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SEMIMARTINGALES AND THE STANDARD BROWNIAN MOTION

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INTRODUCTION

Let $(W_z, z \in \mathbf{R}^2_+)$ be a Brownian sheet and let $(b_t, t \geqslant 0)$ be a standard Brownian motion. It was shown in our previous work [3b] that $(f(W_z), z < z_0)$ is a weak submartingale (resp. a planar semimartingale) if and only if $(tf''(b_t)$

$0 \leqslant t \leqslant s_0t_0)$ is a submartingale (resp. $\int_0^{s_0t_0} \frac{1}{q} \operatorname{Var}_0^q (tf''(b_t))\, dq$ is finite), where

$z_0 = (s_0, t_0)$ is a point of $\mathbf{R}^2_+$ with $s_0t_0 > 0$ and f a function belonging to a dense subspace of $C^2(\mathbf{R}^1)$, called $K(\mathbf{R}^1)$ in [3b].

However one expects the above mentioned probabilistic characterizations could be expressed intrinsically as a geometrical property of the given function f.

The purpose of this note is to characterize all functions φ such that $(t\varphi(b_t), t \geqslant 0)$ is a submartingale (resp. a semimartingale). Such a function φ turns out to be a non-negative convex function (resp. a difference of two convex functions). These results are closely connected with those of Cinlar — Jacod — Protter — Sharpe ([2]), and they are used here to give geometrical interpretations of certain results in [3b].

I. BASIC DEFINITIONS AND PRELIMINARY RESULTS

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filter $(\mathcal{F}_t, t \geqslant 0)$, i. e., a family of σ-algebras $(\mathcal{F}_t, t \geqslant 0)$ satisfying the following conditions ;

1) $\mathcal{F}_0$ contains all null sets of $\mathcal{F}$;
2) If $t < t'$ then $\mathcal{F}_t \subset \mathcal{F}_{t'} \subset \mathcal{F}$;

3) For each t: $\mathcal{F}_t = \bigcap_{t'>t} \mathcal{F}_{t'}$.

Let T be a subset of $(0, +\infty)$ and $X = (X_t, t \in T)$ an adapted process contained in $L^1(P)$. Suppose that $[a, b] \subset T$ with $a < b$ is a compact interval and $\Delta = (a = \rho_0 < \rho_1 < .... < \rho_n = b)$ is a partition of $[a, b]$. Put

(a) $|\Delta| = \max_{0 \leqslant i \leqslant n-1} (\rho_{i+1} - \rho_i)$;

(b) $\mathrm{Var}_\Delta(X) = \sum_{i=1}^{n-1} E \mid E\{X_{\rho_{i+1}} - X_{\rho_i} \mid \mathcal{F}_{\rho_i}\} \mid$

(the variation of the process X on the partition Δ);

(c) $\mathrm{Var}_a^b(X) = \mathrm{Sup}_\Delta (\mathrm{Var}_\Delta(X))$, where the supremum is taken over all partitions of $[a, b]$ ($\mathrm{Var}_a^b(X)$ is called the variation of the process X on the interval $[a, b]$).

DEFINITION 1.1. ([4]) *Let T be a subset of $[0, +\infty]$ and $X = (X_t, t \in T)$ an adapted process contained in $L^1(P)$. Then X is said to be*

1) *a submartingale if for all $t > s$ with $t, s \in T$ we have*

$$E\{X_t - X_s \mid \mathcal{F}_s\} \geqslant 0 \quad P\text{-a. s.};$$

2) *a semimartingale if for all $a < b$ such that $[a, b] \subset T$ we have $\mathrm{Var}_a^b(X) < +\infty$.*

Remark. The above concept of semimartingale is weaker than that presented in [7], wheve a semimartingale is defined as the sum of a local martingale and a process of local bounded variation.

The following lemma will be used in the sequel to approximate the variation of a one-parameter semimartingale. Since it is a simple application of the Lebesgue bounded convergence theorem, its proof is omitted.

LEMMA 1. 2. *Let $[a, b]$ be a compact interval and $X = (X_t, t \in [a,b])$ an adapted process contained in $L^1(P)$. For every $t \in (a,b)$, define*

$$\delta(t) = \overline{\lim_{t_1 \uparrow t, t_2 \downarrow t}} [E \mid E\{X_{t_2} - X_t \mid \mathcal{F}_t\} \mid + E \mid E\{X_t - X_{t_1} \mid \mathcal{F}_{t_1}\} \mid - E \mid E\{X_{t_2} - X_{t_1} \mid \mathcal{F}_{t_1}\} \mid]$$

Then,

1) $\mathrm{Var}_a^b(X) = \lim_{|\Delta| \to 0} \mathrm{Var}_\Delta(\overline{X})$ *if $\delta(t) = 0$ for all $t \in (a, b)$.*

In particular if X is continuous in $L^1(P)$ then

$$\mathrm{Var}_a^b(X) = \lim_{|\Delta| \to 0} \mathrm{Var}_\Delta(X).$$

(2) $\delta(t) = 0$ *for all* $t \in (a,b)$ *if*

$$\mathrm{Var}_a^b(X) = \lim_{|\Delta| \to 0} \mathrm{Var}_\Delta(X) < +\infty.$$

Throughout this note it is assumed that $(\Omega, \mathcal{F}, \mathcal{F}_t, b_t, t \geqslant 0, P^x, x \in \mathbf{R}^1)$ is a linear Brownian motion (see for instance [5]). We denote by E^x the expectation of the probability measure P^x and for convenience we write E and P instead of E^0 and P^0, respectively.

For every probability measure μ on $(\mathbf{R}^1, \mathcal{B}^1)$, where $\mathcal{B}^1$ is the Borel σ-algebra of $\mathbf{R}^1$, the law P^μ on $(\Omega, \mathcal{F})$ is defined as follows:

$$P^\mu = \int \mu(dx)\, P^x .$$

PROPOSITION 1.3. *Suppose that the process* $(\varphi_1(b_t), t \geqslant 0)$ *where* φ *is a real function, is continuous in* $L^1(P)$ *and let* T *be an arbitrary positive number. Then*

$$\left| \mathrm{Var}_0^T(t\varphi(b_t)) - \int_0^T \mathrm{Var}_s^T(\varphi(b))\, ds \right| \leqslant T.\left(\mathrm{Sup}_{0 \leqslant t \leqslant T} E\,|\varphi(b_t)|\right) \qquad (1)$$

provided one of the two terms in the left hand side is finite.

Proof. Let $\Delta = (0 = \rho_0 < \rho_1 < \dots < \rho_n = T)$ be a partition of $[0, T]$. Denote

$$I_\Delta = \sum_{i=0}^{n-1} \rho_i \,.\, E\,|\,E\{\varphi(b_{\rho_{i+1}}) - \varphi(b_{\rho_i}) \,|\, \mathcal{F}_{\rho_i}\}\,| .$$

Then

$$\begin{aligned} &|\mathrm{Var}_\Delta(t\varphi(b_t)) - I_\Delta| \\ &\leqslant \sum_{i=0}^{n-1} (\rho_{i+1} - \rho_i)\, E\,|\varphi(b_{\rho_{i+1}})| \qquad (2)\\ &\leqslant T.(\mathrm{Sup}_{0 \leqslant t \leqslant T} E\,|\varphi(b_t)|. \end{aligned}$$

Now put

$$\Delta_i = (\rho_i < \rho_{i+1} < \dots < \rho_n),\ i = 0,1,\dots, n,$$
$$\mathrm{Var}_{\Delta_n}(\varphi(b)) = 0,$$

Then $I_\Delta = \sum_{i=0}^{n-1} \rho_i (\mathrm{Var}_{\Delta_i}(\varphi(b)) - \mathrm{Var}_{\Delta_{i+1}}(\varphi(b)))$

$$= \sum_{i=1}^{n-1} \mathrm{Var}_{\Delta_i}(\varphi(b)).(\rho_i - \rho_{i-1}).$$

By Lemma 1.2, it follows from the continuity in $L^1(P)$ of the process $(\varphi(b_t), t \geqslant 0)$ that

$\mathrm{Var}_{\Delta_i}(\varphi(b)) \to \mathrm{Var}_{\rho_i}^T(\varphi(b))$ as $|\Delta| \to 0$;

furthermore, the convergence is uniform when ρ_i belongs to an arbitrary closed subset of $[0, T_0]$ or of $[T_0, T]$, where

$$T_0 = \inf\{s : 0 \leqslant s \leqslant T,\ \mathrm{Var}_s^T(\varphi(b)) < +\infty\}$$

Therefore,

$$\lim_{|\Delta|\to 0} I_\Delta = \int_0^T \mathrm{Var}_s^T(\varphi(b))\,ds. \tag{3}$$

On the other hand, since $(t\varphi(b_t), t \geqslant 0)$ is also continuous in $L^1(P)$, we have

$$\lim_{|\Delta| \to 0} \mathrm{Var}_\Delta(t\varphi(b_t)) = \mathrm{Var}_0^T(t\varphi(b_t)). \tag{4}$$

Thus if one of the two terms in the left-hand side of (1) is finite then

$$\lim_{|\Delta| \to 0} (\mathrm{Var}_\Delta(t\varphi(b_t)) - I_\Delta) = \mathrm{Var}_0^T(t\varphi(b_t)) - \int_0^T \mathrm{Var}_s^T(\varphi(b))ds.$$

From (2), (3), (4) we obtain (1). Q.E.D.

II. M IN RESULTS

THEOREM 2.1. *Let φ be a real function such that $(\varphi(b_t), t \geqslant 0)$ is continuous in $L^1(P)$.*
(a) If $(t\varphi(b_t), t \geqslant 0)$ is a P-semimartingale then φ is a difference of two convex functions.
(b) Conversely if $\varphi = \varphi_1 - \varphi_2$, where φ_1, φ_2 are convex functions and $(\varphi_1(b_t), t \geqslant 0)$, $(\varphi_2(b_t), t \geqslant 0)$ are contained in $L^1(P)$, then $(t\varphi(b_t), t \geqslant 0)$ is a P-semimartingale.

Proof. (a) Since $(t\varphi(b_t), t \geqslant 0)$ is a P-semimartingale, by Proposition 1.3 it follows that $(\varphi(b_t), t \geqslant 1)$ is also a P-semimartingale.
Thus, if we denote by μ the Gaussian law with zero mean and covariance one then $(\varphi(b_t), t \geqslant 0)$ is a P^μ-semimartingale.
Now, from Theorem 5.5 of [2], it follows that φ is a difference of two convex functions.

b) From the assumptions that $(\varphi_i(b_t), t \geqslant 0)$, $i = 1, 2$, are P-submartingales, it follows immediately that $(\varphi(b_t), t \geqslant 0)$ is a difference of two P-submartingales. Therefore,

1) $(\varphi(b_t), t \geqslant 0)$ is a P-semimartingale,

2) $\mathrm{Sup}_{0 \leqslant t \leqslant T} E|\varphi(b_t)| \leqslant E|\varphi_1(b_T)| + E|\varphi_2(b_T)| < +\infty$.

Thus, by Proposition 1. 3.

$$\mathrm{Var}_0^T(t\varphi(b_t)) \leqslant T\,.\,(\mathrm{Var}_0^T(\varphi(b)) + \mathrm{Sup}_{0 \leqslant t \leqslant T} E|\varphi(b_t)|) < +\infty$$
for all $T > 0$.

In other words $(t\varphi(b_t), t \geqslant 0)$ is a P-semimartingale. Q.E.D.

PROPOSITION 2.2. *Let φ be a real function. If $(\varphi(b_t),\ t \geqslant 0)$ is a P^x-submartingale for all $x \in \mathbf{R}^1$ then φ is a convex function. More strongly, if $(\varphi(b_t),\ t \geqslant 0)$ is a P-submartingale, then φ is a convex function.*

Proof. (a) First, from Theorem 5. 5. of [2], we know that φ is a difference of two convex functions. In particular φ is a continuous function.

Since $(\varphi(b_t),\ t \geqslant 0)$ is a P^x-submartingale for all $x \in \mathbf{R}^1$, by an argument similar to that used in [2], it follows that there exists a convex function h such that $(\varphi(b_t) - h(b_t),\ t \geqslant 0)$ is a P^x-local martingale for all $x \in \mathbf{R}^1$.

For $a > 0$ and $x \in \mathbf{R}^1$, we put

$$\tau_a^x = \inf \{ t : |b_t - x| = a \}.$$

Since φ and h are bounded over $[x - a, x + a]$, the process $(\varphi(b_{\tau_a^x \wedge t}) - h(b_{\tau_a^x \wedge t}),\ t \geqslant 0)$ is a P^x-bounded martingale. Hence

$$E^x(\varphi(b_{\tau_a^x}) - h(b_{\tau_a^x})) = E^x(\varphi(b_0) - h(b_0)).$$

But the left-hand side can be written as

$$\frac{1}{2}(\varphi(x+a) + \varphi(x-a)) - \frac{1}{2}(h(x+a) + h(x-a))$$

and the right-hand side equals $(\varphi(x) - h(x))$.

Therefore $\Delta_a^x\ \varphi = \Delta_a^x\ h$, where

$$\Delta_a^x\ k = \frac{1}{2}(k(x+a) + k(x-a)) - k(x)$$

for any real function k. Hence, $\Delta_a^x\ \varphi \geqslant 0$ for any $x \in \mathbf{R}^1$ and any $a > 0$. The continuity of φ then implies its convexity.

(b) Suppose now that φ is a real function such that $(\varphi(b_t),\ t \geqslant 0)$ is a P-submartingale.

For $t > s \geqslant 1$, we have

$$E\{\varphi(b_t) - \varphi(b_s) \mid \mathcal{F}_s\} \geqslant 0 \qquad P\text{-a.s}$$

On the other hand, by the Markov property of the Brownian motion (see [5])

$$P\{b_T \in dx \mid b_1 = q\} = P^q\{b_{T-1} \in dx\} \qquad \text{for all } T \geqslant 1,\ x \in \mathbf{R}^1.$$

Therefore,

$$E^x\{\varphi(b_{t-1}) - \varphi(b_{s-1}) \mid \mathcal{F}_{s-1}\} \geqslant 0 \qquad P^x\text{-a.s. for all } x \in \mathbf{R}^1.$$

In other words,

$$(\varphi(b_t),\ t \geqslant 0) \text{ is a } P^x\text{-submartingale for all } x \in \mathbf{R}^1.$$

Hence, from the proof (a), φ is a convex function. Q.E.D.

COROLLARY 2.3. *Let φ be a real function. If $(\varphi(b_t),\ t \geqslant 0)$ is a P-martingale then φ is an affine function, i.e, $\varphi(x) = ax + b$, where a, b are constants.*

THEOREM 2.4. *Let φ be a real function. Then $(t\,\varphi(b_t),\ t \geqslant 0)$ is a P-submartingale if and only if φ is a non-negative convex function such that $E\,|\varphi(b_t)| < +\infty$ for all $t \geqslant 0$.*

Proof. Suppose that φ is a non-negative convex function such that

$$E\,|\,\varphi(b_t)\,| < +\infty \text{ for all } t \geqslant 0.$$

By the Jensen inequality we have

$$E\,\{\,t\,\varphi\,(b_t)\mid \mathcal{F}_s\} \geqslant E\,\{s\,\varphi\,(b_t)\mid \mathcal{F}_s\} \geqslant s\,\varphi\,(b_s) \quad P\text{-a.s} \quad \text{for all } t > s \geqslant 0.$$

Therefore $(t\,\varphi\,(b_t),\ t \geqslant 0)$ is a P-submartingale.

Conversely, suppose that $(t\,\varphi\,(b_t),\ t \geqslant 0)$ is a P-submartingale. It is clear that $E\,|\,\varphi\,(b_t)\,| < +\infty$ for all $t \geqslant 0$. For $t > s \geqslant 0$ we have

$$E\,\{\,t\,\varphi\,(b_t) - s\,\varphi\,(b_s)\mid \mathcal{F}_s\}$$
$$= s\,.\,E\,\{\,\varphi\,(b_t) - \varphi\,(b_s)\mid \mathcal{F}_s\} + (t-s)\,.\,E\,\{\,\varphi\,(b_t)\mid \mathcal{F}_s\}$$
$$= s\,.\,\Big(\int_{-\infty}^{\infty}\varphi\,(x+y)\,\mu\,(dy) - \varphi(x)\Big) + (t-s)\,.\int_{-\infty}^{\infty}\varphi\,(x+y)\,\mu\,(dy) \geqslant 0$$

for all $x \in \mathbf{R}^1$, $x = b_s$. Here μ denotes the Gaussian law with zero mean and covariance $(t-s)$.

Let $s \uparrow +\infty$, let $(t-s)$ be constant. Then

$$\int_{-\infty}^{+\infty}\varphi\,(x+y)\,\mu\,(dy) - \varphi\,(x) \geqslant 0 \text{ for all } x \in \mathbf{R}^1.$$

In other words, $(\varphi\,(b_t),\ t \geqslant 0)$ is a P-submartingale, hence by Proposition 2.2 φ is a convex function.

Furthermore, letting $s \downarrow 0$ yields

$$\int_{-\infty}^{\infty}\varphi(x+y)\,\mu\,(dy) \geqslant 0 \text{ for all } x \in \mathbf{R}^1.$$

Since $\int_{-\infty}^{\infty}\varphi(x+y)\,\mu\,(dy)$ converges uniformly to $\varphi(x)$ as $t \downarrow s$ in any bounded subset of $\mathbf{R}^1$, it follows that φ is a non-negative function. Q.E.D.

COROLLARY 2.5. *Let φ be a real function. If $(t\,\varphi\,(b_t),\ t \geqslant 0)$ is a P-martingale then $\varphi \equiv 0$.*

Proof. By Theorem 2.4. both functions φ and $-\varphi$ are non-negative. Hence $\varphi \equiv 0$. Q.E.D.

Recall (see [3b]) that a twice continuously differentiable function $f:\mathbf{R}^1 \to \mathbf{R}^1$ belongs to the class $K(\mathbf{R}^1)$ provided the following conditions are satisfied:

1) $\int_0^t E(f'(b_s)^2)\, ds < +\infty$ for all $t \geqslant 0$;

2) the process $(f''(b_t),\ t \geqslant 0)$ is continuous in $L^1(P)$.

Note that if $(\varphi(b_t),\ t \geqslant 0)$ is continuous in $L^1(P)$, then

1) $\mathrm{Sup}_{0 \leqslant t \leqslant T} E|\varphi(b_t)| < +\infty$ for all $T > 0$,

2) the process $(t\,\varphi(b_t),\ t \geqslant 0)$ is continuous in $L^1(P)$.

PROPOSITION 2.6. *Let $\varphi = \varphi_1 - \varphi_2$, where φ_1, φ_2 are convex functions such that the processes $(\varphi_1(b_t), t \geqslant 0)$, $(\varphi_2(b_t), t \geqslant 0)$ are contained in $L^1(P)$. Then we have*

$$\int_0^T \frac{1}{q} \mathrm{Var}_0^q(t\,\varphi(b_t))\, dq < +\infty \text{ for all } T > 0.$$

Proof. For $T > 0$ and $0 < q \leqslant T$, we have by Proposition 1.3:

$$\mathrm{Var}_0^q(t\,\varphi(b_t)) \leqslant q.(\mathrm{Var}_0^q(\varphi(b)) + \mathrm{Sup}_{0 \leqslant t \leqslant q} E|\varphi(b_t)|).$$

$$\leqslant q.(\mathrm{Var}_0^T(\varphi(b)) + \mathrm{Sup}_{0 \leqslant t \leqslant T} E|(\varphi(b_t)|)$$

$$\leqslant 2q.(E|\varphi_1(b_T)| + E|\varphi_2(b_T)| + |\varphi_1(0)| + \varphi_2(0)|).$$

Therefore,

$$\int_0^T \frac{1}{q} \mathrm{Var}_0^q(t\varphi(b_t))dq$$

$$\leqslant 2T.(E|\varphi_1(b_T)| + E|\varphi_2(b_T)| + |\varphi_1(0)| + |\varphi_2(0)|) < +\infty. \quad \text{Q.E.D.}$$

Combining known results in [3b] with Theorem 2.1, Theorem 2.4 and Proposition 2.6 we obtain the following characterization of the elements in the space $K(\mathbf{R}^1)$ that transform the Brownian sheet into weak submartingales and planar semimartingales.

COROLLARY 2.7. (a) *Let f be a function of the class $K(\mathbf{R}1)$. Then $(f(W_z), z \in \mathbf{R}_+^2)$ is a weak submartingale if and only if f'' is a non-negative convex function, i.e, f and f'' are convex functions.*

(b) *If f is a function of the class $K(\mathbf{R}1)$ such that $(f(W_z), z \in \mathbf{R}_+^2)$ is a planar semimartingale, then f'' is a difference of two convex functions.*

Conversely, if f'' can be expressed as:

$f'' = \varphi_1 - \varphi_2$, where φ_1, φ_2 are convex functions such that $(\varphi_1(b_t),\ t \geqslant 0)$ and $(\varphi_2(b_t),\ t \geqslant 0)$ are contained in $L^1(P)$, then $(f(W_z),\ z \in \mathbf{R}^2_+)$ is a planar semimartingale.

III. APPLICATIONS

It is a well-known fact that, for all $\alpha \in (0,1)$ $(|b_t|^\alpha,\ t \geqslant 0)$ is not a P-semimartingale. We showed in [3b] that for all $\alpha \geqslant 3$,

$(|W_z|^\alpha,\ z \in \mathbf{R}^2_+)$ is a weak submartingale. For $\alpha \in (2,3)$ let us consider the function $f(x) = |x|^\alpha$ for $x \in \mathbf{R}^1$. Then $f \in K(\mathbf{R}^1)$ and

$f''(x) = \alpha(\alpha-1)\ |x|^{\alpha-2}$ is not a difference of two convex functions.

Therefore, by Corollary 2.7 (b):

$(|W_z|^\alpha,\ z \in \mathbf{R}^2_+)$ is not a planar semimartingale for all $\alpha \in (2,3)$.

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