbf{R}^n$ and possing the following properties:

(i) for each $\omega \in \Omega$, u is continuously differentiable in t, twice continuously differentiable in x and satisfies (1) — (2).

(ii) for each $(t, x) \in [0, \infty] \times \mathbf{R}^n$, $u(\cdot, t, x)$ is measurable on $\Omega_0 = \{\omega \in \Omega; t \in [0, T_0(\omega)]\}$

$$(iii)\quad |u(\omega, t, x)| \leqslant K_0(\omega) \exp\{k(\omega)\, |x|^2\} \tag{6}$$

for all $\omega \in \Omega$, $(t, x) \in$ Graph $T_0 \times \mathbf{R}^n$, where $k(\omega)$ are finiteconstants, measurably depending on ω.

We note that the conditions a) b) and the ones made thereabove imply that

c) the function $\alpha(.)$, $B(.)$, $\delta(.)$, $K(.)$, $h(.)$ can be chosen so that they are measurable on Ω.

This follows from the following simple proposition:

Let $f: \Omega \times \mathbf{R}^n \times \mathbf{R}^m \longrightarrow \mathbf{R}$ be a function, measurable in $\omega \in \Omega$ and continuous separately in $x \in \mathbf{R}^n$ and $u \in \mathbf{R}^m$. Suppose that for each $\omega \in \Omega$, there exists $u = u(\omega) \in \mathbf{R}^m$ such that

$$\sup_{x \in R^n} f(\omega, x, u(\omega)) \leqslant 1.$$

(☆) A function $f(.)$ is said to satisfy the Hölder's condition with the index $\alpha > 0$ if

$$|f(x) - f(y)| \leqslant B\, |x - y|^{\alpha}$$

for all x, y in $\mathbf{R}^n$, where B is a constant.

Then we can choose $u(.)$ in such a way that it is measurable.

To see this, set $s(\omega, u) = \sup_{x \in \mathbb{R}^n} f(\omega, x, u)$, $f_i(\omega, u) = f(\omega, x_i, u)$, where $\{x_i\}_{i=1}^{\infty}$ is a sequence dense in $\mathbb{R}^n$. Denote by F_i, F the multifunction defined by

$$F_i(\omega) = \left\{ u \in \mathbb{R}^m : f_i(\omega, u) \leqslant 1 \right\}$$

and

$$F(\omega) = \{ u \in \mathbb{R}^m : s(\omega, u) \leqslant 1 \}$$

Then Graph $F = \bigcap_{i=1}^{\infty}$ Graph F_i.

Since each f_i is $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^m)$ — measurable [4, Lemma III. 14] we have Graph $F_i = \left\{ (\omega, u) \in \mathbb{R}^m : f_i(\omega, u) - 1 \leqslant 0 \right\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^m)$, where $\mathcal{B}(\mathbb{R}^m)$ denotes the Borel σ—field in $\mathbb{R}^m$.

Hence Graph $F \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^m)$. The existence of a measurable function $u(.)$ stated in the proposition follows now from the V. Neumann's selection theorem [4, Theorem III. 22]

We note that the completeness hypothesis on $(\Omega, \mathcal{A})$ is used only in the proof of the proposition above.

The proof of the Theorem is based on the following lemmas:

LEMMA 1. *If $f(\omega, x)$ is a function measurable in ω for each fixed x and has derivative with respect to x for each fixed ω, then for any x, $f'_x(\omega, x)$ is measurable in ω.*

Proof. We have $f'_x(\omega, x) = \lim_{h \to 0} \dfrac{f(\omega, x+h) - f(\omega, x)}{h}$

Let us put $f_n(\omega, x) = \dfrac{f(\omega, x+h_n) - f(\omega, x)}{h_n}$ where $\{h_n\}_{n=1}^{\infty}$, $h_n \neq 0$ is any sequence such that $h_n \to 0$. Since $f_n(\omega, x)$ is measurable in ω and $f_n(\omega, x) \to f'_x(\omega, x)$ as $n \to \infty$, so is the function $f'_x(\omega, x)$.

LEMMA 2. [2, *Lemma* 2. 5]: *Let $(\Omega, \mathcal{A})$ be a measurable space, $(\Sigma, \mathcal{B}, \mu)$ be a measure space with measure $\mu \geqslant 0$, σ – finite, X be a separable Banach space, $f : \Omega \times \Sigma \to X$ be a $\mathcal{A} \otimes \mathcal{B}$ - measurable function such that for each ω in Ω, the function $\| f(\omega, .) \|$ is μ - integrable on Σ.*

Then $\omega \longrightarrow \int_{\Sigma} f(\omega, s)\, \mu(ds)$ is $(\mathcal{A}, \mathcal{B}(X))$ - measurable, where $\mathcal{B}(X)$ denotes the Borel σ - field on X.

In particular, if Σ is a separable metric space equipped with a Radon measure $\mu \geqslant 0$, σ — finite, $f(\omega, s)$ is measurable in ω on Ω for each fixed s and continuous in s for each fixed ω, then $\omega \longmapsto \int_\Sigma f(\omega, s)\, \mu(ds)$ is $(\mathcal{A}, \mathcal{B}(X))$ — measurable if $\int_\Sigma f(\omega, s)\, \mu(ds)$ is defined for all $\omega \in \Omega$.

LEMMA 3. *Let $f(\omega, t)$ be a function continuous in t on $[0, T(\omega)]$ for each fixed ω and measurable in ω on Ω_t for each fixed t, and let $T : \Omega \to \mathbf{R}_+$ be a measurable function. Then the function*

$$\omega \longmapsto \sup_{t \in [0, T(\omega)]} f(\omega, t)$$

is measurable.

Proof. It is easy to see that $\omega \longmapsto F(\omega) = [0, T(\omega)]$ is a measurable multifunction. Hence by [3, Theorem 5, 6] there exists a sequence $\{u_n\}_{n=1}^{\infty}$ of measurable selections of F such that $F(\omega) = \mathrm{Cl}\,\{u_n(\omega)\}_{n=1}^{\infty}$. Clearly, the function $M(\omega) = \sup_{t \in F(\omega)} f(\omega, t) = \sup_n f(\omega, u_n(\omega))$ is measurable.

LEMMA 4. Suppose:

a) $a_{kj}(\omega, t)$ are continuous in t on $[0, T(\omega)]$ for each fixed ω, measurable in ω on Ω_t for each fixed t and satisfies the condition (3),

b) the coefficients $T(\omega)$, $\delta(\omega)$ are measurable in ω.

Put $Q(\omega, t, \tau, s) = \exp\left\{-\sum_{k,j=1}^{n} \int_\tau^t a_{kj}(\omega, \beta)\, d\beta\, s_k s_j\right\}$ *where* $s_k = \sigma_k + i\gamma_k$ $(k = 1, 2, \ldots, n)$ *are complex variables and*

$G(\omega, t, \tau, x + iy) = (2\pi)^{-n} \int e^{i(x+iy,\, \sigma)}\, Q(\omega, t, \tau, \sigma)\, d\sigma$, $(t > \tau)$ *where* $(\alpha, \beta) = \sum_{i=1}^{n} \alpha_i \beta_i$, $\alpha, \beta \in C^n$.

Then

$$\left|D_x^m G(\omega, t, \tau, x + iy)\right| \leqslant C_m(\omega)\, (t - \tau)^{\frac{-n+m}{2}} \exp\left\{-g(\omega)\, \frac{|x|^2}{t-\tau} + F(\omega)\, \frac{|y|^2}{t-\tau}\right\} \tag{7}$$

where $g(\omega)$, $F(\omega)$ and $C_m(\omega)$ are measurable positive functions.

Proof: Set $M(\omega) = \sup_{t \in [0, T(\omega)]} |a_{kj}(\omega, t)|$

$$k, j = 1, 2, \ldots, n$$

It follows from Lemma 3 that $M(\omega)$ is measurable with respect to ω. We have

$$\varphi = \operatorname{Re}\Big\{-\sum_{k,j=1}^{n}\int_{\tau}^{t} a_{kj}(\omega,\beta)\,d\beta\,(\sigma_k + i\gamma_k)(\sigma_j + i\gamma_j) \leqslant$$

$$\leqslant -\sum_{k,j=1}^{n} \operatorname{Re}\int_{\tau}^{t} a_{kj}(\omega,\beta)\,d\beta\,\sigma_k\sigma_j + M(\omega)(t-\tau)\times$$

$$\times\sum_{k,j=1}^{n}(|\sigma_k||\gamma_j| + |\sigma_j||\gamma_k| + |\gamma_k||\gamma_j|) \leqslant$$

$$\leqslant \Big\{-\delta(\omega)|\sigma|^2 + M(\omega)\Big[2\sum_{k=1}^{n}|\sigma_k|\sum_{k=1}^{n}|\gamma_k| +$$

$$+(\sum_{k=1}^{n}|\gamma_k|)^2\Big]\Big\}(t-\tau) \leqslant \{-\delta(\omega)|\sigma|^2 + 2n^2 M(\omega)|\sigma||\gamma| +$$

$$+ n^2 M(\omega)|\gamma|^2\}(t-\tau).$$

By using the equality $|a||b| \leqslant \varepsilon^2(\omega)a^2 + (2\varepsilon(\omega))^{-2}|b|^2$ where $\varepsilon^2(\omega) = \delta(\omega)/4n^2 M(\omega)$, we have:

$$|\varphi| \leqslant \{[-\delta(\omega) + 2n^2\varepsilon^2(\omega)M(\omega)]|\delta|^2 + \Big[\frac{1}{2}n^2\varepsilon^{-2}(\omega)M(\omega) +$$

$$+ n^2 M(\omega)\Big]|\gamma|^2\}(t-\tau)$$

Set $\delta_1(\omega) = \dfrac{\delta(\omega)}{2}$, $F_1(\omega) = \dfrac{1}{2}n^2\varepsilon^{-2}(\omega)M(\omega) + n^2 M(\omega)$.

Clearly, the functions $\delta_1(\omega)$ and $F_1(\omega)$ are measurable in ω and

$$|Q(\omega), t, \tau, s)| \leqslant \exp\{-\delta_1(\omega)|\sigma|^2 + F_1(\omega)|\gamma|^2](t-\tau) \qquad (9)$$

We have also:

$$G(\omega, t, \tau, x+iy) = (2\pi)^{-n}\int e^{i(x+iy,\sigma)}Q(\omega, t, \tau, \sigma)\,d\sigma =$$

$$= (2\pi\sqrt{t-\tau})^{-n}\int e^{i\left(\frac{x+iy}{t-\tau},\ \beta\right)}Q(\omega, t, \tau), \frac{\beta}{t-\tau})\,d\beta$$

By virtue of (9), $G(\omega, t, \tau, x+iy)$ is an entire function with respect to $x+iy$ and the function under the integral symbol is analytic with respect to $\beta_1, \beta_2, \ldots \beta_n$. Hence by using the Cauchy's integral theorem we obtain:

$$G(\omega, t, \tau, x+iy) = (2\pi\sqrt{t-\tau})^{-n}\int e^{i\left(\frac{x+iy}{\sqrt{t-\tau}},\ \beta+i\eta\right)}\times$$

$$\times Q\left(\omega, t, \tau, \frac{\beta+i\eta}{\sqrt{t-\tau}}\right)d\beta \text{ for all } \eta = (\eta_1, \ldots, \eta_n) \in \mathbf{R}^n.$$

Therefore:

$$| G(\omega, t, \tau, x + iy) | \leqslant (2\pi \sqrt{t-\tau})^{-n} \exp\left\{-\left(\frac{x}{\sqrt{t-\tau}}, \eta\right) + \right.$$

$$\left. + F_1(\omega) | \eta |^2 \right\} \int \exp\left\{- \delta_1(\omega) | \beta |^2 - \left(\frac{y}{\sqrt{t-\tau}}, \beta\right)\right\} d\beta$$

Setting $\eta_o(\omega) = F_1(\omega)/2.$, we see that

$$\eta_o(\omega) - F_1(\omega)\, \eta_o^2(\omega) > 0.$$

Putting $\eta_i(\omega) = \operatorname{sign} x_i \dfrac{| x_i |}{\sqrt{t-\tau}} \eta_o(\omega)$, we then have

$$- \left(\frac{x}{\sqrt{t-\tau}}, \eta\right) + F_1(\omega) | \eta |^2 = \left[- \eta_o(\omega) + F_1(\omega)\, \eta_o^2(\omega)\right] \frac{| x |^2}{t-\tau} =$$

$$= - g(\omega) \frac{| x |^2}{t-\tau} \text{ where } g(\omega) = \eta_o(\omega) - F_1(\omega)\, \eta_o^2(\omega)$$

Applying now (8) with $\varepsilon(\omega) = \delta_1(\omega)/2$, $a = \beta$, $b = \dfrac{y}{\sqrt{t-\tau}}$ yields:

$$| G(\omega, t, \tau, x + iy) | \leqslant (2\pi \sqrt{t-\tau})^{-n} \exp\left\{- g(\omega) \frac{| x |^2}{t-\tau} + \right.$$

$$\left. + \frac{1}{(2\varepsilon(\omega))^2} \frac{| y |^2}{t-\tau}\right\} \int \exp\left\{(- \delta_1(\omega) + \varepsilon^2(\omega)) | \beta |^2\right\} d\beta$$

It suffices now to put $F(\omega) = 2\, \delta_1(\omega)$

and $C_o(\omega) = (2\pi \sqrt{t-\tau})^{-n} \int \exp\left\{- \dfrac{\delta_1(\omega)}{2} | \beta |^2\right\} d\beta$

to complete the proof for the case $m = 0$.

The proof for the case $m \geqslant 1$ is analogous (see I, p. 20).

PROOF OF THEOREM.

We shall find $u(\omega, t, x)$ in the form:

$$u(\omega, t, x) = \int z(\omega, t, 0, x, \xi)\, \varphi(\omega, \xi)\, d\xi +$$

$$+ \int_\tau^t d\tau \int z(\omega, t, \tau, x, \xi)\, f(\omega, \tau, \xi)\, d\xi = P + Q \qquad (10)$$

Where $Z(\omega, t, \tau, x, \xi)$ is the elementary solution of the equation $Lu = 0$.

1. PROPOSITION 1. Let $a_{kj}(\omega, t)$ be functions, continuous in t on $[0, T(\omega)]$ for each fixed ω and measurable in ω on Ω_t for each fixed t. Let $T(\omega)$ be a measurable function. Then the equation:

$$L_0 u \equiv \frac{\partial u}{\partial t} - \sum_{k,j=1}^{n} a_{kj}(\omega, t) \frac{\partial^2 u}{\partial x_k \partial x_j} = 0 \qquad (11)$$

admits the elementary solution $G_0(\omega, t, \tau, x)$ which is measurable in ω on Ω_t for each fixed t, τ, x with $t > \tau$ and satisfies the estimation:

$$\left| D_x^m G_0(\omega, t, \tau, x) \right| \leqslant C_m(\omega)(t-\tau)^{\frac{-n+m}{2}} \exp\left\{ -g(\omega)\frac{|x|^2}{t-\tau} \right\} \tag{12}$$

where $C_m(\omega)$, $g(\omega)$ are positive and measurable.

Proof. It is known that for each fixed parameter ω, the equation (11) admits as elementary solution the function:

$$G_0(\omega, t, \tau, x) = (2\pi)^{-n} \int e^{i(x, б)} Q(\omega, t, \tau, б)\, dб$$

where $Q(\omega, t, \tau, б) = \exp\left\{ -\sum_{k,j=1}^{n} \int_{\tau}^{t} a_{kj}(\omega, \beta)\, d\beta\, б_k\, б_j \right\}$ (see [1]).

By Lemm 4, there exist positive measurable constants $C_m(\omega)$- and $g(\omega)$ satisfying the estimation (12).

Applying Lemma 2 to the case $\Sigma = (t, \tau)$ (equipped with the Lebesgue's measure), we infer that $Q(\omega, t, \tau, б)$ and $G_0(\omega, t, \tau, x)$ are measurable in ω on Ω_t for fixed $t, \tau, б, x$ and $t > \tau$.

Consider now the equation

$$Lu = 0 \tag{13}$$

2. PROPOSITION 2. Suppose that the coefficients of (13) are continuous on $[0, T(\omega)] \times \mathbf{R}^n$ for each fixed ω and measurable on Ω_t for each fixed (t, x). Suppose, furthermore, that for each $\omega \in \Omega$ these coefficients satisfy the Holder's condition w th respect to x with the index $\alpha(\omega)$ and the coefficient $B(\omega)$, both being measurable in ω.

Then (13) admits the elementary solution $Z(\omega, t, \tau, x, \xi)$ which is measurable on Ω_t for each fixed t, τ, x, ξ with for $t > \tau$ and satisfies the estimation:

$$\left| D_x^m Z(\omega, t, \tau, x, \xi) \right| \leqslant C_m(\omega)(t-\tau)^{\frac{-n+m}{2}} \exp\left\{ -g_1(\omega)\frac{|x-\xi|^2}{t-\tau} \right\} \tag{14}$$

where $C_m(\omega)$ and $g_1(\omega)$ are positive and measurable.

Proof: For each $\omega \in \Omega$ let $G_0(\omega, t, \tau, x, y)$ be the elementary solution of the equation:

$$\frac{\partial u}{\partial t} = \sum_{k,j=1}^{n} a_{kj}(\omega, t, x) \frac{\partial^2 u}{\partial x_k \partial x_j}. \tag{15}$$

We shall find the elementary solution of (13) in the form:

$$Z(\omega, t, \tau, x, \xi) = G_0(\omega, t, \tau, x - y, y) + \int_\tau^t d\beta \int G_0(\omega, t, \tau, x - y, y)\, \varphi(\omega, \beta, \tau, y, \xi)\, dy = G_0 + W \tag{16}$$

where $\varphi(\omega, t, \tau, x, \xi) = \sum_{m=1}^{\infty} (-1)^m K_m(\omega, t, \tau, x, \xi)$ (17)

$$K_1(\omega, t, \tau, x, \xi) = K(\omega, t, \tau, x, \xi) = L(\omega, G_0(\omega, t, \tau, x, \xi))$$

$K_m(\omega, t, \tau, x, \xi) = \int_\tau^t d\beta \int K(\omega, t, \tau, x, y)\, K_{m-1}(\omega, \beta, \tau, y, \xi)\, dy$ (see [I, p. 28, 29]).

We now show the measurability of W.

First, by Lemma 1, $K(\omega, t, \tau, x, \xi)$ is measurable in ω on Ω_t for each fixed t, τ, x, ξ with $t > \tau$. Next, set $M_1(\omega) = \sup_{t \in [0, T(\omega)]} \{ |b_k(\omega, t, x)|, |c(\omega, t, x)| \}$.

By Lemma 3, $M_1(\omega)$ is measurable and

$$K(\omega, t, \tau, x, \xi) = \Big| \sum_{k,j=1}^{n} [a_{kj}(\omega, t, \xi) - a_{kj}(\omega, t, x)] \times$$

$$\times \frac{\partial^2}{\partial x_k \partial x_j} G_0 - \sum_{k=1}^{n} b_k(\omega, t, x) \frac{\partial}{\partial x_k} G_0 - c(\omega, t, x)\, G_0 \Big| \leqslant$$

$$\leqslant \Big[c_2(\omega) n\, B(\omega) |x - \xi|^{\alpha(\omega)} (t - \tau)^{\frac{-n-2}{2}} + c_1(\omega) M_1(\omega)\, n (t - \tau)^{\frac{-n-1}{2}} +$$

$$+ c_0(\omega) M_1(\omega) (t - \tau)^{-\frac{n}{2}} \Big] \exp\Big\{ - g(\omega) \frac{|x - \xi|^2}{t - \tau} \Big\}$$

where $g(\omega)$, $c_m(\omega)$ $(m = 0, 1, 2)$ are the constants appearing in Proposition 1.

Set now $N(\omega) = \sup_{u \in \mathbf{R}} \dfrac{u^{\alpha(\omega)/2}}{e^{c(\omega)u/2}}$. By replacing $\mathbf{R}$ by a sequence dense in it we see that $N()$ is a measurable (finite) function. Further, set

$$C_3(\omega) = C_2(\omega) \cdot N(\omega)/2. \tag{18}$$

We have:

$$|K| \leqslant \Big[C_3(\omega) n B(\omega) + C_1(\omega) M(\omega) n T(\omega)^{\frac{1-\alpha(\omega)}{2}} + C_0(\omega) M_1(\omega) \times$$

$$\times T(\omega)^{\frac{2-\alpha(\omega)}{2}} \Big] \cdot (t - \tau)^{-\frac{n+2-\alpha(\omega)}{2}} \exp\Big\{ - \frac{g(\omega)}{2} \frac{|x - \xi|^2}{t - \tau} \Big\} =$$

$$= A(\omega) (t - \tau)^{-\frac{n+2-\alpha(\omega)}{2}} \exp\Big\{ - \frac{g(\omega)}{2} \frac{|x - \xi|^2}{t - \tau} \Big\} \tag{19}$$

It is clear that $A(\omega)$ is measurable.

The function $K_2(\omega, t, \tau, x, \xi) = \int_\tau^t d\beta \int K(\omega, t, \beta, x, y) \times K_1(\omega, \beta, \tau, y, \xi)\, dy$ is continuous in the variables t, τ, x, y when $t > \tau$ and the function $F(\omega, \beta, y) = K(\omega, t, \tau, x, y) \times K(\omega, \beta, \tau, y, \xi)$ is by (19) integrable with respect to y, hence by Lemma 2 for each fixed β, the function $\omega \mapsto \int F(\omega, \beta, y)\, dy$ is measurable. Moreover, for each ω the integral $\int F(\omega, \beta, y)\, dy$ is by (19) uniformly convergent with respect to β on $[\tau, t]$, hence $\int F(\omega, \beta, y)\, dy$ is continuous in β. Thus $\omega \mapsto K_2(\omega, t, \tau, x, \xi)$ is by Lemma 2 measurable on Ω_t.

The proof of the measurability of $K_3, K_4, \ldots$ with respect to ω on Ω_t for each fixed t, τ, x, ξ with $t > \tau$ is analogous.

Now let us estimate φ. For each ω we have (see [1, p. 30]:

$$| K_m(\omega, t, \tau, x, \xi) | \leqslant \frac{\Gamma^m\left(\frac{\alpha(\omega)}{2}\right)}{\Gamma\left(\frac{m\alpha(\omega)}{2}\right)} A^m(\omega) \left(\frac{2\pi}{g(\omega)}\right)^{\frac{n(m-1)}{2}} \times$$

$$\times (t-\tau)^{\frac{m\alpha(\omega)-n-2}{2}} \exp\left\{-\frac{g(\omega)}{2} \frac{|x-\xi|^2}{t-\tau}\right\} \qquad (20)$$

Therefore, (17) is uniformly and absolutely convergent for $t-\tau \geqslant \varepsilon > 0$ and

$$| \varphi | \leqslant C(\omega) (t-\tau)^{-\frac{2-\alpha(\omega)+n}{2}} \exp\left\{\frac{-g(\omega)}{2} \cdot \frac{|x-\xi|^2}{t-\tau}\right\} \qquad (21)$$

where $C(\omega)$ is a measurable positive function.

Since $K, K_1, K_2, \ldots$ are measurable with respect to ω for each fixed t, τ, x, ξ and for $t > \tau$, it follows that φ is measurable with respect to ω for each fixed t, τ, x, ξ with $t > \tau$.

Since φ and K are measurable, by a method analogous to that used for proving the measurability of K_2 we can show that W is measurable in ω on Ω_t for each fixed t, τ, x, ξ with $t > \tau$. This and Proposition 1 imply the measurability of Z in ω on Ω_t for each fixed t, τ, x, ξ with $t > \tau$.

Moreover, φ satisfies the Hölder's condition with respect to x (see [1]). Further, just as in the proof of Lemma 4 and formula (19), we can show the measurable dependence on ω of the constants $\alpha_1(\omega)$, $g'_1(\omega)$ and $C(\omega)$ in the estimation:

$$|\Delta \varphi| = |\varphi(\omega, t, \tau, x, \xi) - \varphi(\omega, t, \tau, x', \xi)| \leqslant$$

$$\leqslant C(\omega) |x - x'|^{\alpha_1(\omega)} (t-\tau)^{-\frac{n+2-\alpha_2(\omega)}{2}} \max\left\{\exp\left(-g'_1(\omega) \frac{|x-\xi|^{\alpha(\omega)}}{t-\tau}\right),\right.$$

$$\left.\exp\left(-g'_1(\omega) \frac{|x'-\xi|^{\alpha(\omega)}}{t-\tau}\right)\right\}$$

$$\alpha_1(\omega) < \alpha(\omega), \ \alpha_2(\omega) = \alpha(\omega) - \alpha_1(\omega) \qquad (22)$$

To omplete the proof of (14) it only remains to estimate the derivatives of W. Consider, for example, the derivative $\frac{\partial^2}{\partial x_k \partial x_j} W$.

Setting $t_1 = \frac{t+\tau}{2}$ we get by [1, p. 32]:

$$\left|\frac{\partial^2 W}{\partial x_k \partial x_j}\right| \leqslant C_2(\omega) C(\omega) \int_\tau^t \frac{d\beta}{(t-\beta)(\beta-\tau)^{\frac{2-\alpha(\omega)}{2}}} \times$$

$$\times \exp\left\{-g_1(\omega)\frac{|x-y|^2}{t-\beta} - g_1(\omega)\frac{|y-\xi|^2}{\beta-\tau}\right\}\frac{dy}{(t-\beta)(\beta-\tau)^{n/2}} +$$

$$+ C_2(\omega) C(\omega) \int_{t_1}^t d\beta \int \frac{|x-y|^{\alpha_1(\omega)}}{(t-\beta)^{\frac{n+2}{2}}} \exp\left\{-g_1(\omega)\frac{|x-y|^2}{t-\tau}\right\} \times$$

$$\times \exp\left\{-g'(\omega)\frac{|y-\xi|^2}{t-\tau}\right\}\frac{dy}{(\beta-\tau)^{\frac{n+2-\alpha(\omega)}{2}}} +$$

$$+ C'(\omega) C(\omega) \int_{t_1}^t (t-\tau)^{-\frac{2-\alpha(\omega)}{2}} (\beta-\tau)^{-\frac{2-\alpha(\omega)}{2}} \times$$

$$\times \exp\left\{-g_1(\omega)\frac{|x-\xi|^2}{\beta-\tau}\right\} d\beta = I_1 + I_2 + I_3,$$

Where the constants $C_2(\omega)$, $C(\omega)$, $C'(\omega)$, $g_1(\omega)$ and $g'(\omega)$ are measurable and defined by the formulas (12), (21) and (22).

Now let us estimate I_1, I_2 and I_3. We have

$$|I_1| \leqslant C_2(\omega) C(\omega) T(\omega)^{\frac{1-\alpha(\omega)}{2}} \frac{\Gamma^2\left(\frac{\alpha}{2}\right)}{\Gamma(\alpha)} \sqrt{\frac{\pi}{g_1(\omega)}} \times$$

$$\times (t-\tau)^{\frac{2\alpha(\omega)-n-2}{2}} \exp\left\{-g_1(\omega)\frac{|x-\xi|^2}{t-\tau}\right\} \leqslant$$

$$\leqslant C_1(\omega)(t-\tau)^{-\frac{n+2-\alpha(\omega)}{2}} \exp\left\{-g_1(\omega)\frac{|x-\xi|^2}{t-\tau}\right\},$$

Where $C_1(\omega)$ is measurable. The estimation for I_2 and I_3 can be obtained in a similar way.

Finally

$$\left|\frac{\partial^2 W}{\partial x_k \partial x_j}\right| \leqslant \tilde{C}(\omega)\cdot(t-\tau)^{-\frac{n+2-\alpha(\omega)}{2}} \exp\left\{-g_2(\omega)\frac{|x-\xi|^2}{t-\tau}\right\}$$

Similarly, we can obtain (14) for all $m > 2$. Thus Proposition 2 is proved.

3. It remains to show the convergence of P and Q in (10), and the measurability of u with respect to ω and (6).

We have $P(\omega, t, x) = \int Z(\omega, t, 0, x, \xi)\, \varphi(\omega, \xi)d\xi$, $|P(\omega, t, x)| \leqslant$

$$\leqslant C_o(\omega)\, K(\omega) \int \exp\left\{-g(\omega)\frac{|x-\xi|^2}{t-\tau} + h(\omega)|\xi|^2\right\} t^{-\frac{n}{2}}\, d\xi =$$

$$= C_o(\omega)\, K(\omega) \sqrt{\pi}\, [g(\omega) - h(\omega)t]^{-\frac{1}{2}} \exp\left\{\frac{g(\omega)\, h(\omega)}{g(\omega) - h(\omega)t}|x|^2\right\}$$

If $0 < t \leqslant \dfrac{g(\omega) - \varepsilon_1(\omega)}{h(\omega)} = T_o(\omega)$, $0 < \varepsilon_1(\omega) < g(\omega)$, then P is convergent and satisfies:

$$|P| \leqslant K_1(\omega) \exp\{k(t, x)\,|x|^2\} \tag{23}$$

where $C_o(\omega)$, $K(\omega)$, $g(\omega)$, $h(\omega)$ are measurable and $\varepsilon_1(\omega)$ can be chosen in such a way that it is measurable. Thus $T_o(\omega)$ is measurable. Further $k(\omega, t) = \dfrac{g(\omega)\, h(\omega)}{g(\omega) - h(\omega)t}$ is measurable in ω on $\Omega_o = \{\omega \in \Omega : t \in [0, T_o(\omega)]\}$ and continuous in t on $[0, T_o(\omega)]$.

On the other hand,

$$Q(\omega, t, x) = \int_o^t d\tau \int Z(\omega, t, \tau, x, \xi)\, f(\omega, \tau, \xi)\, d\xi,$$

$$|Q| \leqslant C_o(\omega)\, K(\omega) \int_o^t d\tau \int \exp\left\{-g(\omega)\frac{|x-\xi|^2}{t-\tau} + h(\omega)|\xi|^2\right\}(t-\tau)^{-\frac{n}{2}}\, d\xi =$$

$$= C_o(\omega)\, K(\omega) \int_o^t (g(\omega) - h(\omega)(t-\tau))^{-\frac{1}{2}} \exp\left\{\frac{g(\omega)\, h(\omega)}{g(\omega) - h(\omega)(t-\tau)} \times |x|^2\right\} d\tau$$

For $0 \leqslant \tau < t \leqslant T_o(\omega)$ the expression under the integral symbol is well defined and attains its maximal value when $\tau = 0$. Hence

$$|Q| \leqslant C_o(\omega)\, K(\omega) \sqrt{\pi}\, t(g(\omega) - h(\omega)t)^{-\frac{1}{2}} \exp\left\{\frac{g(\omega)\, h(\omega)}{g(\omega) - h(\omega)t}|x|^2\right\} \leqslant$$

$$\leqslant C_o(\omega)\, K(\omega) \sqrt{\pi}\, T_o(\omega)\, \varepsilon_1(\omega) \exp\left\{\frac{g(\omega)\, h(\omega)}{g(\omega) - h(\omega)t}|x|^2\right\} =$$

$$= K_2(\omega) \exp\{K(\omega, t)\,|x|^2\}$$

Where $K_2(\omega)$ is measurable. Setting $K_0(\omega) = \max\{K_1(\omega), K_2(\omega)\}$ yields (6).

The estimation (23) and (24) imply the integrability of the function under the integral symbol on the right hand side of (10).

Applying Lemma 2 we infer that $u(\omega, t, x)$ is measurable on Ω_0 for each fixed (t, x).

Finally, analogously to what has been done in [1] we have also:

$$\lim_{t \to 0} u(\omega, t, x) = \varphi(\omega, x),$$

$$L(u(\omega, t, x)) = f(\omega, t, x).$$

The proof of the theorem is complete.

Remark. The hypothesis (5) can be replaced by (5') $|f(\omega, t, x)| \leqslant K(\omega) \exp\{k(\omega, t) \; |x|^2\}$

Hence, the conclusion is still valid if (6) is replaced by

$$|u(\omega, t, x)| \leqslant K_0(\omega) \exp\{p(\omega) k(\omega, t) \, |x|^2\}. \tag{6'}$$

The proof is analogous to that of the theorem.

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