ACTA MATHEMATICA VIETNAMICA

Vol. 12 No 2 (1987)

THE ALGEBRA* H GL(4, Z_2); Z_2)

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INTRODUCTION

The purpose of the present paper is to determine the mod 2 cohomology algebra of the general linear group $GL_4 = GL(4, Z_2)$ over the prime field Z_2 of two elements. The main tools are the cohomology of its Sylow 2-subgroup $GL_{4,2}$ (see [5]) and the modular invariant theory for two sets of variables. We first recall the result of [5].

THEOREM A. $H^*(GL_{4,2}; Z_2) = Z_2[v_{12}, v_{23}, v_{34}, v_{13}, v_{24}, v_{14}, z_1, z_2]/I$ with $|v_{i,i+k}| = k$, $1 \leqslant k \leqslant 3$, $1 \leqslant i \leqslant 4-k$, $|z_i| = i+1$, $1 \leqslant i \leqslant 2$ and I is the ideal generated by the elements:

$$v_{12}v_{23},\ v_{23}v_{34},\ v_{12}v_{24}+v_{34}v_{13},\ v_{12}^2v_{24}+v_{34}^2v_{13},\ v_{12}v_{24}^2+v_{34}v_{13}^2,$$
$$z_1^2+v_{23}z_2+v_{13}v_{24},\ z_2^2+v_{23}^2v_{14}+v_{23}z_1z_2+(v_{13}+v_{24})\ z_1^2,$$
$$(v_{12}+v_{34})\ z_1,\ v_{12}(z_1+v_{24})\ v_{12}z_2,\ v_{34}z_2,$$

Here $|x|$ denotes the degree of an element x in the graded algebra $H^*(GL_{4,2}; Z_2)$. The specification of the elements v_{ij}, z_2 will be recalled in § 3.

Set $w_1 = z_1 + v_{12}v_{34}$, $w_2 = z_2 + v_{12}v_{34}(v_{12}+v_{34})$, $w_3 = v_{12}v_{24}$

$$w_4 = v_{14} + z_1(v_{12}^2 + v_{13} + v_{34}^2 + v_{24} + v_{23}^2) +$$
$$+ v_{12}v_{34}(v_{12}^2 + v_{13} + v_{34}^2 + v_{24}) + v_{12}^4 + v_{13}^2 + v_{34}^4 + v_{24}^2 + v_{23}^4,$$
$$w_5 = v_{14}v_{23} + z_2(v_{12}^2 + v_{13} + v_{34}^2 + v_{24} + v_{23}^2) + z_1v_{23}(v_{13}+v_{24}) +$$
$$+ v_{12}v_{34}(v_{12}+v_{34})\ (v_{12}^2 + v_{13} + v_{34}^2 + v_{24}),$$

$$w_6 = v_{14}(v_{12}^2 + v_{13}) + z_2 v_{23} v_{24} + v_{23}^2 v_{24}^2 + v_{12}^2 v_{13}^2,$$

$$w_7 = v_{14}(v_{34}^2 + v_{24}) + z_2 v_{23} v_{13} + v_{23}^2 v_{13}^2 + v_{34}^2 v_{24}^2,$$

$$w_8 = v_{14} v_{12} v_{13}, \quad w_9 = v_{14} v_{34} v_{24}, \quad w_{10} = v_{14} v_{23} v_{13} v_{24}.$$

Our main result in the sequel is :

THEOREM B. *The algebra $H^*(GL_4 : \mathbf{Z}_2)$ is isomorphic to the subalgebra of $H^*(GL_{4,2} ; \mathbf{Z}_2)$ generated by the elements $w_1, w_2, \dots, w_{10}$.*

Since $w_1, w_2, \dots, w_{10}$ are stable elements (see [6]), Theorem B is a direct consequence of the following (see [1]).

PROPOSITION C. *Let W be the subalgebra of $H^*(GL_{4,2} : \mathbf{Z}_2)$ generated by the elements $w_1, w_2, \dots, w_{10}$. Then*

$$\text{In res}\,(GL_{4,2}, GL_4) \subset W,$$

here res (S,G) denotes the restriction homomorphism from a group G to a subgroup S.

The cohomology of the group GL_4 has been also considered by M. Tezuka and N. Yagita [6]. Comparing their result with Theorem B, we observe that the generators w_3 and w_5 of $H^*(GL_4; \mathbf{Z}_2)$ have been neglected. This is due to the fact is that they have neglected an invariant in [6, § 5. 13] and a stable element in [6, § 5. 14].

The paper consists of 3 sections. The two first sections are devoted to the study of the modular invariants of two sets of variables. In Section 1 we shall state the Theorem on the modular invariants and its Main Lemma. The proof of the Main Lemma will be presented in Section 2. In Section 3 we shall establish Proposition C by considering the images of the restrictions on the maximal elementary abelian 2-subgroups of the group GL_4.

1. MODULAR INVARIANTS OF TWO SETS OF VARIABLES

In this section we shall compute the invariants of (GL_2, GL_2). We first recall the definition given in [5, § 1]. Let $R = \mathbf{Z}_2\,[x_{11}, x_{12}, x_{21}, x_{22}]$ be the polynomial algebra generated by $x_{11}, x_{12}, x_{21}, x_{22}$ over $\mathbf{Z}_2$ and (G_1, G_2) be a pair of groups of linear transformations, $G_i \subset GL_2 = GL(2, \mathbf{Z}_2)$, on the 2-dimensional vector space $\mathbf{Z}_2^2$. There is a natural action of (G_1, G_2) on R defined as follows.

For every $(w_1, w_2) \in (G_1, G_2)$ and $f \in R$,

$$(w_1, w_2)\, f(x_{11}, x_{12}, x_{21}, x_{22}) = f(x'_{11}, x'_{12}, x'_{21}, x'_{22}),$$

where x'_{ij}, $1 \leqslant i, j \leqslant 2$ are given by

$$\begin{bmatrix} x'_{21} & x'_{22} \\ x'_{11} & x'_{12} \end{bmatrix} = w_1 \begin{bmatrix} x_{21} & x_{22} \\ x_{11} & x_{12} \end{bmatrix} w_2^{-1}.$$

If $(w_1, w_2)\, f = f$ for all $(w_1, w_2) \in (G_1, G_2)$, f is called an invariant of (G_1, G_2) or a (G_1, G_2)-invariant. We shall denote by $R(G_1, G_2)$ the algebra of (G_1, G_2)-invariants. The invariants of $(GL_2, 1)$ have been computed by W. C. Krathwohl [2] those of $(GL_{2,2}, GL_{2,2})$ have been computed by M. Tezuka and N. Yagita [6] and by the author in [5] by a different method. Here $GL_{2,2}$ is the Sylow 2-subgroup of GL_2 consisting of all upper matrices with 1 in the diagonal. Let us recall the result of W. C. Krathwohl [2]. Let

$$L_i = \begin{vmatrix} x_{1i} & x_{2i} \\ x_{1i}^2 & x_{2i}^2 \end{vmatrix}, \quad Q_i = \begin{vmatrix} x_{1i} & x_{2i} \\ x_{1i}^4 & x_{2i}^4 \end{vmatrix} \Big/ L_i;\ i = 1, 2$$

$$M = \begin{vmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{vmatrix}, \quad M_1 = \begin{vmatrix} x_{11}^2 & x_{21}^2 \\ x_{12} & x_{22} \end{vmatrix}, \quad M_2 = \begin{vmatrix} x_{11} & x_{21} \\ x_{12}^2 & x_{21}^2 \end{vmatrix}$$

THEOREM 1. 1. (W. C. Krathwohl [2]). *The algebra* $_R(GL_2, 1)$ *is generated by the elements* $L_1, L_2, Q_1, Q_2, M, M_1$ *and* M_2.

We consider now the invariants of (GL_2, GL_2). Suppose that f is a (GL_2, GL_2)—invariant. Then f is also a $(GL_2, 1)$ invariant and by Theorem 1. 1., f can be expressed in terms of $L_1, L_2, Q_1, Q_2, M, M_1$ and M_2. In fact we have

MAIN LEMMA 1. 2. $_R(GL_2, GL_2)$ *in the subalgebra of the algebra R generated by the elements* L_1, L_2, Q_1, Q_2, M *and* $K = M_1 + M_2$.

(The proof of this lemma will be given in Section 2)

From this, f can be expressed in terms of L_1, L_2, Q_1, Q_2, M and K. Set $A_{i1} = L_i + K$, $A_{i2} = Q_i + M$, $i = 1, 2$.

Then, for each $w = (1, \begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1}) \in (1, GL_2)$ we have

$$wA_{11} = aA_{11} + bA_{21}, \quad wA_{12} = aA_{12} + bA_{22}$$
$$wA_{21} = cA_{11} + dA_{21}, \quad wA_{22} = cA_{12} + dA_{22}.$$
$$wM = M, \quad wK = K.$$

Similarly to the proof of Theorem 1.1 we can show that f can be expressed in terms of M, K and $\mathcal{L}_1$ $\mathcal{L}_2$, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{M}$, $\mathcal{M}1$, $\mathcal{M}2$, where

$$\mathcal{L}_i = \begin{vmatrix} A_{1i} & A_{2i} \\ A^2_{1i} & A^2_{2i} \end{vmatrix}, \mathcal{Q}_i = \begin{vmatrix} A_{1i} & A_{2i} \\ A^4_{1i} & A^4_{2i} \end{vmatrix} / \mathcal{L}i, \quad i\ 1,=2,$$

$$\mathcal{M} = \begin{vmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{vmatrix}, \mathcal{M}_1 = \begin{vmatrix} A^2_{11} & A^2_{21} \\ A_{12} & A_{22} \end{vmatrix}, \mathcal{M}_2 = \begin{vmatrix} A_{11} & A_{21} \\ A^2_{12} & A^2_{22} \end{vmatrix}.$$

Since $\mathcal{M}_1 = \mathcal{Q}_2\ M^2 + \mathcal{Q}_1\ M, \mathcal{M}_2 = \mathcal{Q}_2\ K + \mathcal{M} M$, we have proved

THEOREM 1.3. *The algebra* $R^{(GL_2,\ GL_2)}$ *is generated by the elements* $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{Q}_1$, $\mathcal{Q}_2$, M *and* K

2. PROOF OF THE MAIN LEMMA

Let $W_{11} = x_{11}$, $W_{12} = x^2_{12} + x_{12}x_{11}$, $W_{21} = x^2_{21} + x_{21}x_{11}$,
$W_{22} = x_{22}(x_{22} + x_{12})(x_{22} + x_{21})(x_{22} + x_{12} + x_{21} + x_{11})$.

It is easy to see that W_{ij} are $(GL_{2,2}, GL_{22})$ - invariants. Namely, in [5] we have proved (see also [6]).

PROPOSITION 2.1 $R^{(GL_{2,2},\ GL_{2,2})}$ *is a free module over* $\mathbf{Z}_2[W_{11}, W_{12}, W_{21}, W_{22}]$ *generated by* $1, M, K, MK$. *Furthermore, its algebra structure is given by the identities:*

$$M^2 = W_{11}K + W^2_{11}M + W_{12}W_{21}, \tag{2.1}$$
$$K^2 = W^2_{11}W_{22} + W_{11}MK + (W_{12} + W_{21})\ M^2.$$

Suppose that f is a homogeneous (GL_2, GL_2) — invariant. Clearly f is also a $(GL_{2,2}, GL_{2,2})$ — invariant. By 2.1 f can be written uniquely as follows.

$$f = \sum_{n_1, \dots, n_6} m(n_1, \dots, n_6)\ W^{n_1}_{22} K^{n_2} M^{n_3} W^{n_4}_{11} W^{n_5}_{12} W^{n_6}_{21}, \tag{2.2}$$

where n_2, $n_3 = 0$ or 1, $n_i \in \mathbf{Z}_+$, $m(n_1, \dots, n_6) = 0$ or 1. Here Z_+ denotes the set of non-negative integers. On $\mathbf{Z}^6_+ = \underbrace{Z_+ \times \dots \times Z_+}_{6 \text{ times}}$ we have as usual the left lexicographic order, namely, $(n_2, \dots, n_6) > (n'_1, \dots, n'_6)$ if and only if there exists an integer k, $1 \leqslant k \leqslant 6$, such that $n_i = n'_i$, $0 \leqslant i \leqslant k - 1$ and $n_k > n'_k$.

We need an another total order on Z_+^6. Define

$$(n_1, \dots, n_6) \succ (n'_1, \dots, n'_6) \text{ if}$$
$$(4n_1 + 3n_2 + 2n_3 + n_4, n_1, n_2, n_3, n_5 + n_6, n_5) >$$
$$> (4n'_1 + 3n'_2 + 2n'_3 + n'_4, n'_1, n'_2, n'_3, n'_5 + n'_6, n'_5).$$

Let f be a (GL_2, GL_2) — invariant of the form (2.2).
Set $I_f = \{(n_1, \dots, n_6) \mid m(n_1, \dots, n_6) = 1\}$. We say that f has the index $(\alpha_1, \dots, \alpha_6)$ if $(\alpha_1, \dots, \alpha_6)$ is the maximum element in I_f with respect to the order $\succ$.

On the other hand, since f is also a $(GL_2, 1)$ — invariant, by Theorem 1.1, f can be expressed in terms of $L_1, L_2, Q_1, Q_2, M, M_1, M_2$. By a direct verification, we have

$$L_2 M_1 = M_1^2 + K^2 + M^2 Q_2,$$
$$L_1 M_1 = L_1 K + M_1^2 + M^2 Q_1,$$

Hence, f can be written as follows

$$f = f_0(Q_1, Q_2, M, M_1, K) + f_1(L_1, L_2, Q_1, Q_2, M, K) \tag{2.3}$$

with f_0 free of L_1, L_2.

LEMMA 2.4. *Let f be a (GL_2, GL_2)-invariant of the form. If $f_0 \neq 0$ and $(\alpha_1, \dots, \alpha_6)$ is the index of f then $\alpha_5 = \alpha_6 = 0$ and $\alpha_4 - \alpha_2 \leqslant 4\alpha_1$.* (2. 3)

Proof. Let f be a (GL_2, GL_2)-invariant as in Lemma 2.4. First we prove that $\alpha_5 = \alpha_6 = 0$. Let us express f_0 and f_1 as a sum of non-zero monomials of x_{ij}. Since $f_0 \neq 0$, f_0 has a term of the form $x_{11}^a x_{22}^b$ with $a + b = |f_0| = |f|$. On the other hand, by definition of $L_1, L_2, Q_1, Q_2, M, M_1, M_2$, any other term of f is of the form $x_{11}^i x_{12}^j x_{21}^k x_{22}^l$ with $j + k \neq 0$. This means that f contains the term $x_{11}^a x_{22}^b$.

Suppose that $\alpha_5 + \alpha_6 > 0$. By definition of the index any term of f is of the form $x_{11}^i x_{12}^j x_{21}^k x_{22}^l$ with $j + k \geqslant \alpha_5 + \alpha_6 > 0$. Hence f can not contain the term $x_{11}^a x_{22}^b$. This contradiction shows that $\alpha_5 = \alpha_6 = 0$.

We prove now that $\alpha_4 - \alpha_2 \leqslant 4\alpha_1$. Write

$$f = W_{22}^{\alpha_1} K^{\alpha_2} M^{\alpha_3} W_{11}^{\alpha_4} + \dots$$
$$= x_{22}^{4\alpha_1 + 2\alpha_2 + \alpha_3} x_{11}^{\alpha_2 + \alpha_3 + \alpha_4} + \dots$$

Let

$$w = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \in (GL_2, GL_2).$$

Since $wx_{11} = x_{22}$, $wx_{22} = x_{11}$, $wx_{12} = x_{21}$, $wx_{21} = x_{12}$, any **sum of all the** terms of f with the same homogeneous degree in $\{x_{11}, x_{22}\}$ is an invariant of w. Let S be such a sum which contains the term $x_{22}^{4\alpha_1 + 2\alpha_2 + \alpha_3} x_{11}^{\alpha_2 + \alpha_3 + \alpha_4}$. By definition of the index, $4\alpha_1 + 2\alpha_2 + \alpha_3$ is the highest power of x_{22} in f and so in S. Since S is an invariant of w, $4\alpha_1 + 2\alpha_2 + \alpha_3 > \alpha_2 + \alpha_3 + \alpha_4$. Consequently, $\alpha_4 - \alpha_2 \leqslant 4\alpha_1$. The Lemma is thus proved.

LEMMA 2. 5. *Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be non-zero integers such that $\alpha_4 - \alpha_2 \leqslant 4\alpha_1$, $\alpha_2, \alpha_3 = 0$ or 1. There is a (GL_2, GL_2) — invariant of the index $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, 0, 0)$ which can be expressed in terms of Q_2, μ, M, and K.*

Proof. Set $U_0 = Q_2$, $U_1 = \mathcal{M}$, $U_2 = K_2$, $U_3 = KM^2$, $U_4 = M^4$ and $U = M^2$.

By a direct verification we can show that U_i is an invariant of the index $(1, 0, 0, i, 0, 0)$, $0 \leqslant i \leqslant 4$ and U is one of the index $(0, 1, 0, 1, 0, 0)$. From this, if we take

$$H = \begin{cases} U_0^{\alpha_1} K^{\alpha_2} M^{\alpha_3} & \text{if} \quad \alpha_4 - \alpha_2 < 0, \\ U_0^{\alpha_1} U^{\alpha_2} M^{\alpha_3} & \text{if} \quad \alpha_4 - \alpha_2 = 0, \\ U_0^{\alpha_1 - (k+1)} U_r \; U_4^k U^{\alpha_2} M^{\alpha_3} & \text{if} \quad 0 < \alpha_4 - \alpha_2 < 4\alpha_1, \\ & \qquad \alpha_4 - \alpha_2 = 4k + r,\ 0 \leqslant r \leqslant 3, \\ U_4^{\alpha_1} U^{\alpha_2} M^{\alpha_3} & \text{if} \quad \alpha_4 - \alpha_2 = 4\alpha_1, \end{cases}$$

then H is an invariant of the index $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, 0, 0)$ consisting of the terms of Q_2, $\mathcal{M}$, M, and K. The Lemma is proved.

Proof of the Main Lemma.

Let f be a homogeneous (GL_2, GL_2) — invariant. Write f in the form (2. 3). The case $f_0 = 0$ is trivial. Suppose that $f_0 \neq 0$. By Lemma 2 . 4, f has the index $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, 0, 0)$ such that $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfy the conditions of Lemma 2. 5. Hence, there is a (GL_2, GL_2) — invariant H of the index $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, 0, 0)$ which can be expressed in term of Q_2, $\mathcal{M}$, M, $K \in \mathbb{Z}_2[L_1, L_2, Q_1, Q_2, M, K]$. Since the index of $f - H$ is lower than that of f, the Lemma follows by induction of the index of f.

From now on we use the notation $H^*(G) = H^*(G, \mathbf{Z}_2)$ for a group G. Recall that the elementary abelian 2-subgroups A_k, $1 \leqslant k \leqslant 5$, of the group $GL_{4,2}$ are given by

$$A_1 = \langle c_{12}, c_{13}, c_{14} \rangle, \quad A_2 = \langle c_{23}, c_{13}, c_{24}, c_{14} \rangle,$$
$$A_3 = \langle c_{34}, c_{24}, c_{14} \rangle, \quad A_4 \langle c_{12}, c_{34}, c_{14} \rangle,$$
$$A_5 = \langle c_1, c_2, c_3 \rangle.$$

Here $c_{ij} = E + e_{ij}$ with e_{ij} being the elementary matrix i. e. the matrix with 1 in posi ion (i, j) and zeros elsewhere,

$$c_1 = c_{12} c_{\ 4}, \ c_2 = c_{13} c_{24}, \ c_3 = c_{14}.$$

Let x_{ij}, $A_k \to \mathbf{Z}_2$ be the duals of c_{ij} with suitable indices for $1 \leqslant k \leqslant 4$ and x_i, $A^5 \to \mathbf{Z}_2$ be the duals of c_i. We know that

$$H^*(A_k) = \begin{cases} \mathbf{Z}_2 [x_{ij}, \text{ with the suitable indices }], & 1 \leqslant k \leqslant 4, \\ \mathbf{Z}_2 [x_1, x_2, x_3] & , k = 5. \end{cases}$$

In [5] we have shown that the homomorphism

$$\text{Res}: H^*(GL_{4,2}) \to \prod_{i=1}^{5} H^*(A_i)^{W_{GL_{4,2}}(A_i)} \tag{3.1}$$

given by the restriction homomorphisms is injective.

Here $W_G(S) = N_G(S) / C_G(S)$ denotes the Weyl group of a subgroup S in a group G and $H^*(S)$ is considered as a $W_G(S)$ — module via the adjoint isomorphism.

Further, the elements v_{ij}, z_t in Theorem A are defined by

$$v_{i,\,i+1}\big|_{A_k} = \begin{cases} x_{i,\,i+1} & \text{if } c_{i,\,i+1} \in A_k,\ 1 \leqslant k \leqslant 4, \\ x_1 & \text{if i} \neq 2,\ k = 5. \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

$$v_{i,\,i+2}\big|_{A_k} = \begin{cases} x_{i,\,i+2}^2 + x_{i,\,i+2}\, x_{k,\,k+1} & \text{if } k = i \text{ or } i + i,\ k \leqslant 3, \\ x_2^2 + x_2 x_1 & \text{if } k = 5, \\ 0 & \text{otherwise} \end{cases}$$

$$
v_{14}\big|_{A_k} = \begin{cases} x_{14}(x_{14}+x_{13})(x_{14}+x_{12})(x_{14}+x_{13}+x_{12}) & \text{if } k=1, \\ x_{14}(x_{14}+x_{13})(x_{14}+x_{24})(x_{14}+x_{13}+x_{24}+x_{23}) & \text{if } k=2, \\ x_{14}(x_{14}+x_{24})(x_{14}+x_{34})(x_{14}+x_{24}+x_{34}) & \text{if } k=3, \\ x_{14}(x_{14}+x_{12})(x_{14}+x_{34})(x_{14}+x_{12}+x_{34}) & \text{if } k=4, \\ x_3(x_3+x_2)(x_3+x_1)(x_3+x_2+x_1) & \text{if } k=5, \end{cases}
$$

$$z_i\big|_{A_1} = z_i\big|_{A_3} = z_i\big|_{A_4} = 0,\ 1 \leqslant i \leqslant 2,$$

$$z_1\big|_{A_2} = x_{23}\,x_{14} + x_{13}\,x_{24},\quad z_1\big|_{A_5} = x_2^2 + x_2x_1,$$

$$z_2\big|_{A_2} = x_{23}^2x_{14} + x_{23}\,x_{14}^2 + x_{13}^2x_{24} + x_{13}\,x_{24}^2,\ z_2\big|_{A_5} = 0.$$

Let $g = \begin{bmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{bmatrix} \in GL_4$. Then $g\,A_4\,g^{-1} \subset A_2$. From this and (3. 1) we btain:

LEMMA 3.1. *The homomorphism*

$$\mathrm{Res}: H^*(GL_4) \to \bigcap_{i\neq 4} H^*(A_i)^{W_{GL_4}(A_i)}$$

given by the restriction homomorphisms s injective.

Consider now the images of the restrictions on the maximal elementary abe an 2-s bgroups A_k, $1 \leqslant k \leqslant 5$, of the group GL_4.

LEMMA 3. 4.

$$H^*(A_5)^{W_{GL_4}(A_5)} \subset W\big|_{A_5}.$$

Proof. $W_{GL_4}(A_5) = N_{GL_4}(A_5)\,/\,C_{GL_4}(A_5)$ is isomorphic to the subgroup of GL_4 consisting of all matrices $g \in GL_4$ of the form

$$\begin{bmatrix} 1&0&0&0 \\ 0&*&*&* \\ 0&*&*&* \\ 0&*&*&* \end{bmatrix}.$$

Further, the action of $W_{GL_4}(A_5)$ on $H^*(A_5) = \mathbb{Z}_2[x_1, x_2, x_3]$ is as folows.

$wx_1 = ax_1 + bx_2$, $wx_2 = cx_1 + dx_2$, $wx_3 = ex_1 + fx_2 + x_3$ for each

$$w = \begin{bmatrix} 1&0&0&0 \\ 0&a&c&e \\ 0&b&d&f \\ 0&0&0&1 \end{bmatrix} \in W_{GL_4}(A_5).$$

Suppose that we are given an element $x \in H^*(A_5)^{W_{GL_4}(A_5)}$.
If $x \in \mathbb{Z}_2[x_1, x_2]$, then by L.E. Dickson [3, I, 3. 4], $x \in \mathbb{Z}_2[L_2, Q_{21}]$ with

$$L_2 = x_2 x_1 (x_2 + x_1) = w_3|_{A_5}, \quad Q_{21} = x_2^2 + x_2 x_1 + x_1^2 = w_1|_{A_5}.$$

Suppose that $x \in Z_2[x_1, x_2, x_3]$. We write $x = f_0(x_1, x_2) + f(x_1, x_2, x_3)$ with f_0 free of x_3. Obviously, f_0 and f_1 are elements of

$$\mathbb{Z}_2[x_1, x_2, x_3]^{W_{GL_4}(A_5)}.$$

Since f_1 has the factor x_3, it has the factor

$$x_3(x_3 + x_2)(x_3 + x_1)(x_3 + x_2 + x_1) = W_4 \,|_{A_5}.$$

The Lemma follows by induction on degree of f.

LEMMA 3. 5. $W_{GL_4}(A_2) \cong GL_2 \times GL_2$ *and*

$$H^*(A_2)^{W_{GL_4}(A_2)} \subset W|_{A_2}$$

Proof. Clearly, $W_{GL_4}(A_2) \cong GL_2 \times GL_2$.

Further, we know that $H^*(A_2) = \mathbb{Z}_2[x_{23}, x_{13}, x_{24}, x_{14}]$. The action of $W_{GL_4}(A_2)$ on $H^*(A_2)$ is as follows.

Let $w = (w', w'') \in W_{GL_4}(A_2) = GL_2 \times GL_2$ and

$$f \in H^*(A_2) = \mathbb{Z}_2[x_{23}, x_{13}, x_{24}, x_{14}]. \text{ Then}$$

$$wf(x_{23}, x_{13}, x_{24}, x_{14}) = f(x'_{23}, x'_{13}, x'_{24}, x'_{14}) \text{ with}$$

$$\begin{bmatrix} x'_{13} & x'_{14} \\ x'_{23} & x'_{23} \end{bmatrix} = w' \begin{bmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{bmatrix} w''^{-1}.$$

According to Theorem 1.3. $H^*(A_2)^{W_{GL_4}(A_2)}$ is generated by the elements:

$$\mathcal{L}_1 = (w_{10} + w_2 w_6)|_{A_2}, \quad \mathcal{L}_2 = (w_2^2 + w_1 w_4 + w_7)|_{A_2},$$

$$Q_1 = (w_2^2 + w_6)|_{A_2}, \quad Q_2 = w_4|_{A_2},$$

$$\mathcal{M} = w_5|_{A_2}, \quad M = w_1|_{A_2}, \quad K = w_2|_{A_2}.$$

This completes the proof.

From [3, § 1.3.4] we easily derive

LEMMA 3.6. $W_{GL_4}(A_i) \cong GL_3$, $i = 1$ or 3 *and via these isomorphisms* $H^*(A_i)$ *become a* GL_3 *-module, that is*

$$H^*(A_1)^{W_{GL_4}(A_1)} = \mathbf{Z}_2[w_4|_{A_1}, w_6|_{A_1}, w_8|_{A_1}] \subset W|_{A_1},$$

$$H^*(A_3)^{W_{GL_4}(A_3)} = \mathbf{Z}_2[w_4|_{A_3}, w_7|_{A_3}, w_9|_{A_3}] \subset W|_{A_3}.$$

From Lemmas 3.3 and Lemmas 3.4, 3.5 and 3.6 we easily obtain Proposition C. The proof of Theorem B is complete.

Acknowlegement. I am grateful to Professor Huỳnh Mùi for his generous help and inspiring guidance. I would like also to thank Professors M. Tezuka and N. Yagita for keeping me imformed of their results.

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Received April 4, 1986
Revised June 6, 1986

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