ACTA MATHEMATICA VIETNAMICA
Vol. 12 No 1 (1987)

ON THE LANGLANDS TYPE DISCRETE GROUPS. I
THE BOREL — SERRE COMPACTIFICATION

DO NGOC DIEP

0. INTRODUCTION

Let G be a reductive Lie group, K a maximal compact subgroup of G, Γ an arithmetically defined subgroup and V some Γ-module, i.e. there is a representation σ in the complex vector space V. Our aim is to study the Eilenberg — MacLane cohomology group $H^*(\Gamma; V)$. It is well known that if Γ acts freely, the quotient space X/Γ, where $X := K\backslash G$, is a smooth manifold and, since X is contractible, it is also an Eilenberg-MacLane space $K(\Gamma, 1)$. Therefore if the representation (σ, V) is trivial, $H^*(\Gamma; V)$ is isomorphic to the ordinary De Rham cohomology group $H^*_{DR}(X/\Gamma\ ; V)$. If the representation (σ, V) is non-trivial, the coefficient sheaf $C^\infty(X/\Gamma, V)$ must be replaced by a local coefficient sheaf system F_V associated to this representation (see for example [3, § VII.2.2]).

The homogeneous space X has a natural Riemannian structure with negative defined scalar curvature. Thus for the part of this new kind of de Rham cohomology classes with compact support one can develop the ordinary Hodge theory; in particular find the harmonic representatives for every cohomology class with compact support. By virtue of the long exact sequence for the pair of a space and its boundary, the « suplementary » part is closely connected with the boundary of X/Γ. So, for an algebraic reductive group G and an arithmetically defined subgroup Γ A. Borel and J. -P. Serre [2] have proposed a construction of manifold with corners $\overline{X/\Gamma}$ which is a compactification of X/Γ and the boundary of which is homotopic to the quotient by Γ of the Tits building of parabolic subgroups. The main properties of Γ needed for the reduction theory are abstracted by R. P. Langlands [9] by introducing an essentially larger [11, 15] class of discrete subgroups that we call « of Langlands type». Our early goal is to construct the Borel — Serre compactification for Langlands type discrete subgroups of general Lie group. There is some difference between our situation and the Borel — Serre's: We consider general Lie groups and Langlands type discrete subgroups instead of the algebraic reductive Q-groups and arithmetically defined subgroups. So in our situation the cuspidal and percuspidal subgroups must occur in the place of arbitrary parabolic and minimal parabolic subgroups in Borel — Serre's consideration.

With each cuspidal subgroup, R.P. Langlands [9] associated a family of Eisenstein series. But the Langlands theory was for a long time very difficult to understand and to use. Harish-Chandra in his lectures at the Institute for Advanced Studies [6] refined the theory for semisimple Lie groups in a more comprehensive form by using the Maas-Selberg relations. In an other work [7] he developed an analoguous theory of Eisenstein integrals to study the Plancherele measure. We will in a subsequent paper develop the Harish-Chandra's approach to the theory of Eisenstein series in our situation.

In some further parts of the contribution we certainly apply this theory of Eisenstein series to describe the cohomology classes at infinity of $H^*(\Gamma ; V)$. It will be considered as a theory of harmonic forms which appear as the values of Eisenstein series at some special values of parameters. So we shall have some «Hodge theory» for non-square-integrable differential forms, see [4, 5, 8, 10, 13, 14].

1. CUSPIDAL SUBGROUPS AND THE LANGLANDS' ASSUMPTION

In this section we shall introduce the Langlands' notions of cuspidal, percuspidal and finally Langlands type discrete subgroups. The general references are Langlands [9] Borel-Wallach [3] and Harish-Chandra [7].

1.0. SPLIT COMPONENT. Recall that for any Lie group G we denote by $\mathfrak{X}(G)$ the group of all (continuous) homomorphisms from G into the multiplicative group R^* of positive real numbers and by G^0 the connected component of the identity element of G. By a *vector subgroup* of G we mean a closed subgroup A which is topologically isomorphic to the additive group R^n for some n. We define

$$^{0}G = \bigcap_{\chi \in \mathfrak{X}(G)} \operatorname{Ker} |\chi|$$

Then ${}^0G$ is a closed normal subgroup of G and $G/{}^0G$ is an abelian Lie group. We call dim $G/{}^0G$ the *parabolic rank* of G and denote it by prk G. By a *split component* of G we mean a vector subgroup A of G such that $G = {}^0G.A$ and ${}^0G \cap A = \{1\}$.

1.1. ASSUMPTIONS ON G. Let G be a Lie group with only finite number of connected components and with the Lie algebra $\mathcal{G}$, which is reductive, i. e. $\mathcal{G}$ is the direct sum of an abelian subalgebra $\mathcal{A}$ and a semisimple Lie algebra $\mathcal{G}^s$. It will be also supposed that the center of the connected subgroup G^s of G corresponding to the Lie algebra $\mathcal{G}^s$ is finite and Ad G is contained in the connected complex adjoint group G_c of the complexified Lie algebra $\mathcal{G}_c$. One can deduce (see Harish — Chandra [7, Lemma 4. 11]) that the centralizer $Z(T)$ of an arbitrary maximal vector subgroup T of G meets every connected component of G.

1.2. PARABOLIC OBJECTS. Recall that a subalgebra $\mathcal{J}_c$ of $\mathcal{G}_c$ is called a *Cartan subalgebra* iff it is nilpotent and is its own normalizer in $\mathcal{G}_c$. The centralizer in G of a Cartan subalgebra of $\mathcal{G}$ is called a *Cartan subgroup.* A subalgebra $\mathcal{P}$ of $\mathcal{G}$ is called *parabolic* if its complexification $\mathcal{P}_c$ contains a *Borel subalgebra* (i. e. a maximal solvable subalgebra) of $\mathcal{G}_c$. Then, if $\mathcal{P}$ is a parabolic subalgebra, it is its own normalizer in $\mathcal{G}$. A subgroup P of G is said to be *parabolic* iff it is the normalizer in G of a parabolic subalgebra $\mathcal{P}$ of $\mathcal{G}$. It is easy to see that in this case P is closed and its Lie algebra is just $\mathcal{P}$.

1.3. LANGLANDS DECOMPOSITION. Let $\mathcal{N}$ be a maximal subalgebra of $\mathcal{P}^s = \mathcal{P} \cap \mathcal{G}^s$ consisting of the elements whose adjoints are nilpotent and U be the analytic subgroup of P with the Lie algebra $\mathcal{N}$. It is well known that there exists a maximal subalgebra $\mathcal{M}'$ of $\mathcal{P}$ whose image in ad$\mathcal{G}$ is fully reducible and that $\mathcal{P} = \mathcal{M}' \oplus \mathcal{N}$, and $\mathcal{M}'$ contains a Cartan subalgebra of $\mathcal{G}$. Thus $\mathcal{M}'$ is its normalizer. Let $\mathcal{A}$ be a subalgebra of the center of $\mathcal{M}' \cap \mathcal{G}^S$ whose image in ad$\mathcal{G}$ is diagonalizable. If $\mathcal{M}$ is the orthogonal complement of $\mathcal{A}$ in $\mathcal{M}'$ with respect to the Killing form on $\mathcal{G}$ (see Harish — Chandra [7]), then $\mathcal{A} \cap \mathcal{M} = \{0\}$. Let M' be the normalizer of $\mathcal{M}'$ in P. Then the connected component of M' is of finite index in M', M' and U are closed in P and $P = M'.U$, $M' \cap U = \{1\}$, and finally M' satisfies all the conditions imposed on G in 1. 1.

Let $M = {}^0M'$ and let A be the (connected) vector subgroup of P with the Lie algebra $\mathcal{A}$. Then M is the centralizer of A in G and M meets every connected component of G. We have the decomposition $M' = A \times M$ and we call A *the split component of* P.

The map $(m, a, u) \to mau$ defines an analytic diffeomorphism of $(M \times A).U$ onto P. Moreover ${}^0P = M.U$ and we refer to the decomposition $P = MAU$ as the *Langlands decomposition*.

1. 4. STRUCTURAL THEORY OF PARABOLIC SUBGROUPS. By a *parabolic pair*, or *p-pair* we mean a pair (P, A), where P is a parabolic subgroup of G and A is the split component of P. Let $P = MAU$ and $\mathcal{P} = \mathcal{M} + \mathcal{A} + \mathcal{N}$ be the Langlands decomposition of P and its Lie algebra $\mathcal{P}$. By a root $\alpha \in \Sigma(P, A)$ we mean a root $\alpha \in \Sigma(\mathcal{G}, \mathcal{A})$ such that $\mathcal{G}_\alpha = \mathcal{N}_\alpha \subset \mathcal{N}$. If $l = \operatorname{prk} P - \operatorname{prk} G$, there exist exactly l distinct simple roots $\{\alpha_1, \ldots, \alpha_l\}$ which are R-linearly independent and each $\alpha \in \Sigma(P, A)$ can be written in a unique fashion in the form

$$\alpha = m_1 \alpha_1 + \ldots + m_l \alpha_l$$

with nonnegative integral coefficients m_i.

Fix a subset $E \subset \Delta(P, A) = \{\alpha_1, \ldots, \alpha_l\}$ and put

$$\mathcal{A}_F = \{H \in \mathcal{A} ;\ \alpha(H) = 0 \ \forall\ \alpha \in F\},$$

$$\Sigma_F = \{\alpha \in \Sigma(P, A);\ \alpha|_{\mathcal{A}_F} = 0 \quad\},$$

$$\Sigma'_F = \Sigma(P, A) - \Sigma_F ,$$

$$\mathcal{N}_F = \sum_{\alpha \in \Sigma'_F} \mathcal{N}_\alpha ,\ \mathcal{P}_F = \mathcal{Z}_{\mathcal{G}}(\mathcal{A}_F) + \mathcal{N}_F.$$

Then $\mathcal{P}_F$ is a parabolic subalgebra of $\mathcal{G}$ and the corresponding parabolic subgroup $P_F = \mathcal{N}_G(\mathcal{P}_F)$ has the Langlands decomposition $P_F = M_F A_F U_F$, where $A_F = \exp \mathcal{A}_F$, $U_F = \exp \mathcal{N}_F$. Moreover $P_F \supset P$, $M_F \supset M$, $A_E \subset A$, $U_F \subset U$.

Given any parabolic subgroup P' of G containing P, there exists a unique subset $F \subset \Delta(P, A)$ such that $P' = P_F$. We write $(P, A)_F = (P_F, A_F)$. In this case the finite set F of simple roots is called *the type* of the parabolic subgroup P.

There is a one-to-one correspondence between the parabolic subgroups P' of G contained in P and the parabolic subgroups ${}^*P$ of M. This correspondence is given by the relation ${}^*P = P' \cap M$. If $P' = M'A'U'$, ${}^*P = {}^*M{}^*A{}^*U$ are the Langlands decompositions, then $M' = {}^*M$, $A' = {}^*A.A$, $U' = {}^*U.U$, ${}^*A = M \cap A'$, ${}^*U = M \cap U'$.

Let (P_i, A_i), $i = 1, 2$, be two p-pairs in G. We write $(P_1, A_1) > (P_2, A_2)$ if $P_1 \supset P_2$ (and hence $A_1 \supset A_2$).

Any two minimal parabolic subgroups of G are conjugate under K^0. Let P_1, P_2 and Q be parabolic subgroups of G. Suppose that $P_1 \cap P_2 \supset Q$ and P_1 is conjugate to P_2. Then $P_1 = P_2$.

Let $P = MAU$ be a parabolic subgroup of G. Then the following three conditions are equivalent:

(1) P is a minimal parabolic subgroup.

(2) $M \subset K$.

(3) prk P = rank G/K.

1. 5. SIEGEL DOMAIN. The map $H \mapsto \exp H$ from $\mathcal{A}$ to $A = \exp \mathcal{A}$ defines a bijection $\alpha \mapsto \xi_\alpha$ between $\mathcal{A}^* = \mathrm{Hom}_c(\mathcal{A}, c)$ and $\mathfrak{X}(A)$ (the character group), such that:

$$\xi_\alpha(\exp H) = \exp \alpha(H).$$

Let us denote

$$A^+(c_1, c_2) = \{a \in A;\ c_1 < \xi_{\alpha_i}(a) < c_2;\ 1 \leqslant i \leqslant l\}.$$

Fix once for all a maximal compact subgroup K of G^0. If (P, A) is a p-pair, if c is a positive number and if ω is a compact subset of ${}^0P$, then the Siegel domain $\mathcal{S}$ associated to (P, A) is

$$\mathcal{S} = \{g = sak;\ s \in \omega,\ a \in A^+(c, +\infty),\ k \in K\}.$$

1. 6. CUSPIDAL SUBGROUPS. Consider a discrete subgroup Γ of G. A parabolic pair (subgroup) (P, A) is said to be *cuspidal* if every majorating parabolic pair (P', A') satisfies the following

1) $\Gamma \cap P' \subset {}^0P'$,

2) $U'/\Gamma \cap U'$ is compact and

3) ${}^0P/\Gamma \cap {}^0P'$ has finite volume.

A cuspidal subgroup is called *percuspidal* if the condition 3) is replaced by the following

3') ${}^0P'/\Gamma \cap {}^0P'$ is compact.

1. 7. GENERAL CASE. Let us now consider the general (not necessarily reductive) Lie group G wilh finite number of connected components. We denote by $G/R_u G$ the Levi subgroup of G and suppose that it satisfies all the conditions imposed on reductive groups in Section 1. 1. It is well known that there is a bijection $\mathcal{P}(G) \leftrightarrow \mathcal{P}(G/R_u G)$ between the sets of parabolic subgroups of G and of $G/R_u G$.

We always consider discrete subgroups with finite (co-) volume, $\mathrm{Vol}(G/\Gamma) < \infty$. Then in general a parabolic pair (P, A) is said to be cuspidal (respectively, percuspidal) if $P/R_u G$ is cuspidal (respectively, percuspidal and $R_u G/\Gamma \cap R_u G$ is compact) in the sense of reductive case.

1.8. LANGLANDS TYPE DISCRETE SUBGROUPS. A set $\mathscr{G}$ of percuspidal subgroups is said to be *complete* if every two elements (P_1, A_1) and (P_2, A_2) are G-conjugate (i. e. if there exists an element g of G such that $gP_1g^{-1} = P_2$ and $gA_2g^{-1} = A_1$) and if $\mathscr{G}$ is Γ-stable (i. e. whenever a p-pair (P, A) belongs to $\mathscr{G}$ then every p-pair $(\gamma P\gamma^{-1}; \gamma A\gamma^{-1})$, $\gamma \in \Gamma$, belongs to $\mathscr{G}$).

Langlands assumption. There is a complete set $\mathscr{G}$ of percuspidal subgroups such that for any cuspidal subgroup P which majorises an element P' of $\mathscr{G}$, one can find a finite subset $P_1, \dots, P_r$ of $\mathscr{G}$ such that $P > P_i$, $i = 1, \dots, r$ and Siegel domains $\mathcal{S}_i$ associated to $(P_i \cap {}^0P/U, A_i/A)$ such that $M = \bigcup_{i=1}^{r} \mathcal{S}_i \Theta$, where Θ is the image of $\Gamma \cap {}^0P$ in M. Moreover there is a finite subset $\mathfrak{F} \subset \mathscr{G}$ such that

$$\mathscr{G} = \bigcup_{\gamma \in \Gamma} \bigcup_{P \in \mathfrak{F}} \gamma^{-1} P \gamma .$$

DEFINITION. *A discrete subgroup Γ of finite volume of a Lie group G which satisfies the Langlands assumption is called a Langlands type discrete group.*

Remark. Every arithmetically defined subgroup is a Langlands type discrete group (Reduction theory). E. B. Vinberg [15] and V. S. Makarov have constructed examples of non arithmetically defined subgroups of finite volume. These groups are also of Langlands type, as was remarked by R. P. Langlands [9]. (see also Harish—Chandra [6]). So the class of Langlands type discrete subgroups is more general than that of arithmetically definite subgroups.

2. THE BOREL SERRE COMPACTIFICATION

In this section we shall construct the so called Borel-Serre compactification of the quotient of a symmetric space X of G by a Langlands type discrete Γ. Our situation is different from that of Borel—Serre in two respects. First, we consider an almost general Lie group (with finite number of connected components, see [9]). Second, we do not restrict ourselves to arithmetically defined subgroups; instead, our objects are the Langlands type discrete groups as defined in § 1.

2. 1. SPACE OF TYPE S. Let us now motivate the Borel-Serre notion of « space of type S» in our situation.

A space of type S for G is, by definition, any pair consisting of a right homogeneous space X and a family $\{L_x\}$ $x \in X$ of Levi subgroups of G satisfying two conditions:

S_1) There exists a connected normal solvable subgroup R_X of G containing the unipotent radical $R_u G$ such that the isotropy subgroups $H_x (x \in X)$ of G are of the form of a semidirect product $K.S$, where S is a split component of R_X and $K = K_x$ is the maximal compact subgroup of G normalizing S.

S_2) The familly $\{L_x\}$ $x \in X$ is G-invariant, i. e. $L_{xg} = L_x^g = g^{-1}Lg$, $g \in G$, and each isotropy subgroup H_x is contained in L_x .

In the case where $R_X = R_d\, G = A.U$, the split radical of G , we shall say that X is of type S/R . In the sequel we shall effectively be concerned with this situation.

2.2. GEODESIC ACTION. Let X be a space of type S under a Lie group G, H_x $(x \in X)$ the isotropy subgroups and L_x the Levi subgroups of G, P a parabolic subgroup of G, Z the center of $P/R_u P$ and finally let π ; $P \to P/R_u P$ be the canonical projection. Let Y be the greatest compact subgroup of Z. By virtue of our convention in 2.1, $R_X = R_d\, G = A.U$, the kernel of the homomorphism $R_X \to P/R_u P$ induced by π is $R_X \cap R_u P = U$. Then $R_X / U \hookrightarrow P/R_u P$ and its image is a vector group which is invariant and hence central in $P/R_u P$, i. e. $\pi(R_X) \subset Z$. If S is a maximal vector subgroup in R_X then $\pi(R_X) = \pi(S)$ and hence $\pi(R_X) = \pi(H_X \cap R_X)$. On the other hand, $H_x = K.(H_x \cap R_X)$ for some maximal compact subgroup K of G, and one has $H_x \cap P = (K \cap P).(H_x \cap R_X)$. We define

$$Z_0 = Z \cap \pi(H_x \cap P) = (\pi(K \cap P) \cap Z) . \pi (H_x \cap R_X).$$

The group $K \cap P$ is maximal compact, hence $Z \cap \pi(K \cap P)$ is maximal compact in Z. Denoting $Y = Z \cap \pi\ (K \cap P)$ we have that $Z_0 = Y.\pi(H_x \cap R_X) = Y.\pi(R_X)$. In particular, $Z_0 = Z \cap \pi(H_x \cap P) = Y . \pi(R_X)$ is independent of $x \in X$.

It is easy to see that X is canonically of type S under P (cf. also [2 §2.3(3)] for the arithmetic case). For $x \in X$ the Levi subgroup L'_x of P associated to x is contained in L_x . Let $Z_x = C(L'_x)$ be the center of L'_x. Then Z_x is the unique lifting of Z in P which is stable under the Cartan involution of L_x with respect to a maximal compact subgroup of H_x. For $z \in Z$ let z_x be its lifting in Z_x.

Fix a point $x \in X$. Every other point $y \in X$ can be obtained from x by the action of some element $g \in P$, $y = xg$. If $g' \in P$ also satisfies the equality $y = xg$, then $g = hg$ for some $h \in H_x \cap P \subset L'_x$. Therefore, the element h commutes with $z_x \in Z_x = C(L'_x)$. Hence

$$x \,.\, z_x \,.\, g = x \,.\, z_x \,.\, hg = x \,.\, h \,.\, z_x \,.\, g = x \,.\, z_x \,.\, g.$$

Thus $x \,.\, z_x \,.\, g$ depends only on x, y, z and we put by definition

$$y \underset{x}{\circ} z = x \,.\, z_x \,.\, g.$$

It is easy to see that this defines an Z-action γ on X and the definition is independent of the choice of the point x. The action γ of Z on X defined above is called the *geodesic action* of P. It is easy to see that $Z_0 = Z \cap \pi(H_x \cap P)$ operates trivially and Z/Z_0 operates freely.

The group Z_0 contains the maximal compact subgroup $Y = Z \cap \pi(K \cap P)$ of Z. We may therefore write $Z = Z_0 \times A$, where A is the identity component of a (vector) subgroup of Z. Hence, for every $x \in X$, $x \circ Z = x \,.\, A$.

2.3 BUNDLES DEFINED BY THE GEODESIC ACTION. Let T be a split component of $C(P/R_u P) = Z$ whose intersection with Z_0 is finite and A be the image of the connected component T^0 of T into Z/Z^0. By [2, § 1.2] there exists a normal subgroup M of P containing R_X and all maximal compact subgroups of P such that $P = T^0 \times M$. Since P commutes with A, the later operating by geodesic action, we have an action of $A \times M$ onto X defined by

$$x \longmapsto (x \,.\, a).\, m, \text{ for all } a \in A, m \in M \text{ and } x \in X.$$

We have $(x, A).\, M = x.\, A_x \,.\, M = x.\, P = X$. Then the $A \times M$ -action on X is transitive.

The space X is of type S under P and in particular the isotropy group $H_x \cap P$ of x under P is generated by a maximal compact subgroup $K'_x = K_x \cap P$ of P and a subgroup of R_X. Since both of these groups are contained in M, we have $H_x \cap P \subset M$. If $(x \circ a).\, m = x$ then $a_x \,.\, m \in H_x \cap P \subset M$. Thus $a_x \in M \cap A = \{1\}$ and we have:

a) For every $x \in X$ the isotropy group of x in $A \times M$ is $\{1\} \times (H_x \cap P)$.

b) The map $\mu_x : A \times (H_x \cap P)\backslash M \to X$ of $A \times M$ -homogeneous spaces defined above is an isomorphism for every $x \in X$.

c) The space X is a trivial principal A-bundle and the orbits of M are the cross-sections of this fibration.

These results are proved in the same manner as in [2].

2.4. MAXIMAL VECTOR SUBGROUPS IN PARABOLIC SUBGROUPS. Let $\mathcal{P}(G)$ be the set of all parabolic subgroups of G, R the (solvable) radical of G, and $\pi : G \to G/R$ the canonical projection. The correspondence $P \mapsto \pi(P)$ gives us a bijection between $\mathcal{P}(G)$ and $\mathcal{P}(G/R)$. Let S be a maximal split vector subgroup of G^0/U, $\Sigma = \Sigma(G^0/U,S)$ the set of roots of G^0/U with respect to S, and Δ a basis of simple roots. There is therefore a natural 1-1 correspondence between the conjugacy classes in $\mathcal{P}(G)$ and the subsets of Δ. The class corresponding to a subset $J \subset \Delta$ is represented by the standard parabolic subgroup P_J: the image P_J/U of P_J in G/U is the semi-direct product of its unipotent radical U_J by the centralizer $\mathcal{Z}(S_J)$ of $S_J = (\bigcap_{\alpha \in J} \operatorname{Ker} \alpha)^0$ and its split radical is $S_J . U_J$. Given $P \in \mathcal{P}(G)$, the only I such that F is conjugate to P_J under G^0 will be denoted by $I(P)$ and called the type of P.

Let $P \in \mathcal{P}(G)$. The quotient $S_P = R_d P/(R_u P.R_d G)$ is a split vector subgroup and it is also the greatest vector subgroup in $C_P = C(P/(R_u P.R_d G))$. We take by definition $A_P = S_P^0$. Let $P' \in \mathcal{P}(G)$ be conjugate to P under G^0, ${}^xP = P'$, $x \in G^0$ Then Int x induces an isomorphism of $C_{P'}$ onto C_P and one can find a natural isomorphism $\sigma_{P',P} : S_{P'} \xrightarrow{\sim} S_P$. In particular, if $P = P_I$ be a standard parabolic subgroup, then $S_{P'} = S_I/S_\Delta$ and $\Delta - I$ defines a basis of $\mathcal{X}^*(S_I/S)$ and finally we have a canonical isomorphism $A_P \xrightarrow{\sim} (R_+^*)^{\Delta - I}$. This isomorphism defines an open embedding of A_P into $R^{\Delta - I}$. The closure of A_P in $R^{\Delta - I}$ will be denoted by $\overline{A}_P$. The action of A_P on itself by means of translation is extended to one on A_P, given by coordinate multiplication.

2. 5. THE CORNER ASSOCIATED TO A PARABOLIC SUBGROUP. From now on we take $R_X = R_d G$. By 2.3, X is a principal A_P-bundle under the geodesic action. By definition, the corner $X(P)$ associated to P is the total space of the associated bundle with typical fibre $\overline{A}_P$

$$X(P) = X \times_{A_P} \overline{A}_P.$$

Let us put $e(P) = X/A_P$. In particular $e(G^0) = X$. Because $\overline{A}_P = \coprod_{L \subset \Delta} A_P(L)$, where $\overline{A}_P(L), = \{x \in \overline{A}_P ; \alpha(x) = 0, \forall \alpha \notin L \text{ and } \alpha(x) \neq 0, \forall \alpha \in L\}$, we have

$$X(P) = \bigsqcup_{\substack{Q \in \mathcal{P}(G) \\ Q \supset P}} e(Q).$$

Let $P \subset Q$ be two parabolic subgroups of G. Then the inclusion $X(Q) \hookrightarrow X(P)$ is an isomorphism of manifolds with corners $X(Q)$ onto an open subset of $X(P)$. We have also

$$Cl_{X(P)}\, e(Q) = \bigsqcup_{\substack{Q \supset R \supset P \\ R \in \mathcal{P}(G)}} e(R) = e(Q)(P),$$

where $e(Q)$ is viewed as a subspace of type S/R under Q and $e(Q)(P)$ is the corner of $e(Q)$ associated to parabolic subgroup P of Q (see [2, § 4,3 (3)] for the arithmetic case).

2.6. THE MANIFOLD WITH CORNERS $\overline{X}$. For two parabolic subgroups P, $Q \in \mathcal{P}(G)$ and the smallest parabolic subgroup R containing both P and Q we have $X(P) \cap X(Q) = X(R)$. Furthermore if two parabolic subgroups $P \cap P' \in \mathcal{P}(G)$, we have also an open inclusion of topological spaces $X(P') \hookrightarrow X(P)$; Hence there exists one and only one structure of manifold with corners on

$$\overline{X} = \bigsqcup_{P \in \mathcal{P}(G)} e(P) = \bigcup_{P \in \mathcal{P}(G)} X(P)$$

such that $X(P)$'s are open submanifolds with corners of $\overline{X}$, $X\, e(G^0)$. The space $\overline{X}$ will be endowed with that structure and $\{X(P)\}_{P \in \mathcal{P}(G)}$ form an open cover of $\overline{X}$.

Considering $e(P)$ as a manifold with corners endowed with its canonical structure of space of type S/R under P one can take $e(P)$ by the general construction. We have $e(P) \hookrightarrow \overline{X}$. Let Z be the closure $e(P)$ in the topology of $\overline{X}$. Then $e(Q)$, $Q \in \mathcal{P}(G)$, meets Z iff $X(Q) \cap e(P) \neq \phi$, i. e. $Q \subset P$ Therefore

$$Z = \bigcup_{\substack{Q \in \mathcal{P}(G) \\ Q \subset P}} X(Q) \cap Z = \bigcup_{\substack{Q \in \mathcal{P}(G) \\ Q \subset P}} e(P)(Q).$$

We have obviously some corollaries:

(1) For $P,Q \in \mathcal{P}(G)$, $\overline{e(P)} \cap \overline{e(Q)} = \overline{e(P \cap Q)}$ iff $P \cap Q \in \mathcal{P}(G)$. In particular, $\overline{e(P)} = \overline{e(Q)} \Leftrightarrow P = Q$.

(2) For all $P,Q \in \mathcal{P}(G)$, $e(P) \cap \overline{e(Q)} \neq \emptyset \Leftrightarrow e(P) \subset \overline{e(Q)} \Leftrightarrow P \subset Q$.

(3) i) $\{g \in G\,;\, P^g = g^{-1}Pg = Q\} =$
$\{g \in G\,;\, e(P)g = e(Q)\} =$
$\{g \in G\,;\, e(P)g \cap e(Q) \neq \phi\}.$

ii) $\{g \in G\,;\, P^g \cap Q \in \mathcal{P}(G)\} = \{g \in G\,;\, \overline{e(P)g} \cap \overline{e(Q)} \neq \phi\}$

iii) $Q = \{g \in G_0\,;\, \overline{e(Q)g} \cap e(Q) \neq \phi\}.$

THEOREM *(Borel-Serre [2]). The manifold with corners $\overline{X}$ is Hausdorff and countable at infinity.*

2.7. THE BOUNDARY $\partial\overline{X}$ AND TITS BUILDING. Recall that the Tits building of G is the simplicial complex whose set of vertices is the set I of maximal parabolic subgroups of G and whose simplexes are the nonempty subsets S of I such that $P_s \overset{\text{def}}{=\!=} \bigcap_{P \in s} P \in \mathcal{P}(G)$. It is canonically isomorphic to the building attached to the Tits system of G^0/RG constructed in N. Bourbaki, LIE, IV, 2, Exerc. 10.

The cover of $\partial\overline{X}$ given by $e(P)$ is locally finite. It is easy to see that for a subset s of I, $\bigcap_{P \in s} \overline{e(P)} \neq \varnothing \Leftrightarrow P_s \in \mathcal{P}(G) \Leftrightarrow s$ is a simplex of the Tits building T.

Let $|T|$ be the geometric realization of T. Then $\partial\overline{X}$ and $|T|$ have the same homotopy type of a bouquet of $(l-1)$-spheres with the weak topology.

2.8. MAIN RESULTS. Let Γ be a Langlands type discrete subgroup of G. We introduce the cuspidal part of X

$$\overline{X}_{\text{cusp}} = \bigcup_{P \in \mathcal{P}_{\text{cusp}}(G)} e(P),$$

where $\mathcal{P}_{\text{cusp}}(G)$ denotes the set of all cuspidal subgroups of G.

THEOREM 1. *Γ acts properly on $\overline{X}_{\text{cusp}}$ and the quotient $\overline{X}_{\text{cusp}}/\Gamma$ is compact.*

THEOREM 2. Let $\pi : \overline{X}_{\text{cusp}} \to \overline{X}_{\text{cusp}}/\Gamma$ *be the natural projection, D the set of all representatives of Γ-conjugation classes of cuspidal subgroup, $D = \mathcal{P}_{\text{cusp}}(G)/\Gamma$, $\Gamma_P = \mathcal{N}_G(P) \cap \Gamma$, $e'(P) = \pi(e(P))$.*

(i) $e'(P) = e(P)/\Gamma_P$ *and* $e'(P) \cap e'(Q) \neq \phi \Leftrightarrow e'(P) = e'(Q) \Leftrightarrow \exists\, \gamma \ni \Gamma;$ $P^\gamma = Q$. *The set D is finite and* $\overline{X}_{\text{cusp}}/\Gamma = \bigsqcup_{P \in \mathcal{P}_{\text{cusp}}(G)/\Gamma} e'(P).$

(ii) $Cl_{\overline{X}_{\text{cusp}}/\Gamma}(e'(P)) = \pi\,\overline{(e(P))}$. *If $\Gamma \subset G^0$, we have $\Gamma_P = \Gamma \cap P$ and*

$$\pi(\overline{e(P)}) = \overline{e(P)}/\Gamma_P \bigsqcup_{\substack{Q \in \mathcal{P}_{\text{cusp}}(G)/\Gamma_P \\ Q \supset P}} e'(Q).$$

In particular $e'(Q) \in \mathrm{Cl}_{\overline{X}_{cusp}/\Gamma}\, e'(P) \Leftrightarrow \exists\, P' \subset P$ *such that* $P' \in \mathcal{P}_{cusp}(G)$ *and* Q *conjugates under* Γ *to* P'.

We shall prove these theorems in the rest of the paper.

2.9. REDUCTION TO THE REDUCTIVE CASE. Assume that $V = R_u\, P \neq \{1\}$. Let $\sigma: G \to G' = G/V$ *and* $\pi: X \to X' = X/V$ be the canonical projections.

LEMMA. $\Gamma' = \sigma(\Gamma)$ *is of Langlands type in* G', $V/\Gamma \cap V$ *is compact, and so* $\Gamma \cap V$ *is of Langlands type in* V.

Proof. We have the canonical bijection $\mathcal{P}(G) \leftrightarrow \mathcal{P}(G')$ $P \leftrightarrow P/V$. By assumption $V/\Gamma \cap V$ is compact, see § 1.7, and $G \in \mathcal{P}_{cusp}(G)$. It is easy to see that there is also a bijection $\mathcal{P}_{cusp}(G) \leftrightarrow \mathcal{P}_{cusp}(G')$ and $\mathcal{P}_{percusp}(G) \leftrightarrow \mathcal{P}_{percusp}(G')$. Thus the assumptions in the definition of Langlands type discrete subgroup sare fulfilled. The Lemma is proved.

The space $\overline{X}_{cusp}$ is a principal V-bundle and $\overline{X}'_{cusp} = \overline{X}_{cusp}/V$. We are now in the same situation as in 9.2 of Borel-Serre [2]. By induction on dim G we can then conclude that if $V = R_u\, P \neq 1$ then Γ acts properly on $\overline{X}_{cusp}$ and $\overline{X}_{cusp}/\Gamma$ is compact. We are thus reduced to the case where G is connected and reductive.

2. 10. REDUCTION TO THE SIEGEL SET CASE. In view of [2, § 9, 1] applied to $Y = \overline{X}_{cusp}$, $z = \partial \overline{X}_{cusp}$ and $L = \Gamma$, it suffices to show that if C, D are compact subsets of $\overline{X}_{cusp}$, then

$$\# \{\gamma \in \Gamma ;\quad C \cdot \gamma \cap D \cap X \neq \phi\} < \infty .$$

The closure $\mathrm{Cl}_{\overline{X}_{cusp}}(\mathcal{S})$ in $\overline{X}_{cusp}$ of a Siegel set $\mathcal{S}$ with respect to a fixed point $x \in X$ and a fixed parabolic subgroup P is compact and each point in the corner $X(P)$ has a neighbourhood of this form. On the other hand, we have

LEMMA. *When* P *runs over the set* $\mathcal{P}_{percusp}(G)$, *the corners* $X(P)$ *form an open cover of* $\overline{X}_{cusp}$.

Proof. $\overline{X}_{cusp} = \bigsqcup_{P \in \mathcal{P}_{cusp}(G)} e(P) = \bigcup_{P \in \mathcal{P}_{cusp}(G)} X(P)$ and $X(Q) = \bigcup_{P \in \mathcal{P}(Q)} X(P)$ and $X(P)$ is open in $X(Q)$. Since every cuspidal subgroup contains some percuspidal one, $\{X(P)\}_{P \in \mathcal{P}_{percusp}(G)}$ form an open cover of $\overline{X}_{cusp}$, the Lemma is proved.

Thus it suffices to check the case where $C = \mathrm{Cl}_{\overline{X}_{\mathrm{cusp}}}(\mathcal{S})$ and $D = \mathrm{Cl}_{\overline{X}_{\mathrm{cusp}}}(\mathcal{S}')$, where $\mathcal{S}$ (respectively, $\mathcal{S}'$) is a Siegel set with respect to X and a percuspidal subgroup P (respectively, P'). Any two percuspidal subgroups are conjugate under G, by assumption. Thus there exists $g \in G$ such that $P' = P^g = g^{-1}Pg$. Then $\mathcal{S}g^{-1}$ is a Siegel set with respect to P, x. Since any two Siegel sets are contained in a bigger one, we may also assume the set ω occuring in the definition of Siegel sets to be compact, see § 1. 5. Thus $\mathcal{S}$ is closed in $\overline{X}_{\mathrm{cusp}}$, hence equal to $\mathrm{Cl}_{\overline{X}_{\mathrm{cusp}}}(\mathcal{S}) \cap X$. Under this condition, we must prove

$$\# \{\gamma \in \Gamma ;\ \mathcal{S}.\gamma \cap \mathcal{S}.g \neq \phi\} < \infty.$$

2. 11. THE ACTION PROPERNESS. Let $\pi : G \to X$ be the map $g \to x.g$ and let $\mathcal{S}'' = \pi^{-1}(\mathcal{S})$. Then $\mathcal{S}''$ is a Siegel set in the group G with respect to a maximal compact subgroup K of H_x, P and a suitable maximal vector subgroup of $R_d G$. It suffices thus to prove

$$\# \{\gamma \in \Gamma ; \mathcal{S}''.\gamma \cap \mathcal{S}''.g \neq \phi\} < \infty.$$

Thus the finiteness follows from Langlands' work [9], see also A. Borel [1].

2.12. THE COMPACTNESS OF $\overline{X}_{\mathrm{cusp}} / \Gamma$. In view of the relation between Siegel sets in X and in G, there exists a Siegel in X wi h respect to a percuspidal subgroup P and a finite subset C of G such that $X = \mathcal{S}.C.\Gamma$. By virtue of [2, §7.9], the closure M of $\mathcal{S}.C$ in $\overline{X}_{\mathrm{cusp}}$ is compact. Since Γ acts properly on $\overline{X}_{\mathrm{cusp}}$, the family of sets $\{M.\gamma\}_{\gamma \in \Gamma}$ is locally finite in $\overline{X}_{\mathrm{cusp}}$, hence is closed in $\overline{X}_{\mathrm{cusp}}$. On the other hand $X \subset M.\Gamma$ (by assumption) and X is dense in $\overline{X}_{\mathrm{cusp}}$. Then $M.\Gamma = \overline{X}_{\mathrm{cusp}}$ and M is mapped onto $\overline{X}_{\mathrm{cusp}} / \Gamma$ under the natural projection. So $\overline{X}_{\mathrm{cusp}} / \Gamma$ is compact and the theorem is proved.

Remark. $\overline{X} = \overline{X}_{\mathrm{cusp}} \cup$ open subset, therefore $\overline{X}_{\mathrm{cusp}}$ is Hausdorff and closed in the Hausdorff space $\overline{X}$.

2.13. PROOF OF THEOREM 2. (i) By virtue of 2.6, we have immediately that

$$e'(P) = \pi(e(P)) = e(P)/\Gamma_P,$$

where $\Gamma_P = \Gamma \cap \mathcal{N}_G(P)$. We also have

$$e'(P) \cap e'(Q) \neq \emptyset \Leftrightarrow e'(P) = e'(Q) \Leftrightarrow P^\gamma = Q$$

for some $\gamma \in \Gamma$. Since $D = \left\{ \begin{array}{l} \text{representatives of percuspidal} \\ \text{sugrboups of } G \end{array} \right\}$, by the Langlands assumption D is finite, and

$$\overline{X}_{\mathrm{cusp}} / \Gamma = \Big(\bigsqcup_{P \in \mathcal{P}_{\mathrm{cusp}}(G)} e(P)\Big)/\Gamma = \bigcup_{P \in \mathcal{P}_{\mathrm{cusp}}(G)} e(P)/\Gamma_P = \bigcup_{P \in \mathcal{P}_{\mathrm{cusp}}(G)} e'(P).$$

(ii) Since $e(P)\,.\,\gamma = \overline{e(\gamma^{-1}P\gamma)}$, $\gamma \in \Gamma$, it follows that $\overline{e(P)}\,.\,\Gamma$ forms a locally finite family in $\overline{X}_{\text{cusp}}$ and $\overline{e(P)}.\,\Gamma$ is closed in $\overline{X}_{\text{cusp}}$, $\pi^{-1}\pi\,(\overline{e(P)}) = \overline{e(P)}.\,\Gamma$. Hence $\pi(\overline{e(P)})$ contains the closure of $e'(P)$ which is dense in $\pi(\overline{e(P)})$ we have then $\mathrm{Cl}_{\overline{X}_{\text{cusp}}/\Gamma}(e'(P)) = \pi(\overline{e(P)})$. From (2.6), we obtain

$$\overline{\pi(e(P)} = \overline{e(P)}/\Gamma_P = \bigsqcup_{\substack{Q\in\mathcal{P}_{\text{cusp}}(G)/\Gamma_P \\ Q\supset P}} e(Q).$$

The proof of our theorem is complete.

REFERENCES

[1] A. Borel, *Introduction aux groupes arithmétiques*, Hermann ed., Paris, 1969.

[2] A. Borel and J.-P. Serre, *Corners and arithemtic groups*, Comment. Math. Helvet., 48 (1973), 435 – 442.

[3] A. Borel and. N. Wallach, *Continuous cohomology, discrete subgroups and representations of reductive groups*, Princeton Univ. Press and Univ. of Tokyo Press, 1980.

[4] G. Harder, *On the cohomology of SL_2 (O)*, in « *Lie groups and their representations* ». Proc. of the Summer School on group representations, I.M. Gelfand ed., London, 1975, 139 – 150.

[5] G. Harder, *On the cohomology of discrete arithmetically defined groups*, in Proc. Int. Colloq. on Discrete subgroups of Lie groups and application to Moduli, Bombay, 1973, 129 – 160.

[6] Harish-Chandra, *Automorphic forms on semisimple Lie groups*, Lecture Notes in Math. Springer Verlag, 63 (1968).

[7] Harish-Chandra, *Harmonic analysis on real reductive groups. I*, J. Funct. Anal. 19 (1975), 104–204.

[8] K.E. LAI, *On the cohomology of congruence subgroups of symplectic groups*, Nagoya Math J., 85 (1982), 155 – 174.

[9] R. P. Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Math. Vol. No 544, Springer Verlag, 1976.

[10) R. Lee and J. Schwermer, *Cohomology of arithmetic subgroups of SL_3 at infinity*, J. Fur die Reine under angewandte mathematik, Bd 330, 100 – 131.

[11] V. S. Makarov, *On a class of discrete subgroups of Lobachevski space having a fundamental domain of finite volume*, Dokl. AN SSSR, 167 (1966), 30 — 33.

[12] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer Verlag, 1972.

[13] J. Schwermer, *Eisensteinreinheit und Lie kohomologie von kongruenzuntergruppen von SL_n* (Z), Bonner Mathmatische Schriften, 99 (1977).

[14] J. Schwermer, *Sur la cohomologie des sous-groupes de congruence de SL_3 (Z)*, C. R. Acad. Sci. Paris, 283 (1976), 817 — 820.

[15] E. B. Vinberg, *Discrete groups generated by reflections in Lobachevski space*, Mat. Sbornik, 72 (114), No 3 (1967), 471–488.

Received February 28, 1986

INSTITUTE OF MATHEMATICS, P. O. BOX 631,
10 000 BO HO, HANOI, VIETNAM