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AN IMPLICIT SPACE COVERING METHOD WITH APPLICATIONS TO FIXED POINT AND GLOBAL OPTIMIZATION PROBLEMS

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1. INTRODUCTION

We shall be concerned with the following problem:

(P) Let there be given in R^n a convex set C with nonempty interior and an (arbitrary) set D.

Find an element of the intersection $C \cap D$.

This problem is frequently encountered in numerical analysis. In fact, problems such as: finding a fixed point of a mapping, solving a nonlinear equation, finding the global optimum of a function over a set, can all be reduced to problem (P) with suitable C and D.

Unless the set D is also convex, this problem is computationally very difficult. In the most general case, it is essentially intractable. As far as deterministic methods are concerned, there seems to be no other way for solving this problem than by an exhaustive search throughout the set C.

In the sequel, however, we shall show that under suitable assumptions, it is possible, by systematically exploiting the convexity of the set C, to construct an implicit space covering method which is guaranteed to produce a solution after finitely many steps when the problem is solvable.

2. BASIC ASSUMPTIONS

We shall make the following assumptions:

(I) A point w is available such that

$$w \in (\text{int } C) \setminus D; \tag{1}$$

(II) The set $C \cap D$ is bounded; for any given point $u \in \partial C =$ the boundary of C, one can compute a supporting hyperplane to C at u.

(III) For any $z \in R^n$ one can determine whether an element of $C \cap D$ exists in the halfline from w through z and compute such an element if it exists.

Assumption (I) is quite natural: let w be any interior point of C; if $w \in D$ the problem is solved, otherwise it satisfies (1). Assumption (II) is self-explanatory. If we denote by $\Gamma(z)$ the intersection of C with the halfline from w through z, then Assumption (III) means that for any $z \in R^n$ one can solve the one-dimensional « section » of problem (P) of $\Gamma(z)$, namely: find an element of $\Gamma(z) \cap D$ or else establish its emptiness. Though this Assumption does restrict the class of problems under consideration, it is fulfilled in many cases of interest. If, as it often happens, the set D is given by a function $g: R^n \to R$ such that

$$D = \{ x : g(x) \geqslant 0 \}, \tag{2}$$

then it is easily seen that Assumption (III) will be fulfilled in each of the following cases:

1) g is convex (so that the set D is *complementary convex*, i.e. is the complement to an open convex set);

2) g is the pointwise minimum of a finite family of convex and concave functions (D is the intersection of a finite number of convex and complementary convex sets);

3) g is piece-wise affine;

4) g is a d. c. function, i. e. a function of the form $g(x) = g_1(x) - g_2(x)$, where both g_1, g_2 are convex.

Aside from the above restrictive assumptions, we shall have to make two more technical ones:

(IV) The set C is *strictly convex*, in the sense that its boundary ∂C contains no line segment with distinct endpoints;

(V) The set D is *robust*, by which we mean that

$$D \subset \mathrm{cl}(\mathrm{int}\, D), \tag{3}$$

i. e. every point of D is the limit of a sequence of interior points of D.

In actual practice these last two assumptions can be enforced by a small perturbation of the data of the original problem. Indeed, if B denotes the unit ball around the origin 0 and ε a positive number, then for any set D the set $D + \varepsilon B$ is robust. When

$$C = \{ x : f(x) \leqslant 0 \}, \tag{4}$$

where $f(x)$ is a convex function, while D is defined by (2), then to make C and D satisfy Assumptions (IV) and (V) it suffices to add the term $\varepsilon \| x \|^2$ to each of the functions $f(x)$, $g(x)$.

A function $\rho : R^n \rightarrow R$ is called a separator for C, with respect to D, if

1) $\rho(x) = 0$ for $x \in C$ and whenever $z^\nu \rightarrow z \in \partial C$, $z^\nu \notin C$, $\Gamma(z^\nu) \cap D = \phi$ then $\rho(z^\nu) \rightarrow 0$;

2) For any bounded sequence z^ν, $\rho(z^\nu) \rightarrow 0$ implies $d(z^\nu, C) \rightarrow 0$, where $d(x, C)$ denotes the distance from x to C.

Examples of separators :

If C and D are given by (4) and (2) resp., where $f(x)$ is a convex and $g(x)$ is a continuous function, it is easy to verify that each of the following functions can serve as a separator :

I. $\rho(x) = d(x, C)$ (obvious).

II. $\rho(x) = \| x - u(x) \|$ $(x \notin C)$, where, for $x \notin C$, $u(x)$ denotes the intersection of ∂C with the halfline from w through x (obvious).

III. $\rho(x) = \max \{f(x), 0\}$. Indeed, the properties 1) and 2) of a separator easily follow from the Lipschitz property of f on a bounded set.

The above functions do not depend upon D. They are in fact separators for C with respect to any subset of D (in particular the empty set). The next functions are more proper separators for C with respect to D.

IV. $\rho(x) = tg^+(x) + f(x)$ $(x \notin C)$, where $t \geqslant 0$ is a constant and $g^+(x) = \max \{0, g(x)\}$. To check 1) observe that whenever $z^\nu \rightarrow z \in \partial C$ and $\overline{\lim}\, g^+(z^\nu) > 0$, then it follows from the continuity of g that $g(z) > 0$, and hence, for ν large enough, $\Gamma(z^\nu) \cap D \neq \phi$. On the other hand, since $g^+(z^\nu) \geqslant 0$, if $\rho(z^\nu) \rightarrow 0$ then $f(z^\nu) \rightarrow 0$ and property 2) follows.

V. $\rho(x) = \max \{g(x), f(x)\}$, assuming that $g(x)$ is convex. Just as in case IV, it suffices to observe that $z^\nu \rightarrow z \in \partial C$ and $\lim g(z^\nu) > 0$ imply $\Gamma(z^\nu) \cap D \neq \phi$ for ν large enough.

We now describe the proposed method for solving (P).

Denote by $\widetilde{D}$ the umbra of D relative to w:

$$\widetilde{D} = \bigcup_{t \geqslant 1} ((1-t)\, w + tD).$$

Algorithm A

Select a separator ρ, and a polytope S_1 containing $C \cap \widetilde{D}$, such that no vertex of S_1 lies on the boundary of C. Compute V_1, the set of vertices of S_1. Set $k = 1$.

Step $k = 1, 2, \ldots$: Find

$$z^k \in \arg\max \{\rho(z) : z \in V_k\}. \tag{5}$$

Compute an element of $\Gamma(z^k) \cap D$: if such an element exists, the problem (P) is solved; otherwise, find the point $u^k = u(z^k)$ (for any z, $u(z)$ denotes, as previously, the intersection of ∂C with the halfline from w through z). Construct a supporting halfspace to C at u^k:

$$L_k = \{x : l_k(x) \leqslant 0\}. \tag{6}$$

Set $S_{k+1} = S_k \cap L_k$, compute V_{k+1}, the set of vertices of S_{k+1}, and go to Step $k+1$.

Remark 1. S_{K+1} is obtained by adjoining one additional linear constraint o S_k. This enables us to derive the vertex set V_{k+1} of S_{k+1} from V_k using the scheme developed e.g. in [6]. Clearly the cumulation of constraints as the algorithm proceeds constitutes the major drawback of this kind of outer approximation methods, though several constraint dropping techniques can be applied to partially circumvent this difficulty (see. e.g. [10]).

Remark 2. If C is given in the form (4) with f being a convex function, then a supporting halfspace to C at u^k is obtained by calculating a subgradient, p^k of $f(x)$ at u^k and setting $l_k(x) = \langle p^k, x - u^k \rangle$ ($p^k \neq 0$ because (1) implies $0 > \min\{f(x) : x \in R^n\}$).

4. CONVERGENCE

We shall first prove some lemmas.

Suppose the Algorithm A is infinite.

LEMMA 1. *Any cluster point $\bar{z}$ of the sequence $\{z^k\}$ belongs to ∂C.*

Proof. Observe that $d(z^k, L_k)$ tends to 0 as $k \to \infty$. Indeed, otherwise there would exist an $\varepsilon > 0$ and an infinite subsequence z^{k_ν} such that $d(z^{k_\nu}, L_{k_\nu}) \geqslant \varepsilon$. Since, obviously, $z^k \in L_h$ for $h < k$, we would have $d(z^{k_\mu}, z^{k_\nu}) \geqslant$ $\geqslant d(z^{k_\nu}, L_{k_\nu}) \geqslant \varepsilon$ for all $\mu > \nu$. Thus, z^{k_ν}, $\nu = 1, 2, \ldots$ form an infinite set of points contained in S_1 and mutually at least ε apart. This conflicts with the compactness of S_1. Therefore, $d(z^k, L_k) \to 0$. Since, by (1), $\operatorname{int} C \neq \emptyset$, we may suppose $l_k(x) = \langle a^k, x \rangle + \beta_k$ with $\| a^k \| = 1$, hence $l_k(z^k) = d(z^k, L_k) \to 0$.

Now let $\bar{z} = \lim_\nu z^{k_\nu}$. From $l_k(u^k) = 0$ we have $\beta_k = -< a^k, u^k >$, therefore, by taking subsequences if necessary, we may suppose $a^k \to a$, $\beta_k \to \beta$, $u^k \to u \in \partial C$. Then $l_{k_\nu}(x) \to l(x) = < a, x > + \beta$ for every x and $l(\bar{z}) = \lim l_{k_\nu}(z^{k_\nu}) = 0$, $l(u) = \lim l_{k_\nu}(u^{k_\nu}) = 0$. But for every $x \in \text{cl}\, C$ we have $l(x) = \lim l_{k_\nu}(x) \leqslant 0$. This together with the facts $w \in \text{int}\, C$ and $l(x) \not\equiv 0$ (as $a \neq 0$) implies $l(w) < 0$. Noting that u lies in the line segment joining w with $\bar{z}$ and $l(\bar{z}) = l(u) = 0$, while $l(w) < 0$, we then conclude $\bar{z} = u \in \partial C$.

LEMMA 2. *We have*

$$\bigcap_{k=1}^{\infty} S_k \subset \text{cl}\, C. \tag{7}$$

Proof. Let $\bar{z}$ be any cluster point of the sequence z^k. By the previous Lemma, $\bar{z} \in \partial C$ and since $\Gamma(z^k) \cap D = \phi$ (as Step $k+1$ is performed), it follows from the property 1) of a separator that $\rho(z^k) \to 0$. Therefore, by (5),

$$\max \{ \rho(z) \,;\, z \in V_k \} \to 0.$$

In particular, if $\widehat{z}^k \in \arg\max \{ d(z, C) : z \in V_k \}$ then $\rho(\widehat{z}^k) \to 0$ and hence, in view of the property 2) of a separator, $d(\widehat{z}^k, C) \to 0$, i. e.

$$\max \{ d(z, C) : z \in V_k \} \to 0.$$

The convexity of the distance function then implies

$$\max \{ d(z, C) : z \in S_k \} \to 0,$$

proving (7).

Thus, roughly speaking the nested sequence of polytopes S_k tends to $S_1 \cap (\text{cl}\, C)$ as $k \to \infty$. The next proposition goes in the converse direction of Lemma 1.

LEMMA 3. *Any point $\bar{z} \in S_1 \cap \partial C$ is a cluster point of the sequence $\{ z^k \}$.*

Proof. By construction of S_1, no vertex of S_1 lies on ∂C. Hence $\bar{z}$ is not a vertex of S_1 and there is a line segment Δ contained in S_1 with midpoint $\bar{z}$. From the strict convexity of C it follows that at least one endpoint of Δ, say c, along with all points of Δ between c and $\bar{z}$, except $\bar{z}$, do not belong to C. That is, one can pick a sequence $y^\nu \to \bar{z}$ such that $y^\nu \in S_1 \setminus \text{cl}C$. The relation (7) then implies the existence for each ν of a k_ν such that $y^\nu \in S_{k_\nu - 1} \setminus S_{k_\nu}$, i.e. such that $l_{k_\nu}(y^\nu) > 0$. We may assume that the sequence z^{k_ν} tends to some z^*, while, as in the proof of Lemma 1, $l_{k_\nu}(x) \to l(x)\ \forall x$, with $l(x) \not\equiv 0$ and $l(x) \leqslant 0$ for all $x \in \text{cl}\ C$. Since both $\bar{z}$ and z^* belong to ∂C ($\bar{z}$ by hypothesis, z^* by virtue of Lemma 1), it follows that $l(\bar{z}) \leqslant 0$ and $l(z^*) \leqslant 0$. But $y^\nu \to \bar{z}$ and $l_{k_\nu}(y^\nu) > 0$ imply $l(\bar{z}) \geqslant 0$, hence $l(\bar{z}) = 0$. On the other hand, by construction of l_{k_ν} we have $l_{k_\nu}(z^{k_\nu}) > 0$, hence, as $z^{k_\nu} \to z^*$, $l(z^*) \geqslant 0$ and consequently, $l(z^*) = 0$. This, together with the relation $l(\bar{z}) = 0$ implies $l(x) = 0$, i.e. $x \in \partial C$, for all x in the line segment $[\bar{z}, z^*]$. In view of the strict convexity of C this may occur only if $\bar{z} = z^*$, i. e. $\bar{z} = \lim z^{k_\nu}$, as was to be proved.

Finally, before establishing the convergence theorem, we note the following

LEMMA 4. *The robustness condition (3) implies*

$$C \cap (\text{int}\ D) = \phi \Leftrightarrow (\text{int}\ C) \cap D = \phi.$$

Proof. Suppose there is $x \in C \cap (\text{int}\ D)$. Then some neighbourhood U of x is entirely contained in D. Since C is convex and has nonempty interior, U must intersect int C, i. e. $(\text{int}\ C) \cap D \neq \phi$. Conversely, if there is $x \in (\text{int}\ C) \cap D$, then since D is robust, $(\text{int}\ C) \cap (\text{int}\ D) \neq \phi$ and hence, $C \cap (\text{int}\ D) \neq \phi$.

We are now in a position to prove

THEOREM. *If* (int $C \cap D \neq \phi$, *Algorithm A finds a point of* $C \cap D$ *after finitely many steps.*

Proof. By the previous Lemma there is an $\bar{x} \in C \cap (\text{int}\ D)$ and by slightly moving x if necessary, we may assume $\bar{x} \in (\text{int}\ C) \cap (\text{int}\ D)$. Let U be a neighbourhood of $\bar{x}$ entirely contained in $(\text{int}\ C) \cap D$. Let $\bar{z}$ be the point where ∂C meets the halfline from w through $\bar{x}$. Then $\bar{z} \in C \cap \widetilde{D} \subset S_1 \cap \partial C$ and by Lemma 3, if the Algorithm is infinite $\bar{z}$ is the limit of some subsequence $\{z^{k_\nu}\}$. Therefore, for large enough ν, the halfline from w through z^{k_ν} intersects $U \subset (\text{int}\ C) \cap D$, i. e. $\Gamma(z^{k_\nu}) \cap D \neq \phi$, and the Algorithm must have stopped at step k_ν.

Thus, Algorithm A can be infinite only if $(\text{int}\ C) \cap D = \phi$.

5. APPLICATION TO GLOBAL OPTIMIZATION PROBLEMS

Consider the global minimization problem

$$\min f(x) \text{ subject to } g(x) \geqslant 0, \tag{8}$$

where $f : R^n \to R$ is a strictly convex function with bounded level sets and $g : R^n \to R$ is a continuous function.

Following the approach adopted in [12], in order to find an α-optimal solution to (8) (i. e. a point x satisfying the constraint $g(x) \geqslant 0$ and such that $f(x) \leqslant f(x) + \alpha$ for all x satisfying this constraint), we perform a sequence of cycles of computations consisting each of two phases: a *local phase*, where, starting from the current feasible solution, say y^0, we use local optimization techniques to find a local minimizer x^0 (or a feasible solution x^0 close to a local minimizer) such that $f(x^0) \leqslant f(y^0)$; then a *global phase*, where, starting from the current local minimizer x^0 (resulting from the local phase) we attempt to find a feasible solution y^1 such that $f(y^1) < f(x^0) - \alpha$. The latter problem, which constitutes the crux of the whole scheme, is thus a problem of type (P), with

$$C = \{ x : f(x) < f(x^0) - \alpha \} \quad , \quad D = \{ x : g(x) \geqslant 0 \}.$$

As a point w satisfying (1) we can always choose a point such that

$$g(w) < 0;\ f(w) < \min f(D) - \alpha \tag{9}$$

(provided, of course, the constraint $g(x) \geqslant 0$ is essential and α is appropriate). For any z denote by $\pi(z)$ the element of $\Gamma(z) \cap D$ that is nearest to w. In view of (9) and the convexity of f it follows that

$$\pi(z) \in \arg\min \{ f(x) : x \in \Gamma(z) \cap D \}.$$

Let us assume that the set D is bounded and robust in the sense (3) and that for any $z \in R^n$ one can determine whether or not $\Gamma(z) \cap D = \phi$ and compute $\pi(z)$ if $\Gamma(z) \cap D \neq \phi$. Then, since C is strictly convex, all conditions (I) through (V) are satisfied. Therefore, under the specified assumptions Algorithm A can be applied in the global phase to find after finitely many steps a feasible solution y^1 such that $f(y^1) < f(x^0) - \alpha$. This procedure is infinite only if x^0 is already an α-optimal solution. Note, however, that the Algorithm is unable to recognize an α-optimal solution in a finite number of steps.

Clearly, the efficiency of the procedure depends upon various factors. In particular, the choise of an appropriate separator may be crucial. With $\rho(x) = f(x)$ the method was first proposed by Ng. V. Thuong in his dissertation (see [14]), for minimizing a convex function subject to reverse convex constraints. Preliminary computational experiments have shown its practicability for problems of small size (the method is not sensitive to the number of constraints). It seems, however, that the method could perform better with $\rho(x) = tg^+(x) + f(x)$.

Convex minimization under reverse convex constraints (or *reverse convex programming*, following a now established terminology) has been the subject of an increasing number of researches in recent years, due to its importance in many applications (see e.g. [1, 2, 3, 5, 11—15]). Previously published methods for dealing with these problems either use cuts in a complicated manner which can be practical only for small numbers of constraints and often are not guaranteed to converge, or require additional variables in order to reduce problems with many reverse convex constraints to the case of one single reverse convex constraint. Furthermore, when applied to problems with piecewise affine (but neither convex nor concave) constraints, or d.c. constraints, these methods require the knowledge of an effective representation of the constraint functions as differences of convex functions.

The above proposed method, based on Algorithm A, is free from these limitations. Also, it could equally be applied to problems with convex constraint and nonconvex objective function. For instance, problems of the form (8), where f is nonconvex but g is concave, can be treated in an analogous manner by taking

$$C = \{x : g(x) \geqslant 0\}, \quad D = \{x : f(x) < f(x^0) - \alpha\}.$$

6. APPLICATION TO FIXED POINT PROBLEMS

It has long been noticed that finding a fixed point of a mapping $F : \Omega \subset R^n \to R^n$ can be viewed as a nonlinear optimization problem:

$$\min \| F(x) - x \| \text{ subject to } x \in \Omega. \tag{10}$$

But, to the best of our knowledge, so far no serious attempt has been made to solve fixed point problems by this approach (though some work has been done in the converse direction): In fact, the global optimization problem (10) to which the fixed point problem has been reduced is itself inherently difficult and cannot be handled by local optimization (convex optimization) methods which have been the main concern of optimization theory over the past decades.

Let C be a compact strictly convex set containing Ω in its interior. Setting $D = \{x \in \Omega : \| F(x) - x \| < \varepsilon\}$, where $\varepsilon > 0$ is a given tolerance, it is easily seen that finding an ε-approximate fixed point of F is a problem of type (P).

If F is continuous and Ω is robust in the sense (3) then D is robust too. Let w be an interior point of C which does not belong to D (see (1)) and denote as previously by $\Gamma(z)$ the intersection of C with the halfline from w through z. Suppose that

(*) For any $z \in R^n$ one can find a point x in the line segment $\Gamma(z)$ such that $x \in \Omega$, $\| F(x) - x \| < \varepsilon$, or else establish that no such point exists.

Then all conditions (I) through (V) as stated in Section 2 are fulfilled. Therefore, applying Algorithm A with these data will produce an ε-approximate fixed point of F after finitely many steps, provided the mapping F has a fixed point.

Sometimes it may be useful to generate not just one approximate fixed points with a prescribed accuracy, but a sequence of points converging to a fixed point. We may then proceed as follows.

Apply Algorithm A with $D = \Omega$ but at each Step k compute

$$x^k \in \arg\min \{ \| F(x) - x \| : x \in \Gamma(z^k) \cap \Omega \},$$

and record

$$\bar{x}^k \in \arg\min \{ \| F(x^i) - x^i \| : i = 1, \ldots, k \}.$$

Moreover, stop the Algorithm *only* when $\bar{x}^k$ is a fixed point of F.

PROPOSITION. *Any cluster point $\bar{x}$ of the generated sequence $\bar{x}^k$ achieves the minimum of $\varphi(x) = \| F(x) - x \|$ over Ω.*

Proof. Let $x^* \in \arg\min \{ \varphi(x) : x \in \Omega \}$, and let z^* be the point where ∂C intersects the halfline from w through x^*. For any $\varepsilon > 0$, since $x^* \in \Omega$ and Ω is robust, there exists an interior point x' of Ω such that $\varphi(x') \leqslant \varphi(x^*) + \varepsilon/2$. Let U be a neighbourhood of x' entirely contained in Ω such that $\varphi(x) \leqslant \varphi(x') + \varepsilon/2 \leqslant \varphi(x^*) + \varepsilon$ for all $x \in U$. Denote by z' the intersection of ∂C with the halfline from w through x'. By Lemma 3, there exists a subsequence $z^{k_\nu} \to z'$. Observe that $x' \in \operatorname{int} C$. Therefore, for ν large enough the halfline from w through z^{k_ν} will meet U. This implies $\varphi(\bar{x}^{k_\nu}) \leqslant \varphi(x^*) + \varepsilon$, and hence, $\varphi(x^*) \leqslant \varphi(\bar{x}^k) \leqslant \varphi(x^*) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary and the sequence $\varphi(\bar{x}^k)$ is nonincreasing, we conclude $\varphi(\bar{x}^k) \to \varphi(x^*)$, proving the Proposition.

Therefore, if F has a fixed point, any cluster point of the sequence $\bar{x}^k$ must be such a fixed point.

The same method could be used to find the global minimum (or maximum) of an (arbitrary) continuous function $\varphi(x)$ over a compact robust set $\Omega \subset R^n$. Needless to say, if no further information is given on the structure of φ and Ω, the algorithm can be practical only for small problems. In the general case while being far better than the rudimentary grid search method, it would still require a prohibitive amount of computations and storage. Nevertheless, there are instances, e.g. in design calculations, where the models contain only small numbers of variables but are highly nonlinear. In other circumstances, though the models are large only a few variables are highly nonlinear. In such cases, the above presented method, either used in a direct way or combined with other existing methods (including heuristic ones), could help to effectively solve a number of problems which otherwise are intractable.

Also it should be emphasized that the purpose of any deterministic global optimization technique is not so much to solve the problem to the end as to provide a procedure for transcending local optimality.

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