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SEMI-ATTRACTION DOMAINS OF SEMISTABLE LAWS ON TOPOLOGICAL VECTOR SPACES

HO DANG PHUC

1. INTRODUCTION AND NOTATION

Let E be a locally convex (l. c.) topological vector space (TVS) with the dual E'. Throughout this paper (a_k), (b_k) and (n_k), also with other subscripts or indexes, will denote a sequence of positive real numbers tending to zero, a sequence of elements from E, and a strictly increasing sequence of positive integers, respectively. If p and q are the laws of independent random vectors (r. v. 's) X and Y, n is a natural number and $a \neq 0$, then pq denotes the law of r. v. $X+Y$, a-p denotes the law of aX, and p^n is defined recursively by $p^n = p^{n-1}\,p$. Futhermore, if p is infinitely divisible (inf. div.) then following Siebert ([9]), p. 243), we can define p^t for all $t > 0$.

Let $\Rightarrow$ denote the weak convergence of laws and $\delta(b)$ denote the law concentrated at the point $b \in E$. If

$$(a_k \cdot p^{n_k})\ \delta(b_k) \ \Rightarrow\ q, \tag{1}$$

when $k \to \infty$, then we say that p belongs to the *domain of partial attraction of* q, *(DPA(q))*. If we assume in addtion that

$$(n_k / n_{k+1}) \ \to\ r > 0, \tag{2}$$

when $k \to \infty$, then we say that q is *semistable* and p belongs to the *domain of semi-attraction of* q, *(DSA(q))*, or more exactly, p belongs to the *domain of r-semi-attraction of* q, *(DSA(r, q))*. Further, we say that q is *stable* and p belongs to the *domain of attraction of* q, *(DA(q))*, if in (1), (n_k) coincides with the sequence of all natural numbers, i. e.

$$(a_k \cdot p^k)\ \delta(b_k) \ \Rightarrow\ q.$$

For a real sequence (c_k) let $\mathrm{LIM}(c_k)$ denote the set of all limit points of (c_k). Then it is easy to see that

(*) If (1) holds for some sequences (a_k), (b_k) and (n_k), then there exist sequences (a'_k), (b'_k) and (n'_k) satisfying (1) and such that $(1) \in \mathrm{LIM}(n'_k / n'_{k+1})$.

Let p and q be Radon laws on E, let q be convexly tight and let $H = \{t > 0 : p \in DSA(t, q)\}$. By virtue of Theorem 3 and Lemma 4 in [1], if $H \neq \emptyset$ then H is a closed multiplicative subgroup of $R^+ = \{r : r > 0\}$. Thus either $H = R^+$ (and then q is stable) or H is generated by s, the largest element in H less than 1. In the latter case we say that q is (s)—semistable and p belongs to the *domain of* (s) — *semi-attraction of* q, $(DSA((s), q))$.

The concept of semistable laws was introduced by Lévy [7] in 1937. The characterization of semistable laws on the real line was first given by Kruglov [3] in 1972. The characterization of semistable laws on a Hilbert space was studied in [4], [5] and [6]. Recently, in 1982, the problem for semistable laws on a l.c. TVS has been solved by D. M. Chung, B. S. Rajput and A. Tortrat [1]. In this paper we shall study the relationship between p and q satisfying (1) and (2). We shall also show that in the definitions of semistability and of domains of semi — attraction, condition (2) can be replaced by weaker ones.

2. RESULTS AND PROOFS

Let p and q be infinitely divisible laws. We say that p and q are *equivalent*, $(p \sim q)$, if there exist numbers $a > 0$, $t > 0$ and an element $b \in E$ such that

$$p = (a \cdot q^t)\,\delta(b)$$

THEOREM 1. *Let p, q_1 and q_2 be Radon laws on E, q_1 and q_2 be convexly tight. Assume that $p \in DPA(q_1)$ and there exist sequences (a_k), (b_k) and (n_k) such that*

$$(a_k \cdot p^{n_k})\,\delta(b_k) \Rightarrow q_2 \tag{3}$$

and

$$(n_k / n_{k+1}) \geqslant c \tag{4}$$

for all k, where c is a positive number. Then

$$q_1 \sim q_2 .$$

Proof. By assumption we can find sequences (a'_k), (b'_k) and (n'_k) such that

$$(a'_k \cdot p^{n'_k})\,\delta(b'_k) \Rightarrow q_1 \tag{5}$$

Without loss of generality one can suppose that there exists a subsequence of positive-integers $(k(m))$ such that

$$n_{k(m)-1} \leqslant n'_m \leqslant n_{k(m)}.$$

Then for all $m = 1, 2, \dots$ we have

$$c \leqslant n_{k(m)-1} / n_{k(m)} \leqslant n'_m / n_{k(m)} \leqslant 1.$$

Hence, one can assume moreover that

$$n'_m / n_{k(m)} \to s \text{ as } m \to \infty, \tag{6}$$

with $c \leqslant s \leqslant 1$.

Let $y \in E'$. If p is the law of the r. v. X, then p_y denotes the law of the random variable y_0X. From (5) we have

$$(a'_m \cdot p_y^{n_m}) \, \delta \, (y(b'_m)) \Rightarrow (q_1)_y. \tag{7}$$

On the other hand, the left side of (7) can be written as

$$((a'_m / a_{k(m)}) \cdot ((a_{k(m)} \cdot p_y^{n_{k(m)}}) \;\; \delta \; (y \, (b_{k(m)})))^{n'_m / n_{k(m)}} \cdot$$

$$\cdot \, \delta \, (y \, (b'_m - (a'_m \; n'_m \; / \; n_{k(m)}) \;\; b_{k(m)})).$$

Hence and by the type convergence theorem on the real line we have $a'_m / a_{k(m)} \to a > 0$, $y \, (b'_m - (a'_m \; n'_m \; / \; n_{k(m)}) \, b_{k(m)} \to b_y$ which together with (3), (6), (7) imply the equ tion

$$(q_1)_y = (a \cdot (q_2)^s_y) \, \delta \, (b_y). \tag{8}$$

From this and Corollary 1 of Lemma 2 in [10] we con lude that there exists $b \in E$ such that $y(b) = b_y$ for a l $y \in E'$ and $q_1 = (a \cdot q_2^s) \, \delta \, (b)$, i. e. $q_1 \sim q_2$. The theorem is proved.

THEOREM 2. *Let p, q_1 and q_2 be as in Theorem 1 and let $p \in DSA(r, q_2)$ with $r \in [0, 1]$. Then $q_1 \sim q_2$ if and o ly if $p \in DSA(r, q_1)$.*

Proof. The « if » part follows from Theorem 1, so we need only prove the « only if » part. Assume that (2) and (3) hold and $q_1 \sim q_2$, i.e. $q_1 = (a.q_2^s) \, \delta \, (b)$ with $a > 0$, $s > 0$ and $b \in E$. Put

$$a'_k = a \cdot a_k,$$

$$b'_k = b + asb_k,$$

$$n'_k = [n_k \cdot s],$$

where $[t]$ means the integer part of real number t. Then by (2) we have

$$\lim_{k \to \infty} (n'_k / n'_{k+1}) = \lim_{k \to \infty} ([n_k \cdot s] / [n_{k+1} \cdot s]) =$$

$$= \lim_{k \to \infty} (n_k / n_{k+1}) = r, \tag{9}$$

and

$$n'_k/n_k = [n_k . s]/n_k \to s,$$

as $k \to \infty$. Therefore, it is easy to verify that (5) holds. Hence from (9) one has $p \in DSA(r, q_1)$. The proof is complete.

From the above theorems we infer that the r-semistability is invariant under the equivalence relation $\sim$. On the other hand, the DSA's of r-semistable and — semistable laws are disjoint provided $v \neq s$. Moreover, we get the following theorem:

THEOREM 3. *Let p and q be Radon laws on E, q be convexly tight. Assume that (1) holds and the following condition is satisfied:*

$$LIM(n_k / n_{k+1}) \cap (0, 1) \neq \phi. \tag{10}$$

Then q is semistable.

Proof. By virtue of (10) we can find a number $c \in (0, 1)$ and a sequence $(k(m))$ of natural numbers such that

$$n_{k(m)} / n_{k(m)+1} \to c \text{ when } m \to \infty,$$

On the other hand, we have the equality

$$(a_{k(m)+1} . p^{n_{k(m)+1}}) \delta(b_{k(m)+1}) =$$

$$= ((a_{k(m)+1} / a_{k(m)}) . ((a_{k(m)} . p^{n_{k(m)}}) \delta(b_{k(m)}))^{n_{k(m)+1} / n_{k(m)}} . \delta(b_{k(m)+1} - (a_{k(m)+1} / a_{k(m)}) . (n_{k(m)+1} / n_{k(m)}) b_{k(m)}).$$

Then using the same technique as in the proof of Theorem 1 we can show by (1) that there exist $a > 0$ and $b \in E$ such that

$$(a_{k(m)+1} . p^{n_{k(m)+1}}) \delta(b_{k(m)+1}) \Rightarrow (a . q^c) \delta(b).$$

Hence from (1) we have

$$q = (a . q^c) \delta(b),$$

which together with Theorem 3 of [1] implies the semistability of q. The theorem is proved.

It should be noted that in the one-dimensional case this theorem was proved by F. Misheikis ([8, Theorem 12]). In view of this theorem, one can ask the following.

Question. Assume that p and q satisfy the conditions in Theorem 3. Does p belong to the DSA of q?

A partial answer to this question is contained in the following:

THEOREM 4. *Let p and q be as in Theorem 3. Then (1) together with (4) implies*
(a) *If q is (r)—semistable then $p \in DSA((r), q)$,*
(b) *If q is stable then $p \in DA(q)$.*

To establish Theorem 4 we need two lemmas.

LEMMA 1. *Let $0 < r < 1$ and q be an (r)—semistable law. Suppose that there exist sequences (a_k), (b_k), (n_k) and a real number c, $0 < c \leqslant 1$ satisfying (1) and (4). Then there exist sequences (a'_k), (b'_k) and (n'_k) such that*

$$(a'_k \cdot p^{n'_k})\ \delta\ (b'_k) \Rightarrow q \tag{1'}$$

and

$$LIM\ (n'_k/n'_{k+1}) = \{r, 1\}. \tag{2'}$$

Proof. Let α be the semistability exponent of q, $\gamma = 1/\alpha$ and N be a natural number satisfying

$$r^N \geqslant c > r^{N+1}$$

Let sequences $(a_k^{(m)})$, $(b_k^{(m)})$ and $(n_k^{(m)})$, $m = 1, 2, \dots, N+1$, be defined by

$$a_k^{(m)} = a_k \cdot r^{(m-1)\gamma},$$

$$b_k^{(m)} = b_k \cdot (r^{(m-1)\gamma}\ [n_k/r^{(m-1)}]/n_k),$$

$$n_k^{(m)} = [n_k/r^{(m-1)}].$$

Then for $m = 1,2,\dots, N+1$ there is an element $b^{(m)} \in E$ such that

$$(a_k^{(m)} \cdot p^{n_k^{(m)}})\ \delta\ (b_k^{(m)}) \Rightarrow q\ \delta\ (b^{(m)}). \tag{11}$$

Indeed, the left side of (11) can be written as

$$r^{(m-1)\gamma}((a_k \cdot p^{n_k})\ \delta(b_k))^{[n_k/r^{(m-1)}]/n_k} \Rightarrow r^{(m-1)\gamma} \cdot q^{1/r^{(m-1)}}$$

when $k \to \infty$, because of (1) and

$$[n_k/r^{(m-1)}]/n_k \to 1/r^{(m-1)} \text{ as } n_k \to \infty.$$

But q being (r)—semistable; by virtue of Lemma 4 in [1], we have

$$r^{(m-1)\gamma} \cdot q^{1/r^{(m-1)}} = q\ \delta(b^{(m)})$$

with $b^m \in E$. Thus (11) is true.

Let $h(k)$, $k = 1,2,\dots$, be natural numbers such that

$$n_k/r^{h(k)-1} \leqslant n_{k+1} < n_k/r^{h(k)}. \tag{12}$$

Then from (4) we have for all k

$$1 \leqslant h(k) \leqslant N+1. \tag{13}$$

We shall show that

$$\mathrm{LIM}(n_k^{(h(k))}/n_{k+1}) = \{r, 1\}, \tag{14}$$

Indeed, (11) implies

$$(a_k^{(m)} . p^{n_k^{(m)}}) \delta(b_k^{(m)} - b^{(m)}) \Rightarrow q \text{ as } k \to \infty$$

for $m = 1, 2, \ldots, N+1$. Therefore, by setting

$$p_{2k-1} = (a_k^{(h(k))} . p^{n_k^{(h(k))}}) \delta(b_k^{(h(k))} - b^{(h(k))}),$$

$$p_{2k} = (a_{k+1} p^{n_k+1}) \delta(b_{k+1})$$

for $k = 1, 2, \ldots$, we have from (1)

$$p_k \Rightarrow q \text{ as } k \to \infty.$$

If $s \in \mathrm{LIM}(n_k^{(h(k))}/n_{k+1})$, $s \neq 1$, then (12) implies $r \leqslant s < 1$.
On the other hand, from (4) and the definition of p_k, by just the same way as in the proof Theorem 3 we can see that q is s-semistable. But q is(r)-semistable. Consequently, $s=r$, proving (14).

The sequences (a'_k), (b'_k) and (n'_k) are constructed as follows:

$$a'_k = a_j^{(m)},$$

$$b'_k = b_j^{(m)} - b^{(m)},$$

$$n'_k = n_j^{(m)}$$

if $k = h(1) + h(2) + \ldots + h(j-1) + m$, $1 \leqslant m \leqslant h(j)$, $j = 2, 3, \ldots$. Then by virtue of (11) and (13) we can easily verify that (1') holds. Besides, for $k = h(1) + h(2) + \ldots + h(j-1) + m$,

(a) If $1 \leqslant m < h(j)$ then

$$n'_k/n'_{k+1} = n_j^{(m)}/n_j^{(m+1)} = [n_j/r^{(m-1)}]/[n_j/r^m] \to r \tag{15}$$

as $j \to \infty$.

(b) If $k = h(1) + h(2) + \ldots + h(j)$ then

$$n'_k/n'_{k+1} = n_j^{(h(j))}/n_{j+1}.$$

This together with (14) and (15) yields (2'). The proof is complete.

LEMMA 2. *Let p and q be laws on E, q be convexly tight and $0 < r < 1$.*

(i) *If there exist sequences (a_k), (b_k) and (n_k) satisfying (1) and (2), then we can find sequences (a'_k), (b'_k) and (n') such that (1') and (2') hold.*

(ii) *Conversely, if there exist sequences (a'_k), (b'_k) and (n'_k) such that (1') and (2') hold with q non-stable, then we can construct sequences (a_k), (b_k) and (n_k) satisfying (1) and (2).*

Proof. (i) Let us put for $m = 1, 2, \ldots$

$$a'_{2m-1} = a'_{2m} = a_m ,$$

$$b'_{2m-1} = b'_{2m} = b_m ,$$

$$n'_{2m-1} = n_m , \quad n'_{2m} = n_m + 1.$$

Then $n'_{2m} / n'_{2m-1} \to 1$ and, by the assumption, $n'_{2m-1} / n'_{2m+1} \to r$. Therefore $n'_{2m} / n'_{2m+1} \to r$. Consequently we have (2').

On the other hand, $(a'_k \cdot p) \Rightarrow \delta(0)$ because $a'_k \to 0$. Then (1') holds by virtue of (1).

(ii) Now suppose that (1') and (2') are satisfied. By an argument analogous to that used for the proof of Theorem 3 we see that q is r-semistable. Then, since q is non-stable, by virtue of Lemma 4 in [1] there exists a positive number $r_0 < 1$ such that q is (r_0)—semistable and $r = r_0^m$ for some natural m. Under these conditions:

(a) If $m = 1$ then q is (r)—semistable. Let α be the semistability exponent of q and $\gamma = 1/\alpha$. For every $k = 1, 2, \ldots$ let $h(k)$ be a natural number such that

$$n'_{h(k)-1} \leqslant n'_k / r < n'_{h(k)} . \tag{16}$$

Then by virtue of (1') and Lemma 6 in [1] we see that

$$\lim_{k \to \infty} (a'_k \cdot p^{[n'_k / r]}) \, \delta(b'_k / r) =$$

$$= \lim_{k \to \infty} ((a'_k \cdot p^{n'_k}) \, \delta(b'_k))^{1/r} =$$

$$= p^{1/r} = (r^{-\gamma} \cdot q) \, \delta(b^0_r)$$

with $b^0_r \in E$. Hence

$$((a'_k \, r^{\gamma}) \cdot p^{[n'_k / r]}) \, \delta(b'_k \cdot r^{\gamma - 1} - b^0_k) \Rightarrow q. \tag{17}$$

We now show that

$$\mathrm{LIM}([n'_k / r] / n'_{h(k)}) = \{r, 1\}. \tag{18}$$

Indeed, it follows from (16) that

$$1 > [n'_k / r] / n'_{h(k)} \geqslant n'_{h(k)-1} / n'_{h(k)}$$

and by virtue of (2')

$$\mathrm{LIM}(n'_{h(k)-1}/n'_{h(k)}) \subset \{r, 1\}.$$

Consequently,

$$\mathrm{LIM}([n'_k/r]/n'_{h(k)}) \subset [r, 1].$$

Thus, if $s \in \mathrm{LIM}([n'_k/r]/n'_{h(k)})$ and $s \neq 1$ then by virtue of (1') and (17), just as in the proof Lemma 1 we have $s = r$, proving (18).

Define

$$K_1 = \{k : [n'_k/r]/n'_{h(k)} \geqslant (1+r)/2\},$$

$$K_2 = \{k : [n'_k/r]/n'_{h(k)} < (1+r)/2\}.$$

Then (18) implies

$$\begin{aligned} &\lim_{k\to\infty,\ k\in K_1} ([n'_k/r]/n'_{h(k)}) = 1, \\ &\lim_{k\to\infty,\ k\in K_2} ([n'_k/r]/n'_{h(k)}) = r. \end{aligned} \tag{19}$$

Moreover, it is obvious that

$$n'_k/[n'_k/r] \to r, \text{ as } k \to \infty. \tag{20}$$

We shall construct the sequences (a_k), (b_k) and (n_k) by induction:

Let $a_1 = a'_1$, $b_1 = b'_1$, $n_1 = n'_1$. Further we set

$$a_2 = a'_{h(1)},\ b_2 = b'_{h(1)},\ n_2 = n'_{h(1)}$$

if $1 \in K_1$, and

$$a_2 = a'_1 r^{\gamma},\ b_2 = b'_1 r^{\gamma-1} - b^0_r,\ n_2 = [n'_1/r].$$

$$a_3 = a'_{h(1)},\ b_3 = b'_{h(1)},\ n_3 = n'_{h(1)}$$

if $1 \in K_2$.

Suppose that a_i, b_i, n_i have been constructed for $i = 1, 2, \dots, k$ and

$$a_k = a'_{h(j)},\ b_k = b'_{h(j)},\ n_k = n'_{h(j)}$$

for some natural j. Then we set

$$a_{k+1} = a'_{h(h(j))},\ b_{k+1} = b'_{h(h(j))},\ n_{k+1} = n'_{h(h(j))}$$

if $h(j) \in K_1$, and

$$a_{k+1} = a'_{h(j)} \cdot r^{\gamma},\ b_{k+1} = b'_{h(j)} \cdot r^{\gamma - 1} - b^0_r,\ n_{k+1} = [n'_{h(j)}/r],$$

$$a_{k+2} = a'_{h(h(j))},\ b_{k+2} = b'_{h(h(j))},\ n_{k+2} = n'_{h(h(j))}$$

if $h(j) \in K_k$, etc.

It follows from (1') and (17) that (1) is true for the new sequences (a_k), (b_k), and (n_k). Moreover (2) follows immediately from (19) and (20).

(b) In the case $m > 1$, by virtue of (2'), we can suppose that $n'_{k+1} \geqslant r_o^{m+1} > 0$ for all k. Then the conditions of Lemma 1 are satisfied with r_o in place of r and (a'_k), (b'_k), (n'_k) playing the role of (a_k), (b_k), (n_k) respectively. Thus, with the new sequences constructed by using Lemma 1, we are reduced to the case $m = 1$ and can apply the above part to complete the proof.

Proof of Theorem 4.

(a) Let q be (r) —semistable with $0 < r < 1$. Then we can suppose in addition that $0 < c \leqslant r$ and by applying Lemma 1 reduce the situation to the case when $\mathrm{LIM}\,(n_k / n_{k+1}) = \{r, 1\}$. Thus, by virtue of Lemma 2 we have $p \in DSA((r), q)$.

(b) Let q be stable of exponent α and let $\gamma = 1/\alpha$. Define $\beta(s) \in E$ for $s \in (0,1)$ by the equality

$$(s^{\gamma} \cdot q^{1/s})\,\delta(\beta(s)) = q, \tag{21}$$

(see Lemma 4 and Lemma 6 of [1]). Then is is clear that for $s, t \in (0,1]$ the following is true

$$\beta(s) \to \beta(t) \text{ as } s \to t. \tag{22}$$

Let sequences (a^o_m) and (b^o_m) be defined as follows:

$$a^o_m = (n_j / m)^{\gamma} \cdot a_j, \tag{23}$$

$$b^o_m = (n_j / m)^{\gamma - 1} \cdot b_j + \beta(n_j / m),$$

for $n_j \leq m < n_{j+1}$, $j = 1, 2 \ldots$ We shall show that.

$$(a^0_m \cdot p^m)\,\delta(b^0_m) \Rightarrow q \tag{24}$$

when $m \to \infty$.

Indeed, let (m') be any subsequence of natural numbers. Then for all $m' \in (m')$ one can find a natural number $j(m')$ such that

$$n_{j(m')} \leqslant m' < n_{j(m')+1}.$$

Hence, from (4),

$$1 \geq n_{j(m')} / m' > n_{j(m')} / n_{j(m')+1} > c.$$

One can pick from (m') another subsequence (m'') such that

$$n_{j(m'')} / m'' \to s, \tag{25}$$

with $1 \geqslant s > 0$. Now, from (1), (21), (22), (23), and (25), we have

$$(a^0_{m''} \cdot p^{m''})\,\delta(b^0_{m''}) =$$
$$= ((n_{j(m'')}/m'')^\gamma \cdot a_{j(m'')} \cdot p^{m''}) .$$
$$\cdot\, \delta((n_{j(m'')}/m'')^{\gamma-1} \cdot b_{j(m'')} - \beta(n_{j(m'')}/m''))$$
$$= ((n_{j(m'')}/m'')^\gamma \cdot ((a_{j(m'')} \cdot p^{n_{j(m'')}}) .$$
$$\cdot\, \delta(b_{j(m'')}))^{m''/n_{j(m'')}})\ \delta(\beta(n_{j(m'')}/m''))$$
$$\Rightarrow (s^\gamma \cdot q^{1/s})\,\delta(\beta(s)) = q.$$

Thus (23) holds, i. e. $p \in \mathrm{DA}(q)$, completing the proof.

As an immediate consequence of Theorem 4, we have

COROLLARY. *If q is a stable law then $DA(q) = DSA(r,q)$ for every $r \in (0,1)$.*

It is worth noticing that conditions (4) and (10) are weaker than condition(2) and that (10) is the weakest of these conditions. Thus, Theorem 3 gives a new characterization for semistability while Theorem 4, gives new characterizations for domains of attraction and domains of semi-attraction.

Finally, the following example will explain why in Theorem 4 we can not replace (4) by (10):

Example. Let E be a separable Fréchet space and q be an r-semistable law on E with $0 < r < 1$. By virtue of Theorem 1 in [2] there exists an universal law p on E which belongs to DPA's of all inf. div. laws. Thus, we can find sequences (a'_k), (b'_k), and (n'_k) satisfying (1'). Then, by taking a subsequence if necessary, we can assume in addition that

$$n'_k < [n'_k / r] < n'_{k+1}\,. \tag{26}$$

Let α be the semistability exponent of q and $\gamma = 1/\alpha$. Then, in follows from Lemma 6 in [1] and (1') that

$$\lim_{k\to\infty} (a'_k \cdot p^{[n'_k/r]})\ \delta(b'_k / r) =$$

$$= \lim_{k\to\infty} ((a'_k \cdot p^{n'_k})\ \delta(b'_k))^{1/r} =$$

$$= q^{1/r} = (r^{-\gamma} \cdot q)\ \delta(b^0_r)$$

with $b^0_r \in E$. Therefore

$$(a'_k \cdot r^\gamma) \cdot p^{[n'_k/r]}) \delta(b'_k \cdot r^{\gamma-1} - b^0_r) \Rightarrow q. \qquad (27)$$

Now we define sequences (a_k), (b_k), and (n_k) by

$$a_{2m-1} = a'_m, \; a_{2m} = a'_m \cdot r^\gamma,$$

$$b_{2m-1} = b'_m, \; b_{2m} = b'_m \cdot r^{\gamma-1} - b^o_r,$$

$$n_{2m-1} = n'_m; \; n_{2m} = [n'_m/r].$$

Then, by virtue of (26), (n_k) is a strictly increasing sequence and (1') together with (27) implies (1). Besides,

$$n_{2k-1} / n_{2k} = n'_k / [n'_k / r] \to r \text{ as } k \to \infty.$$

Thus, we have (10).

On the other hand, if $p \in \mathrm{DSA}(q)$ then, according to Theorem 1, every inf. div. law is equivalent to q, because q is an universal law. This is not true since the class of all inf. div. law. is evidently larger than the class of all semistable laws. Thus, $p \notin \mathrm{DSA}\ (q)$ although (1) and (10) hold for q.

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INSTITUTE OF MATHEMATICS, P. O. BOX 631 BO HO, HA NOI