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EXACT BOUNDS FOR DERIVATIVES OF CLASSICAL SOLUTION OF CAUCHY'S PROBLEM FOR A QUASILINEAR EQUATION

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1. INTRODUCTION

1. It has been proved in [1] that in general the following Cauchy's problem

$$\frac{\partial u(x,t)}{\partial t} + a(u)\frac{\partial u(x,t)}{\partial x} = 0, \; -\infty < x < \infty, \; t \geqslant 0, \tag{1}$$

$$u(x,0) = u_o(x), \tag{2}$$

has no classical solution (c. s.) for large t. Here $u(x,t)$ is the unknown function of two variables, $a(u)$ and $u(x)$ are given smooth functions of one variable. On the other hand, for small t c. s. always exists. This solution may be implicitly determined by the formula (see [2], p. 18)

$$u(x,t) = u_o(x - ta(u)) \tag{3}$$

In this paper we shall construct an increasing sequence $[t_i]_{i=0}^{\infty}$, $t_o = 0$ such that in each segment $[t_i, t_{i+1}]$, $i = 0, 1, \ldots$ a c. s. $u(x, t)$ is implicitly determined by formulas of the form (3) with exact bounds for the derivatives $\frac{\partial}{\partial x}u(x,t)$ and $\frac{\partial}{\partial t}u(x,t)$. We shall prove that the interval $[0, T]$, $T = \lim_{i\to\infty} t_i$, is the existence interval for c. s. of problem (1), (2). Finally, we shall show that in the case $T = \infty$, $\frac{\partial}{\partial x}u(x,t)$ uniformly converges to zero as $t \to \infty$, while in the case $T < \infty$ the derivative converges to ∞ as $t \to T-0$ at some point x, i.e. there is no c.s. for $t \geqslant T$. The results in this paper may be used to study the convergence of the method of splines for solving the problem (1), (2) (see [5]).

2. EXISTENCE INTERVAL FOR CLASSICAL SOLUTION

Let us first deal with the question in which interval $0 \leqslant t < T$ a c.s. of the problem (1) (2) can be determined by formula (3)?

Let $\mathcal{C} = \hat{C}_{(-\infty, \infty)}$ be the Banach space of bounded and continuous functions $f(x)$, $-\infty < x < \infty$, with the norm (see [3], p. 29)

$$\| f(x) \| = \sup_{-\infty<x<\infty} | f(x) | .$$

Now suppose that the values of the unknown function $u(x, t)$ are contained in the segment $[u_1, u_2]$,

$$u_1 \leqslant u(x, t) \leqslant u_2 \tag{4}$$

and that the functions $u_0(x)$ and $a(\xi)$ have the continuous derivatives $u'_k(x) \in \mathcal{C}$. Define

$$K = \max_{u_1 \leqslant \xi \leqslant u_2} | a'(\xi) | , \tag{5}$$

$$K_0 = \sup_{-\infty<x<\infty} | u'_0(x) | , \ t_1 = \frac{1}{\beta K_0 K}, \ \beta > 1^*. \tag{6}$$

We begin with proving that in the segment $[0, t_1]$ a c.s. $u(x, t)$ of the problem (1), (2) is implicitly determined by (3). In fact, let $C_1 = C([0, t_1], \mathcal{C})$ denote the Banach space of continuous functions $f(x, t) = f(\cdot, t)$, defined on the segment $[0, t_1]$ and taking values in $\mathcal{C}$. C_1 is endowed with the norm (see [4], p. 14)

$$\| f(x, t) \|_{C_1} = \max_{0 \leqslant t \leqslant t_1} \| f(\cdot, t) \| = \max_{0 \leqslant t \leqslant t_1} \sup_{-\infty<x<\infty} | f(x, t) |$$

Then the map

$$u(x, t) \rightarrow u_0(x - ta(u))$$

is a contraction in C_1. Hence the equation (3) has a unique solution in C_1. Moreover if $a(u)$ and $u_1(x)$ have continuous derivatives, then $u(x, t)$ has continuous partial derivatives and

$$\frac{\partial u(x, t)}{\partial t} = -a(u) \frac{u'(\xi)}{1 + tu'(\xi)\, a'(u)}, \ \xi = x - ta(u) \tag{7}$$

$$\frac{\partial u(x, t)}{\partial x} = \frac{u'(\xi)}{1 + tu'(\xi)\, a'(u)} . \tag{8}$$

From (7) and (8) it follows that $u(x, t)$ satisfies (1). It is easy to show that $u(x,t)$ satisfies (2) too. Hence, the function $u(x, t)$, determined by (3), is the c.s. of the problem (1), (2) for $0 \leqslant t \leqslant t_1$.

We now extend this solution for $t > t_1$. To this end define (see [5])

$$u_i(x) = u(x, t_i), \ K_i = \sup_{-\infty<x<\infty} | u'(x) | ,$$

$$t_{i+1} = t_i + \frac{1}{\beta K_i K}, \ \beta > 1, \ t_0 = 0. \tag{9}$$

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*) We shall suppose that $K_0 \neq 0$. The case $K_0 = 0$ is trivial because $u(x, t)$ is then constant.

From the formulas for K_i to be established later (15) it will follows that if $K_o \neq 0$ then $K_i \neq 0$.

For $t_i \leqslant t \leqslant t_{i+1}$ we implicitly determine the solution $u(x, t)$ of equation (1) by

$$u(x, t) = u_i (x - (t - t_i) a(u)) \tag{10}$$

Let $C_{i+1} = C([t_i, t_{i+1}], \mathcal{C})$ be the Banach space of bounded and continuous functions $f(x, t) = f(., t)$, defined on the segment $[t_i, t_{i+1}]$, and taking values in $\mathcal{C}$. C_{i+1} is endowed with the norm

$$\| f(x, t) \|_{C_{i+1}} := \max_{t_i \leqslant t \leqslant t_{i+1}} \| f \| = \max_{t_i \leqslant t \leqslant t_{i+1}} \sup_{-\infty < x < \infty} | f(x,t) |$$

As shown above in the case $0 \leqslant t \leqslant t_1$, the map

$$u(x, t) \to u_i (x - (t - t_i) a(u))$$

is a contraction in C_{i+1}. Therefore, (10) has unique solution $u(x,t)$ in C_{i+1} and if $u_o(x)$ has a continuous derivative, then

$$\frac{\partial u(x, t)}{\partial t} = - a(u) \frac{u'(\xi)}{1 + (t - t_i) u'_i(\xi) a'(u)}, \xi = x - (t - t_i) a(u) \tag{11}$$

$$\frac{\partial u(x, t)}{\partial x} = \frac{1}{1 + (t - t_i) u'_i(\xi) a'(u)} \tag{12}$$

i.e. $u(x, t)$ satisfies (1) for $t_i \leqslant t \leqslant t_{i+1}$.

Let

$$T = \lim_{i \to \infty} t_i . \tag{13}$$

PROPOSITION 1. *The function $u(x, t)$, determined by (3) and (10), is the c.s. of problem (1), (2) for $0 \leqslant t < T$.*

Proof. It is easy to see that $u(x, t)$ is a continuous function of two variables $-\infty < x < \infty$, $0 \leqslant t < T$. For $\beta > 1$ the denominators in (7), (8) and (11), (12) are always positive, i. e.

$$1 + (t - t_i) u'_i(\xi) a'(u) > 0, \xi = x - (t - t_i) a(u) \tag{14}$$

and from (8) and (12) it follows that

$$K_{i+1} \leqslant \frac{\beta}{\beta - 1} K_i \quad , i = 0, 1, \ldots \tag{15}$$

Later we shall see that the equality is possible in (15).

If $u'_0(x) \in \mathcal{C}$, then the function $u(x, t)$ has continuous derivatives at $t \neq t_i$. Now we prove that they are continuous at every t, $0 \leqslant t < T$. Indeed,

$$\frac{\partial u(x,t)}{\partial x}\Big|_{t=t_i-0} = -\frac{u'_{i-1}(\xi_{i-1})}{1+(t_i-t_{i-1})u'_{i-1}(\xi_{i-1})\,a'(u_i(x))},$$

$$\xi_{i-1} = x-(t-t_{i-1})\,a(u_i(x)).$$

On the other hand, since $u'_i(x) = \frac{\partial}{\partial x}u(x,t_i)$ we can write

$$\frac{\partial u(x,t)}{\partial x}\Big|_{t=t_i+0} = u'_i(x) = \frac{\partial}{\partial x}u(x,t_i) = \frac{u'_{i-1}(\xi_{i-1})}{1+(t_i-t_{i-1})u'_{i-1}(\xi_{i-1})a'(u_i(x))} =$$

$$= \frac{\partial u(x,t)}{\partial x}\Big|_{t=t_i-0}$$

i. e. the function $\frac{\partial u(x,t)}{\partial x}$ is continuous at point $t=t_i$.

Further, for $t \neq t_i$, we have

$$\frac{\partial u(x,t)}{\partial t} = -a(u)\frac{u(x,t)}{x}.$$

The continuity of the function $\frac{\partial u(x,t)}{\partial t}$ at $t=t_i$ then follows from that of $a(u)$ and $\frac{\partial u(x,t)}{\partial x}$. Hence the function $u(x,t)$ is the c.s. of problem (1), (2) for $0 \leqslant t < T$. Proposition 1 is proved.

Since each t_i depends on β, one might think that T itself depends on β. However, we have:

PROPOSITION 2. *The value T in (13) is independent of* β.

Proof. We first observe that if the problem (1), (2) has a.c.s. $u(x,t)$ for $0 \leqslant t \leqslant T$, $1 < T$, then this solution is unique. This fact can be derived from the uniqueness theorem for the general solution of problem (1), (2) for $t \geqslant 0$ (see for example [10]). But here it can also be easily proved by the method of characteristics. In fact, let $\tilde{u}(x,t)$ be a c.s. of the problem (1), (2) for $0 \leqslant t \leqslant T_1$.

Then we determine the functions $x(t)$ and $\tilde{x}(t)$ by means of the system of ordinary differential equations (see for example [6], p.71)

$$\frac{dx}{dt} = a(u(x,t)), \qquad x(0)=y, \tag{16}$$

$$\frac{d\tilde{x}}{dt} = a(\tilde{u}(\tilde{x},t)), \qquad \tilde{x}(0)=y. \tag{17}$$

Here y is a parameter $-\infty < y < \infty$. Because the function $a(\xi)$ has a continuous derivative on $[u_1, u_2]$ and $\frac{\partial u(x,t)}{\partial x} \in C([0,T_1], \mathscr{C})$, $\frac{\partial \tilde{u}(x,t)}{\partial x} \in C([0,T_1], \mathscr{C})$, the problems (16), (17) have unique solutions $x(t)$ and $\tilde{x}(t)$, $0 \leqslant t \leqslant T_1$ and these solutions satisfy the following integral equalities

$$x(t) = y + \int_0^t a(u(x(s),s)ds, \quad \widetilde{x}(t) = y + \int_0^t a(\widetilde{u}(\widetilde{x}(s), s)ds.$$

Now define the functions $v(t)$, $\widetilde{v}(t)$ by

$$v(t) = u(x(t), t), \ \widetilde{v}(t) = \widetilde{u}(\widetilde{x}(t), t), 0 \leqslant t \leqslant T_1 .$$

Then

$$x(t) = y + \int_0^t a(v(s))ds, \quad \widetilde{x}(t) = y + \int_0^t a(\widetilde{v}(s))ds. \tag{18}$$

Noting that $u(x, t)$ and $\widetilde{u}(x, t)$ satisfy (1) and (2), we have

$$\frac{dv(t)}{dt} = 0, \ v(0) = u_0(y)$$

$$\frac{d\widetilde{v}(t)}{dt} = 0, \ \widetilde{v}(0) = u_0(y)$$

From these equalities it follows that $v(t) = \widetilde{v}(t)$, $0 \leqslant t \leqslant T$ and using (18) we obtain $u(x, t) = \widetilde{u}(x, t)$, $-\infty < x < \infty$, $0 \leqslant t \leqslant T_1$.

This being so, from (3) and (10) we have for the unique solution $u(x, t)$ of the problem (1), (2):

$$\inf_{-\infty < x < \infty} u_0(x) \leqslant u(x, t) \leqslant \sup_{-\infty < x < \infty} u_0(x)$$

Hence, if u_1 and u_2 in (4) satisfy

$$u_1 \leqslant \inf_{-\infty < x < \infty} u_0(x), \quad \sup_{-\infty < x < \infty} u_0(x) \leqslant u_2$$

then the c.s. of the problem (1), (2), determined by (3) and (10), satisfies the bounding conditions (4).

Now let us prove that the value T from (13) is independent of β. Take any $\widetilde{\beta} > 1$ such that $\widetilde{\beta} \neq \beta$, and let $\widetilde{t}_i$ be the value obtained from (9) when β is replaced by $\widetilde{\beta}$ and let $\widetilde{T} = \lim_{i \to \infty} \widetilde{t}_i$. We shall show that $T = \widetilde{T}$. In fact, first note that $(t_i)_{i=0}^{\infty}$ and $(\widetilde{t}_i)_{i=0}^{\infty}$ are increasing sequences. Consider now any fixed t_i. Since for $0 \leqslant t \leqslant t_i$ the problem (1), (2) has a unique c. s. $u(x, t)$ with $\frac{\partial u(x, t)}{\partial x} \in C([0, t], \mathscr{C})$ we have

$$\left|\frac{\partial u(x, t)}{\partial x}\right| \leqslant \widetilde{K} = \widetilde{K}(t_i) < \infty, \ -\infty < x < \infty, \ 0 \leqslant t \leqslant t_i .$$

Therefore, for $\tilde{t}_n \leqslant t_i$ we have

$$\tilde{K}_n = \sup_{-\infty<x<\infty} \left| \frac{\partial u(x, t)}{\partial x} \right| \leqslant \tilde{K}.$$

The equality

$$\tilde{t}_{n+1} = \tilde{t}_n + \frac{1}{\beta \tilde{K}_n K}$$

then implies

$$\tilde{t}_{n+1} \geqslant \tilde{t}_n + \frac{1}{\tilde{\beta} \tilde{K} K}.$$

Hence, there must exist a j such that $\tilde{t}_j > t_i$. This being true for any i, we have $T \leqslant \tilde{T}$. In the same manner, $\tilde{T} \leqslant T$ and therefore $T = \tilde{T}$, Proposition 2 is proved.

From now on we shall suppose that $\beta > 2$ and be fixed. Then for every i, $i = 0, 1, \ldots$, the equation

$$x - \frac{1}{\beta K_i K} a(u_{i+1}(x)) = y, \; -\infty < y < \infty$$

has a unique solution. In fact, for $\beta > 2$, from (15) we see that the map

$$x \to \frac{1}{\beta K_i K} a(u_{i+1}(x))$$

is a contraction.

Remark 1. In the case where the condition

$$u(x, t_0) = u_0(x), \; t_0 \neq 0 \tag{2'}$$

hold instead of (2), we can substitute $\tau = t - t_0$, $v(x, \tau) = u(x, t)$ and obtain the following problem

$$\frac{\partial v(x, \tau)}{\partial \tau} + a(v) \frac{\partial v(x, \tau)}{\partial x} = 0, \tag{1'}$$

$$v(x, 0) = u_0(x). \tag{2''}$$

So, if the problem (1'), (2'') has a c. s. $v(x, \tau)$ for $0 \leqslant \tau < T$, then the problem (1'), (2') has a c. s. $u(x, t)$ for $t_0 \leqslant t < t_0 + T$.

3. ESTIMATES FOR T.

Let us now estimate a bound for T. For that we have to estimate a bound for $u_i'(x)$.

First consider the case (see [7], p. 354)

$$a(u) = u \tag{19}$$

Then $K = 1$, where K is determined from (5). From now on, denote

$$K_0^- = \inf_{-\infty<x<\infty} u_0'(x), \quad K_0^+ = \sup_{-\infty<x<\infty} u_0'(x) \tag{20}$$

THEOREM 1. *Let* $a(u) = u, u'(x) \in \mathcal{C}$ *and* $K_0^- > 0$. *Then* $T = \infty$, *and for any* t, $0 \leqslant t < \infty$ *we have* $\frac{\partial u(x,t)}{\partial x} \in \mathcal{C}$ *and*

$$\lim_{t\to\infty} \left\| \frac{\partial u(x,t)}{\partial x} \right\|_{\mathcal{C}} = 0. \tag{21}$$

Proof. Denote

$$K_i^- = \inf_{-\infty<x<\infty} u_i'(x), \qquad K_i^+ = \sup_{-\infty<x<\infty} u_i'(x). \tag{22}$$

By induction we can show that

$$K_i^- > 0, \tag{23}$$

$$K_i^+ = \left(\frac{\beta}{\beta+1}\right)^i K_0^+ . \tag{24}$$

In fact, for $i = 0$ (23) holds by assumption and (24) is trivial. Now suppose that (23) and (24) hold for $i = j$. Then $K_j = K_j^+$. From (8) and (12) we have

$$u'_{j+1}(x) = \frac{u_j'(\xi_j)}{1 + \frac{1}{\beta K_j} u_j'(\xi_j)}, \ \xi_j = x - \frac{1}{\beta K_j} u_{j+1}(x) \tag{2}$$

Since $u_j'(x) > K_j^- > 0$, it follows from (14) that

$$K_{j+1}^- > 0.$$

But clearly, the function

$$g_\alpha(x) = \frac{x}{1+\alpha x}, \ x \neq -\frac{1}{\alpha}$$

increases with α for each fixed x. Therefore, from (25) it follows that

$$u'_{j+1}(x) = g_{\frac{1}{\beta K_j}}(u_j'(\xi)).$$

and hence, taking account of the induction assumption,

$$K_{j+1}^+ = \frac{K_j^+}{1+\frac{1}{\beta}} = \frac{\beta}{\beta+1} K_j^+ = \left(\frac{\beta}{\beta+1}\right)^{j+1} K_0^+.$$

Thus, (23) and (24) hold for every i. Further

$$t_{j+1} = t_j + \frac{1}{\beta K_j} = t_j + \frac{1}{K_0}\left(\frac{\beta+1}{\beta}\right)^j = \frac{1}{K_0}\left[1 + \left(\frac{\beta+1}{\beta}\right) + \dots + \left(\frac{\beta+1}{\beta}\right)^j\right] =$$

$$= \frac{1}{K_0}\left[\left(\frac{\beta+1}{\beta}\right)^{j+1} - 1\right].$$

Hence

$$T = \lim_{j\to\infty} t_j = \infty.$$

For

$$t_j = \frac{1}{K_0}\left[\left(\frac{\beta+1}{\beta}\right)^j - 1\right] \leqslant t \leqslant t_{j+1} = \frac{1}{K_0}\left[\left(\frac{\beta+1}{\beta}\right)^{j+1} - 1\right],$$

we get from (8), (12) and (23)

$$0 \leqslant \frac{\partial u(x,t)}{\partial x} \leqslant K_j^+ = K_0\left(\frac{\beta}{\beta+1}\right)^j \tag{26}$$

whence (21) follows. Theorem 1 is proved.

Remark 2. In view of (1) and (4), we have

$$\left|\frac{\partial u(x,t)}{\partial t}\right| \leqslant \max(|u_1|, |u_2|)\left|\frac{\partial u(x,t)}{\partial x}\right| .$$

So under the hypotheses of Theorem 1, for every fixed t, $\frac{\partial u(x,t)}{\partial t} \in \mathcal{C}$ and

$$\lim_{t\to\infty} \left\|\frac{\partial u(x,t)}{\partial t}\right\|_{\mathcal{C}} = 0 .$$

Turning to the case $K_0 < 0$, we can prove the following

THEOREM 2. *Let* $a(u) = u$, $u_0'(x) \in \mathcal{C}$ *and* $K_0^- < 0$. Then

$$T = \frac{1}{-K_0^-} < \infty ,$$

and for every fixed t, $0 \leqslant t < T$, $\frac{\partial u(x,t)}{\partial x} \in \mathcal{C}$ *and*

$$\lim_{t\to T-0} \left\|\frac{\partial u(x,t)}{\partial x}\right\|_{\mathcal{C}} = \infty . \tag{27}$$

The equality (27) shows that, for $t \geqslant T$ the problem (1), (2) has no c. s., for these t only a general solution can be constructed (see [1]).

Proof. We first show that for fixed β, there is a number j such that

$$K_j = -K_j^- , \tag{28}$$

where K_i^- and K_i^+ are determined by (22). In fact, if $-K_0^- \geqslant K_0$ then $j = 0$. Now suppose

$$0 < -K_0^- < K_0 .$$

Assume the contrary to (28), i.e.

$$K_i = K_i^+ > -K_i^- , \tag{29}$$

for all i, $i = 0, 1, \ldots$ Then from (14) and (25) we have

$$K_{i+1} = K_{i+1}^{+} = \frac{K_i^{+}}{1 + \frac{1}{\beta K_i} K_i^{+}} = \frac{\beta}{\beta + 1} K_i^{+} = \ldots = \left(\frac{\beta}{\beta + 1}\right)^{i+1} K_o^{+} .$$

Hence

$$\lim_{i \to \infty} K_i = 0. \tag{30}$$

On the other hand, (14) and (25) imply

$$K_{i+1} \geqslant -K_{i+1}^{-} = \frac{-K_i^{-}}{1 - \frac{1}{\beta K_i}(-K_i^{-})} \geqslant -K_i^{-} \geqslant \ldots \geqslant -K_o^{-} > 0. \tag{31}$$

The relation (31) conflict with (30). Therefore, there must be a number j such that

$$K_i = K_i^{+} > -K_i^{-}, \; i = 0, 1, \ldots, j-1,$$

$$K_j = -K_j^{-} > K_j^{+}.$$

We now prove Theorem 2 by induction on this number j.

For $j = 0$ we have $K_o = -K_o^{-}$. For simplicity, let

$$K_0 = -u'_o(x_o) .$$

From (8) it follows that

$$| u'(x) | < \frac{| u'(\xi) |}{1 - \frac{1}{\beta K_o} | u'(\xi) |} \leq \frac{\beta}{\beta - 1} K_c . \; \xi = x - \frac{1}{\beta K_o} u_1(x).$$

Let x_1 be the solution of the equation $x_1 - \frac{1}{\beta K_o} u_1(x_1) = x_o$.

Then

$$-u'_1(x_1) = \frac{-u'(x_o)}{1 - \frac{1}{\beta K_o} - u'_o(x_o)} = \frac{\beta}{\beta - 1} K_o$$

So

$$K_1 = -K_1^{-} = \left(\frac{\beta}{\beta - 1}\right)\left(-K_o^{-}\right). \tag{32}$$

Further, from (12) and (32),

$$K_i = -K_i^{-} = \left(\frac{\beta}{\beta - 1}\right)^i \left(-K_0^{-}\right) = \left(\frac{\beta}{\beta - 1}\right)^i K_o . \tag{33}$$

Therefore

$$t_{i+1} = t_i + \frac{1}{\beta K_i} = t_o + \frac{1}{\beta K_o}\left(\frac{\beta-1}{\beta}\right)^i = \frac{1}{K_o}\left[1-\left(\frac{\beta-1}{\beta}\right)^{i+1}\right]$$

and hence,

$$T = \lim_{i\to\infty} t_i = \frac{1}{K_o} = \frac{1}{-K_o^-},$$

From (33) then deduce (27). Thus, if $j = 0$ then (27) holds.

Suppose now that (27) holds for $k = 0,1,\ldots, j-1$, $j \geq 1$, and let us prove it for $k = j$. For this purpose, consider the Cauchy's problem for equation (1) ($t \geq t_1$), with the initial condition

$$u(x, t_1) = u_1(x). \tag{34}$$

Then $K_{j-1} = -K_{j-1}^-$. Using Remark 1 we obtain the c. s. of the problem (1) and (34) for

$$t_1 \leq t < t_1 + T_1, \tag{35}$$

with

$$T_1 = \frac{1}{-K_1^-}.$$

Since $t_1 = \dfrac{1}{\beta K_o}$, we get from (31) and (35)

$$T = t_1 + T_1 = \frac{1}{\beta K_o} + \frac{\beta K_o + K_o^-}{-K_o^- \beta K_o} = \frac{1}{-K_o^-} \tag{36}$$

For every t, $t_i \leq t < t_{i+1}$, we have $\dfrac{\partial u(x,t)}{\partial x} \in \mathcal{C}$ and $K_i' < \| \dfrac{\partial u(x,t)}{\partial x} \| \leq K_i''$,

where

$$K_i' = \min(K_i, K_{i+1}) \qquad K_i = \begin{cases} K_i^+ = \left(\dfrac{\beta}{\beta+1}\right)^i K_0^+, \; i = 0,1,\ldots\, j-1 \\ -K_i^- = \dfrac{-K_{i-1}^-}{1+\dfrac{1}{\beta K_{i-1}} K_{i-1}^-}, \; i > j \end{cases} \tag{37}$$

$$K_i'' = \max(K_i, K_{i+1})$$

But, for $i > j$,

$$K_i = \left(\frac{\beta}{\beta-1}\right)^{i-j} K_j$$

Hence

$$\lim_{t\to T-0}\left\|\frac{\partial u(x,t)}{\partial x}\right\|_C=\infty,$$

and using (37) we can calculate all K_i, $i=0, 1,\ldots$ Theorem 2 is proved.

Remark 3. From Theorems 1 and 2 we see that on the segment $[t_i, t_{i+1}]$ the bounds for $\frac{\partial u(x,t)}{\partial x}$ are K_i and K_{i+1}, and when $t=t_i$ and $t=t_{i+1}$ these bounds are exact.

Now consider the case $a(u)\neq u$. Suppose that the equation (1) is strictly hyperbolic (see [1]), i.e.

$$a'(\xi)\neq 0, \quad u_1\leqslant\xi\leqslant u_2. \tag{38}$$

Let us set

$$v(x,t)=a(u(x,t)).$$

Then $v(x,t)$ is the solution of the following Cauchy's problem

$$\frac{\partial v(x,t)}{\partial t}+v\frac{\partial v(x,t)}{\partial x}=0, \ -\infty<x<\infty, \ t\geqslant 0.$$

$$v(x,t)=v_0(x),$$

where $v_0(x)=a(u(x,0))=a(u_0(x))$. Therefore using Theorems 1 and 2 we can determine the interval in which the c. s. of the problem (1), (2) exist . In fact, when

$$\frac{d}{dx}[a(u_0(x))]\geqslant 0, \tag{39}$$

the c. s. $u(x,t)$ exists for all t, $0\leqslant t<\infty$. In the case when (39) does not hold, the c. s. $u(x,t)$ exists only on the finite interval $[0,T]$. This criterion for the existence of c. s. of (1), (2) has been stated in [8].

Remark 4. In [9], the case

$$c(u)=\sigma u$$

has been considered for all t, $-\infty<t<\infty$. We note that if $u'(x)\leqslant 0$ (or $u'(x)\geqslant 0$), the c. s. $u(x,t)$ exists for all $t\geqslant 0$ ($t\leqslant 0$ respectively).

Finally if the equation (1) is not strictly hyperbolic, then from (8) and (12) we get

$$K_{i+1}\leqslant\frac{K_i}{1-\frac{1}{\beta}}\leqslant\frac{\beta}{\beta-1}K_i\leqslant\ldots\leqslant\left(\frac{\beta}{\beta-1}\right)^{i+1}K_0 .$$

Further,

$$t_{i+1}=t_i+\frac{1}{\beta K_i K}\geqslant t_i+\frac{1}{\beta K_0 K}\left(\frac{\beta-1}{\beta}\right)^{i+1}\geqslant\frac{1}{K_0 K}\left[1-\left(\frac{\beta-1}{\beta}\right)^{i+2}\right].$$

Hence

$$T=\lim_{i\to\infty}t_i\geqslant\frac{1}{K_0 K}.$$

For $t \leqslant t_i$ we have

$$\left|\frac{\partial u(x,t)}{\partial x}\right| \leqslant K_i \leqslant \left(\frac{\beta}{\beta-1}\right)^i K_0 .$$

However, we are not able to find the exact bounds in this case.

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