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LOCAL MODELS OF THE GREATEST CHARACTERISTIC EXPONENT OF DIFFERENTIAL EQUATIONS DEPENDING ON PARAMETERS.

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§ 1. INTRODUCTION

Let us consider the differential equation

$$\frac{dx}{dt} = A(\lambda)x$$

where $x \in \mathbf{R}^n$, $A(\lambda)$ is a $n \times n$ – matrix depending differentiably on a parameter λ that belongs to a differentiable manifold Λ of finite dimension. We put

$$f(\lambda) = \lim_{t \to +\infty} \frac{1}{t} \ln \| e^{A(\lambda)t} \| .$$

Then $f(\lambda)$ is equal to the greatest real part of all eigenvalues of $A(\lambda)$ [1]. Generally speaking, $f(\lambda)$ is a continuous but not differentiable function of λ.

In the case where dim Λ is less than 2, V. I. Arnold [2] has classified singularities of $f(\lambda)$ by using the versal deformation of matrices. Our purpose is to consider the problem in the case where dim Λ is arbitrary and the equation has the form

$$a_0(\lambda)\, y^{(n)} + a_1(\lambda)\, y^{(n-1)} + \ldots + a_n(\lambda)\, y = 0$$

where $a_i \in C^\infty(\Lambda)$.

Our method is to use the technique of stratification, Weierstrass Preparation Theorem and Mather Division Theorem, Thom Transversality Theorem and Lemma on a family of Morse functions [3]. Our paper is divided into several sections corresponding to the following steps:

— stratifying the space of polynomials;

— describing the behaviour of polynomials in a neighbourhood of a stratum;

— determining equations and calculating the codimension of strats;

— using Thom Transversality Theorem and Lemma of a family of Morse functions to find the local models of $f(\lambda)$.

It will be proved that in the « general case » $f(\lambda)$ receives only a finite number of local models that can be given in a list. Note that in the case $a_0(\lambda) = 1$ the finiteness of the number of these local models was proved geometrically by L. V. Levantovski [5].

§ 2. STRATIFICATION OF THE SPACE OF POLYNOMIALS

Let us consider the set P of non constant polynomials

$$x_0 t^n + x_1 t^{n-1} + \ldots + x_n \text{ where } x_0^2 + \ldots + x_{n-1}^2 \neq 0.$$

We denote by $S_l(k, k_1, \ldots, k_r)$ the set of polynomials of degree $n-l$ (i. e. such that $x_0 = \ldots x_l = 0$), for which the greatest real part of all the roots is attained just at one real roots with multiplicity k and r pairs of complex roots with multiplicity $k_1, \ldots, k_r$ respectively. Then we have

$$P = \bigcup_{0 \leqslant l < n-1} S_l\ (k; k_1, \ldots, k_r).$$

These sets $S_1(k, k_1, \ldots, k_r)$ are disjoint. Besides they form a stratification of P whose strata are $S_1(k; k_1, \ldots, k_r)$ for all possible $k, k_1, \ldots, k_r$ (that means the closure of each stratum is composed of itself and the finite union of all strata of lower dimension).

§ 3. DESCRIPTION OF THE BEHAVIOUR OF POLYNOMIALS IN THE NEIGHBOURHOOD OF EACH STRATUM

3.1 We use the following notations

C_p^∞ = ring of germs of real (C^∞) differentiable functions at a point p.

$\mathcal{C}_p^\infty$ = ring of germs of complex value (C^∞) differentiable functions at a point p.

$M_p = \{ u \in C_p^\infty \mid u(p) = 0 \}$.

$\mu_p = \{ v \in \mathcal{C}_p^\infty \mid v(p) = 0 \}$.

$C_p^\infty[t]$ = ring of polynomials whose coefficients belong to C_p^∞.

Let us consider the polynomial

$a_0(x)\, t^n + a_1(x)\, t^{n-1} + \dots + a_n(x)$ where $a_i \in C_p^\infty$, $i = 0, 1, \dots, n$.

LEMMA 1. *a) Let the equation* $\sum_{i=1}^{n} a_i(p)\, t^{n-i} = 0$ *have one real root* α *with multiplicity* k. *Then there are* $b_i \in M_p$ $(i = 1, 2, \dots, k)$, $c_j \in C_p^\infty$ $(j = 0, 1, \dots, n-k)$ *such that in a neighbourhood of* $p \in R^1$ *we have*

$$\sum_{i=o}^{n} a_i\, t^{n-i} = \left((t-\alpha)^k + \sum_{j=1}^{k} b_j\, (t-\alpha)^{k-j}\right) \sum_{i=0}^{n-k} c_i\, t^{n-k-i} \quad (1).$$

b) Let the equation $\sum_{i=0}^{n} a_i(p) t^{n-i} = 0$ *have one pair of conjugate complex roots* $\alpha \pm i\omega$ $(\omega \neq 0)$ *with multiplicity* k. *Then there are* $b_j \in \mu_p$ $(j = 1, \dots, k)$, $c_j \in C_p^\infty$ $(j = 0, 1, \dots, n-2k)$ *such that in a neighbourhood of* $p \in R^1$ *we have*

$$\sum_{i=0}^{n} a_i\, t^{n-i} = (\tau^k + \sum_{j=1}^{k} b_j\, \tau^{k-j})\, (\bar{\tau}^k + \sum_{j=1}^{k} \bar{b}_j \bar{\tau}^{k-j}) \times$$

$$\times \sum_{j=0}^{n-2k} C_j t^{n-2k-j} \quad (2)$$

where $\tau = t - \alpha + i\omega$.

Proof. Let us consider the polymomial

$$P(z_0, z_1, z_n, W) = z_0 W^n + z_1 W^{n-1} + \dots + z_{n-1} W + z_n$$

where $(z_0, z_1, \dots, z_n) \in \mathbf{C}^{n+1}$, $W \in \mathbf{C}$.

Suppose that $P(a_0(q), \dots, a_n(p), \beta) = 0$, whes β is real or complex and that $P(a_0(p), \dots, a_n(p), w) \neq 0$. Then from Weierstrass Theorem it follows that in a neighbourhood of $(a_0(p), \dots, a_n(p), \beta) \in \mathbf{C}^{n+1} \times \mathbf{C}$ we have

$$P(z_0, \dots, z_n, w) = \{(w-\beta)^k + p_1(z_0, \dots, z_n)(w-\beta)^{k-1} + \dots +$$

$$+ p_k(z_0, \dots, z_n)\} \times \varphi(z_0, \dots, z_n, w),$$

where $k \geqslant 1$ is the multiplicity of β, $p_i(a_0(p), \dots, a_n(p)) = 0$, $\varphi(z_0, \dots, z_n, w) \neq 0$, p_i, φ are holomorphic in this neighbourhood.

a) Suppose that $\beta = \alpha \in \mathbf{R}$. According to Mather Division Theorem [3], p_i, φ are real differentiable functions if $(z_0, \dots, z_n) \in \mathbf{R}^{n+1}$, $w \in \mathbf{R}$. Besides, p_i, φ satisfy the following conditions

$$p_i(a_0(p), \dots, a_n(p)) = 0, \quad \varphi(a_0(p), \dots, a_n(p)) \neq 0.$$

So we put $p_i(a_0(x), \ldots, a_n(x)) = b_i(x)$, $b_i \in M_p$. It is easy to see that in this case $\varphi(a_0(x), \ldots, a_n(x))$ has the form $\sum_{i=0}^{n-k} c_i(x)\, w^{n-k-i}$ where k is the multiplicity of β, c_i are differentiable functions of $x (i = 0, 1, \ldots, n-k)$. Hence we obtain (1).

b) Suppose that $\beta = \alpha \pm i\omega \in \mathbf{C}\ (\omega \neq 0)$, $(z_0, \ldots, z_n, w) \in \mathbf{R}^{n+1} \times \mathbf{R}$. According to the fundamental theorem of algebra:

$$z_0 t^n + z_1 t^{n-1} + \ldots + z_n = \{(w - \alpha - i\omega)^k + p_1(z_0, \ldots, z_n)(w - \alpha - i\omega)^{k-1} + \ldots + p_k(z_0, \ldots, z_n)\}.\ \{(w - \alpha + i\omega)^k + \overline{p}_1(z_0, \ldots, z_n) \times (w - \alpha + i\omega)^{k-1} + \ldots + \overline{p}_k(z_0, \ldots, z_n)\}.\ q(z_0, \ldots, z_n, w).$$

where $q(z_0, \ldots, z_n, w)$ can be written in the form of a polynomial with respect to w and $q(a_0(p), \ldots, a_n(p), \beta) \neq 0$, p_i are holomorphic $(i = 1, 2, \ldots, k)$.

If we put $z_i = a_i(x)$, $a_i \in C_p^\infty$ $(i = 0, \ldots, n)$ then $p_i(a_0(x), \ldots, a_n(x)) = b_i(x)$; $b_i \in \mathscr{C}_p^\infty$ satisfy (2), where $\tau = t - \alpha - i\omega$, $b_i \in \mathscr{M}_p$, $c_j \in C_p^\infty$.

Remark. In this lemma if $a_0 = 1$ then $c_0 = 1$ in both cases.

3.2 Let us consider $\sum_{j=0}^{n} a_j(p)\, t^{n-j} = 0$ where $\sum_{j=0}^{n-1} a_j^2(p) \neq 0$.

Suppose that the roots of this equation are as follows:

– real roots $\quad \alpha_1$ with multiplicity k_1,

$\quad\quad\quad\quad\quad \ldots\ldots\ldots\ldots$

$\quad\quad\quad\quad\quad \alpha_r$ with multiplicity k_r

– complex roots $\quad \beta_1 \pm i\omega_1$ with multiplicity l_1

$\quad\quad\quad\quad\quad \ldots\ldots\ldots\ldots$

$\quad\quad\quad\quad\quad \beta_s \pm i\omega_s$ with multiplicity l_s

$\quad\quad\quad\quad\quad (\omega_j \neq 0,\ j = 1, 2, \ldots, s)$,

$$\sum_{j=1}^{r} k_j + 2 \sum_{m=1}^{s} l_m \leqslant n\ .$$

Then, by repeatedly applying Lemma 1 we obtain

LEMMA 2. *In a neighbourhood of p each element of $C_p^\infty[t]$ can be decomposed in factors as follows*

$$\sum_{j=1}^{n} a_j t^{n-j} = \prod_{j=1}^{r} P_j(t) \prod_{k=1}^{s} Q_k(t) \prod_{k=1}^{s} \overline{Q}_k(t) \,.\, R(t)$$

where $P_j(t) = (t - \alpha_j)^{k_j} + \sum_{h=1}^{k_j} u_{jh} (t - \alpha_j)^{k_j - h}; u_{jh} \in M_p$

$$h = 1, \dots, k_j;\ j = 1,2, \dots, r;$$

$$Q_k(t) = (t - \beta_k - iw_k)^{l_k} + \sum_{h=1}^{l_k} v_{kh} (t - \beta_k - iw_k)^{l_k - h}$$

$$v_{kh} \in \mu_p\ ;\ h = 1, \dots, l_k\ ;\ k = 1, \dots, s\ ;$$

$$\overline{Q}_k(t) = (t - \beta_k + iw_k)^{l_k} + \sum_{h=1}^{l_k} \overline{v}_{kh} (t - \beta_k + iw_k)^{l_k - h};$$

$$R(t) = \sum_{i=0}^{n-K-2L} q_i t^i\ ;\ K = \sum_{i=1}^{r} k_i\ ;\ L = \sum_{k=1}^{s} l_k\ ;$$

$$q_i \in M_p,\ i = 1, 2, \dots, n - K - 2L,$$

$$q_0 \in C_p^\infty \setminus M_p.$$

Denote by $\mathcal{P}$ the mapping: $\Lambda \longrightarrow R^{n+1} \setminus \{0\} \times \dots \times \{0\} \times R$

$$\lambda \longmapsto x_0(\lambda) t^n + \dots + x_n(\lambda).$$

Suppose that $\mathcal{P}(\lambda_0) = x_0(\lambda_0) t^n + \dots + x_n(\lambda_0) \in S_l(k; k_1, \dots, k_r)$ and that α is the greatest real part of its roots. The following lemma will describe the behaviour of polynomials in a neighbourhood of $\mathcal{P}(\lambda_0)$.

LEMMA 3. *There exists a neighbourhood of λ_0 in which $\mathcal{P}(\lambda)$ has the form*

$$\mathcal{P}(\lambda) = P(\tau) \prod_{j=1}^{r} Q_j(\tau_j) \prod_{j=1}^{r} \overline{Q}_j(\tau_j)\, R(t)\, S(t).$$

where

$$P(\tau) = (\tau - a_1)^k - \sum_{j=2}^{k} a_j (\tau - a_1)^{k-j},\ \tau = t - \alpha,\ \text{if } k \geqslant 2;$$

$$P(\tau) = \tau - a_1 \qquad \text{if} \qquad k = 1; \tag{3}$$

$$Q_j(\tau_j) = (\tau_j - b_{j1})^{k_j} - \sum_{h=2}^{k_j} b_{jh} (\tau_j - b_{j1})^{k_j - h},\ \tau_j = t - \alpha + i\omega_j$$

$$\text{if } k_j \geqslant 2\ ; \tag{4}$$

$$Q_j(\tau_j) = \tau_j - b_{j1} \qquad \text{if} \qquad k_j = 1\ ; \tag{5}$$

$$S(t) = \sum_{j=0}^{l} c_j t^{l-j}\ ; \tag{6}$$

(This polynomial is constant at the point λ_0).

$R(t) = R_1(t).R_2(t)\ldots, R_s(t)$, where the factors $R_i(t)$ have the form

$$R_j(t) = (t-\gamma_j)^{m_j} \sum_{i=1}^{m_j} + d_i (t-\gamma_i)^{m_j - i}, j = 1,\ldots, s, \tag{7}$$

a_j, b_{jh}, c_j, d_i are differentiable functions of λ in the mentioned neighbourhood while γ_j are real or complex satisfying $Re\gamma_j(\lambda_0) < \alpha$.

Proof. At first we notice that if $m \geqslant 2$ then the polynomial $\tau^n + \sum_{i=1}^{n} x_i \tau^{n-i}$ can be written in the form

$$(\tau - y_1)^n - \sum_{i=2}^{n} y_i (\tau - y_i)^{n-i} \tag{8}$$

where $y_1 = -\frac{x_1}{n}$, $y_2 = -x_2 + C_n^2\left(\frac{x_1}{n}\right)^2$, ..., $y_n = x_n + \ldots$

The mapping $\Gamma: \mathbf{R}^n \to \mathbf{R}^n$, $(x_1,\ldots, x_n) \to (y_1,\ldots, y_n)$ and Γ^{-1} are both polynomial mappings. So Γ is a (local) diffeomorphism. Note that polynomials of the form (8) have $\tau = y_1$ as a root with multiplicity n if and only if $y_2 = y_3 \ldots = = y_n = 0$.

Now, by applying Lemma 2 we obtain Lemma 3.

§ 4. EQUATIONS AND CODIMENSION OF THE STRATA

Let us consider the family

$$x_0 t^n + x_1 t^{n-1} + \ldots + x_n, (x_0,\ldots, x_n) \in \mathbf{R}^{n+1} \setminus \{0\}\times\ldots\times\{0\}\times \mathbf{R}.$$

According to Lemma 3, if $\mathcal{P}(x_0^0, \ldots, x_n^0) \in S_l(k; k_1,\ldots, k_r)$ then the stratum $S_l(k; k_1,\ldots, k_r)$ is defined in a neighbourhood of $(x_0^0,\ldots, x_n^0)$ by the equations

$$a_h = 0,\ h = 2,\ldots, k;$$

$$\mathrm{Re}\, b_{jh} = \mathrm{Im}\, b_{jh} = 0,\ h = 2,\ldots, k_j;\ j = 1, \ldots, r;$$

$$a_1 = \mathrm{Re}\, b_{j1},\ j = 1, \ldots, r;$$

$$c_h = 0,\ h = 0,\ldots, l-1.$$

It follows that the codimension of $S_l(k; k_1,\ldots, k_r)$ is

$$l + k + 2\sum_{j=1}^{r} k_j - r - 1.$$

§ 5. APPLICATION OF THOM TRANSVERSALITY THEOREM AND THE LEMMA OF A FAMILY OF MORSE FUNCTIONS TO FIND THE LOCAL MODELS OF $f(\lambda)$

Let Λ be a differentiable manifold of finite dimension. Then from Thom Transversality Theorem it follows that the set of mappings

$$\Lambda \longrightarrow \mathbf{R}^{n+1} \setminus \{0\} \times \dots \times \{0\} \times \mathbf{R}$$

$$\lambda \longmapsto x_0(\lambda)\, t^n + \dots + x_n(\lambda)$$

transversal to every $S_1\ (k;\ k_1, \dots, k_r)$ forms an everywhere dense set in $C^\infty\,(\Lambda, \mathbf{R}^{n+1} \setminus \{0\} \times \dots \times \{0\} \times \mathbf{R})$ with Whitney C^∞ -topology. Then we say that $x_i\,(\lambda)\ (i = 1, \dots, n)$ are generic.

LEMMA 4. *Let $\mathcal{P}$ be transversal to the stratification under consideration at λ_0. Then there exists a system of local coordinates in Λ centred at λ_0 such that if $\mathcal{P}(\lambda_0) \in S_1(k;\ k_1, \dots, k_r)$ we have*

$$f \circ \mathcal{P}(\lambda) = \alpha + g\,(\lambda) + \max_{1 \leqslant j \leqslant r} (\mu,\ \lambda_{j1} + \nu_j,\ \xi - \alpha - g(\lambda))$$

where $1 + k + 2 \sum_{j=1}^{r} k_j - r - 1$ first coordinates of λ are denoted by $\lambda_2, \dots, \lambda_k$, $\lambda_{j1}, \dots, \lambda_{j2\eta-1}, \lambda_{j2\eta}, \lambda_{01}, \dots, \lambda_{0l}\ (j = 1, \dots, r;\ \eta = 2, \dots, k_j)$,

μ is the greatest real part of all the roots of the polynomial

$$\tau^k - \sum_{i=2}^{k} \lambda_i \tau^{k-i} \text{ if } k \geqslant 2,\ \mu = 0 \text{ if } k = 1,$$

ν_j is the greatest real of all the roots of the polynomial

$$\tau^{k_j} - \sum_{\eta=2}^{k_j} (\lambda_{j2\eta-1} + i\lambda_{j2\eta})\, \tau^{k_j - \eta} \quad \text{if } k_j > 2 \text{ and}$$

$\nu_j = 0$ if $k_j = 1$.

ξ is the greatest real part of all the roots of the polynomial

$$\lambda_{01}\, t^l + \dots + \lambda_{0l}\, t + 1 \qquad \text{if } (\lambda_{01}, \dots, \lambda_{0l}) \neq (0, \dots, 0),$$

$$\xi = \alpha - 1 \qquad \text{if } (\lambda_{01}, \dots, \lambda_{0l}) = (0, \dots, 0)$$

Proof. According to Lemma 3, in a neighbourhood of $\lambda_0 \in \Lambda$ we can write $\mathcal{P}(\lambda)$ as

$$\mathcal{P}(\lambda) = P(\tau) \prod_{j=1}^{r} Q_j(\tau_j) \prod_{j=1}^{r} \overline{Q}_j(\tau_j) .\ R(t).\ S(t)$$

where the expressions $P(\tau)$, $Q_j(\tau_j)$, $R(t)$, $S(t)$ are defined just as in Lemma 3 (see (3) — (7)). So

$$f \circ \mathcal{P}(\lambda) = \max_{1 \leqslant j \leqslant r} (f \circ P, f \circ Q_j, f \circ S).$$

According to (3) — (6) we have

$$f \circ P = \alpha + a_1 + \mu(a_2, \dots, a_k) \qquad \text{where } \mu(a_2, \dots, a_k)$$

equals zero if $k = 1$ and equals the greatest real part of all the roots of the equation $P(\tau) = 0$ if $k \geqslant 2$,

$$f \circ Q_j = \alpha + Re\, b_{j1} + \nu_j (Re\, b_{j2}, Im\, b_{j2}, \dots, Re\, b_{jkj}, Im\, b_{jkj}),$$

where $\nu_j(\dots)$ equals zero if $k_j = 1$ and equals the greatest real part of all the roots of the equation $Q_f(\tau_j) = 0$ if $k \geqslant 2$; $f \circ S = \xi(C_0, \dots, C_l)$ where $\xi(\dots)$ is the greatest real part of all the roots of the equation $\sum_{j=1}^{l} c_j t^{l-j} = 0$ $(C_0 \neq 0)$ if $(C_1, \dots, C_1) \neq (0, \dots, 0)$ and $\xi = \alpha - 1$ if $(C_1, \dots, C_1) = (0, \dots, 0)$. Notice that as $C_0(\lambda)$ is different from zero in the neighbourhood under consideration we can put $C_0(\lambda) = 1$.

On the other hand, the condition of transversality of $\mathcal{P}$ to $S_1(k; k_1, \dots, k_r)$ at λ_o is equivalent to the fact that $\varphi \circ \mathcal{P}$ is a submersion in a neighbourhood U of λ_o where

$$\varphi : \mathbf{R}^{n+1} \setminus \{0\} \times \dots \times \{0\} \times \mathbf{R} \supset U(\mathcal{P}(\lambda_0)) \to \mathbf{R}^{l+k+2 \sum_{i=1}^{r} k_i - r - 1},$$

$$x \longmapsto \begin{cases} a_\nu, \ \nu = 2, \dots, k. \\ Re\, b_{jh}, Im\, b_{jh}, \ h = 2, \dots k_j; \ j = 1, \dots, r \\ Re\, b_{j1} - a_1 \\ c_h, \ h = 1, \dots, l \end{cases}$$

Note that in this case

$$U(\mathcal{P}(\lambda_0)) \cap S_l(k; k_1, \dots, k_r) = \varphi^{-1}(0).$$

As

$$\varphi_0 \mathcal{P} : A \supset U(\lambda_0) \longrightarrow \mathbf{R}^{l+k+2 \sum k_i - r - 1}$$

is a submersion at $\lambda_0 \in A$, there exists a system of coordinates centred at λ_0 whose $1 + k + 2 \sum_{i=1}^{r} k_i - r - 1$ first coordinates are $\lambda_2, \dots, \lambda_k, \lambda_{j1}, \lambda_{j2\eta-1}$ $\lambda_{j2\eta}, \lambda_{o1}, \dots, \lambda_{ol}$ $(j = 1, \dots, r; \eta = 2, \dots, k_j)$, such that in this system $\varphi \circ \mathcal{P}$ becomes a canonical projection onto $1 + k + 2 \sum_{i=1}^{r} k_i - r - 1$ first coordinates

$$a_\nu = \lambda_\nu \qquad \nu = 2, \dots, k;$$
$$Re\ b_{j1} - a_1 = \lambda_{j1};$$
$$\left.\begin{array}{l} Re\ b_{j\eta} = \lambda_{j2\eta} - 1 \\ Im\ b_{j\eta} = \lambda_{j2\eta} \end{array}\right\} j = 1, \dots, r;\ \eta = 2, \dots, k_j;$$
$$c_{l-h} = \lambda_{0h}, \qquad h = 0, \dots, l-1.$$

According to Lemma 2 $a_1 = \mathrm{Re}\ b_{j1} = g(\lambda)$ where g is a differentiable function, $g(0) = 0$ and we can write

$$f_0 P = \max_{1 \leqslant j \leqslant r} (f_0 P, f_0 Q_j, f_0 S)$$
$$= \alpha + g(\lambda) + \max_{1 \leqslant j \leqslant r} (\mu, \lambda_{j1} + \nu_j, \xi - \alpha - g(\lambda))$$

where μ, ν_j, ξ are defined as in Lemma 4.

Remark 1. From the conditions of transversality it follows that if $\mathcal{P}$ is transversal to the stratification under consideration then $\mathcal{P}(\lambda)$ belongs to only $S_1(k; k_1, \dots, k_r)$ where

$$S = l + k + 2 \sum_{i=1}^{r} k_i - r - 1 \leqslant \dim \Lambda.$$

2. We can give $g(\lambda)$ a simpler form by using the following lemma. Let $m = \dim \Lambda - S$.

LEMMA 5. (on a family of Morse functions). *There exists an open everywhere dense subset of functions $g \in C^\infty(\mathbf{R}^m \times \mathbf{R}^s, \mathbf{R})$ such that in a neighbourhood of $(0,0) \in \mathbf{R}^m \times \mathbf{R}^s$ there is a change of coordinates preserving y: $(x', y) \to (x, y)$ that reduces g to one of the following forms:*

1. $g(x,y) = const + x_1$;

2. $g(x,y) = const + h(y) + \sum_{i=1}^{m} \varepsilon_i x_i^2$, $\varepsilon_i = \pm 1$;

where $h(y)$ is differentiable $h(0) = 0$, $(Dh)_0 \neq 0$, $(x,y) = (x_1, \dots, x_m, y_1, \dots, y_s)$.

The proof of this lemma is analogous to that of Morse's lemma [3].

THEOREM. *Let us consider the following family of differential equations*

$$a_0(\lambda) y^{(n)} + a_1(\lambda) y^{(n-1)} + \dots + a_n(\lambda) y = 0, \ \lambda \in \Lambda.$$

If the coefficients $a_i(\lambda)$ are generic, then for every $\lambda_0 \in \Lambda$ there exists a local system of coordinates $\lambda = (x,y)$ centred at λ_0 such that $f(\lambda)$ has one of the following normal forms:

(I) $\alpha + x_1 + \max(\mu, y_{11} + \nu_1, \dots, y_{r1} + \nu_r, \xi - \alpha - x_1)$.

$$(II)\quad \alpha + \sum_{i=1}^{m} \varepsilon_i x_i^2 + h(y) + max\,(\mu, y_{11} + \nu_1, \dots, y_{r1} + \nu_r, \xi - \alpha - h(y) - \sum_{i=1}^{m} \varepsilon_i x_i^2)\,.$$

$$(III)\ \alpha + h(y) + max\,(\mu, y_{11} + \nu_1, \dots, y_{r1} + \nu_r, \xi - \alpha - h(y))$$

where $a_0(\lambda_0)t^n + \dots + a_n(\lambda_0) \in S_l(k; k_1, \dots, k_r)$, μ, ν_j, ξ *are defined as in Lemma 4, the coordinates* (x,y) *and* $h(y)$ *are chosen as in Lemma 5.*

LIST OF LOCAL MODELS FOR THE CASE DIM $\Lambda = 1$

Co-dim	*Strata*	*Local models of* $f(\lambda)$
0	$S_0(1)$, $S_0(0;1)$	$\alpha + \lambda$, $\alpha \pm \lambda^2$
1	$S_0(2)$	$\alpha + g(\lambda) + Re\sqrt{\lambda}$
1	$S_0(1;1)$, $S_0(0;1,1)$	$\alpha + g(\lambda) + \lvert\lambda\rvert$
1	$S_1(1)$, $S_1(0;1)$	$\begin{cases} \max\ (\alpha + g(\lambda),\ 1/\lambda) & \text{if } \lambda > 0 \\ \alpha + g(\lambda) & \text{if } \lambda \leqslant 0 \end{cases}$
		where $g(\lambda)$ is a differentiable function, $g(0) = 0$

LIST OF LOCAL MODELS FOR THE CASE DIM $\Lambda = 2$

Co-dim	*Strata*	*Local models of* $f(\lambda)$
0	$S_0(1)$, $S_0(0,1)$	$\alpha + x$, $\alpha \pm x^2 \pm y^2$
1	$S_0(2)$	$\alpha + x + \begin{cases} \sqrt{y} & \text{if } y \geqslant 0 \\ 0 & \text{if } y < 0 \end{cases}$ $\alpha \pm x^2 + h(y) + \begin{cases} \sqrt{y} & \text{if } y \geqslant 0 \\ 0 & \text{if } y < 0 \end{cases}$
1 1	$S_0(1;1)$ $S_0(0;1,1)$	$\alpha + x + y + \lvert y\rvert$ $\alpha \pm x^2 + \lvert y\rvert + h(y)$
1 1	$S_1(1)$ $S_1(0;1)$	$\begin{cases} 1/y & \text{if } y > 0 \\ \alpha + x & \text{if } y \leqslant 0 \end{cases}$ $\begin{cases} 1/y & \text{if } y > 0 \\ \alpha \pm x^2 + h(y) & \text{if } y \leqslant 0 \end{cases}$
2	$S_0(3)$	$\alpha + g(x,y) + \mu(x,y)$

Co-dim	Strata	Local models of $f(\lambda)$
2	$S_0(2;1)$	$\begin{cases} \alpha + g(x,y) + \max(\sqrt{x}, y) \text{ if } x \geqslant 0 \\ \alpha + x + \lvert y \rvert \text{ or } \alpha \pm x^2 + \varphi(y) + \lvert y \rvert \text{ if } x < 0 \end{cases}$
2	$S_0(1;1,1)$, $S_0(0;1,1,1)$	$\max(0,x,y) + \alpha + g(x,y)$
2	$S_0(0;2)$	$\alpha + g(x,y) + \lvert Re\sqrt{x+iy} \rvert$
2	$S_1(2)$	$\begin{cases} 1/y & \text{if } y > 0 \\ \alpha + g(x,0) + \begin{cases} \sqrt{x} & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases} & \text{if } y = 0 \\ \alpha + y + \sqrt{x} \text{ or } \alpha \pm y^2 + \sqrt{x} & \text{if } x \geqslant 0, y < 0 \\ \alpha + x \quad \alpha \pm x^2 \pm y^2 & \text{if } x < 0, y < 0 \end{cases}$
2 2	$S_1(1;1)$ $S_1(0;1,1)$	$\begin{cases} 1/y & \text{if } y > 0 \\ \alpha + g(x,y) + \lvert x \rvert & \text{if } y \leqslant 0 \end{cases}$
2	$S_2(1)$, $S_2(0;1)$	$\max(\alpha + g(x,y), \xi(x,y))$
		where $h(y)$, $\varphi(y)$, $g(x,y)$ are differentiable functions, $\mu(x,y)$ and $\xi(x,y)$ are the greatest real part of all the roots of the polynomials $\tau^3 - x\tau - y$, $xt^2 + yt + 1$ respectively, $h(0) = \varphi(0) = g(0,0) = 0$

LIST OF LOCAL MODELS FOR THE CASE DIM $\Lambda = 3$

codim	Strata	Local models of $f(\lambda)$
0	$S_0(1)$, $S_0(0;1)$	$\alpha + x$, $\alpha \pm x^2 \pm y^2 \pm z^2$
1	$S_0(2)$	$\alpha + y + \begin{cases} \sqrt{z} & \text{if } z \geqslant 0 \\ 0 & \text{if } z < 0 \end{cases}$ $\alpha \pm x^2 \pm y^2 + h(z) + \begin{cases} \sqrt{z} & \text{if } z \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases}$
1 1	$S_0(1;1)$ $S_0(0;1,1)$	$\alpha + y + z + \lvert z \rvert$ $\alpha \pm x^2 \pm y^2 + h(x) + \lvert z \rvert$
1 1	$S_1(1)$ $S_1(0;1)$	$\begin{cases} 1/z & \text{if } z > 0 \\ \alpha + x & \text{if } z \leqslant 0 \end{cases}$ $\begin{cases} 1/z & \text{if } z > 0 \\ \alpha \pm x^2 \pm y^2 + h(z) & \text{if } z \leqslant 0 \end{cases}$
2	$S_0(3)$	$\alpha + x + \mu(y,z)$ $\alpha \pm x^2 + k(y,z) + \mu(y,z)$
2	$S_0(2;1)$	$\begin{cases} \alpha + g(x,y,z) + \max(\sqrt{y}, z) \text{ if } y \geqslant 0 \\ \alpha + x + \lvert z \rvert \text{ or } \alpha \pm x^2 \pm y^2 + h(z) + \lvert z \rvert \text{ if } y < 0 \end{cases}$

codim	Strata	Local models of $f(\lambda)$
2 2	$S_0(1;1,1)$ $S_0(0;1,1,1)$	$\alpha + x + \max(0, y, z)$ $\alpha \pm x^2 + k(y, z) + \max(0, y, z)$
2	$S_0(0;2)$	$\alpha + x + \lvert Re\sqrt{y+ix}\rvert$ $\alpha \pm x^2 + k(y, z) + \lvert Re\sqrt{y+iz}\rvert$
2	$S_1(2)$	$\begin{cases} 1/z & \text{if } z > 0 \\ z + y + \sqrt{x} \text{ or } \alpha \pm y^2 \pm z^2 + \sqrt{x} & \text{if } z \leqslant 0,\ x \geqslant 0 \\ z \pm x^2 \pm y^2 \pm z^2 \text{ or } \alpha + x & \text{if } z \leqslant 0,\ x < 0 \end{cases}$
2 2	$S_1(1;1)$ $S_1(0;1,1)$	$\begin{cases} 1/z & \text{if } z > 0 \\ \alpha + x + \lvert y\rvert \text{ or } \alpha \pm x^2 + k(y,z) + \lvert y\rvert & \text{if } z \leqslant 0 \end{cases}$
2	$S_2(1)$, $S_2(0;1)$	$\max(\alpha + g(x, y, z), \xi(y, z))$
3	$S_0(4)$	$\alpha + g(x, y, z) + \nu(x, y, z)$
3	$S_0(3;1)$	$\alpha + g(x, y, z) + \max(\mu(x, y), z)$
3	$S_0(2;1,1)$	$\alpha + g(x, y, z) + \begin{cases} \max(\sqrt{x}, y, z) & \text{if } x \geqslant 0 \\ \max(0, y, z) & \text{if } x < 0 \end{cases}$
3	$S_0(1;1,1,1)$, $S_0(0;1,1,1,1)$	$\alpha + g(x, y, z) + \max(0, x, y, z)$
3	$S_0(1;2)$, $S_0(0;1,2)$	$\alpha + g(x, y, z) + \max(\lvert Re\sqrt{x+iy}\rvert, z)$
3	$S_1(3)$	$\begin{cases} \max(1/x, \alpha + g(x, y, z) + \mu(y, z)) \text{ if } x \neq 0 \\ \alpha + g(0, y, z) + \mu(y, z) \text{ if } x = 0 \end{cases}$
3	$S_1(2;1)$	$\alpha + g(x, y, z) + \begin{cases} \max(\sqrt{y}, z, 1/x - \alpha - g(x,y,z)) \text{ if } y \geqslant 0 \\ \max(0, x, 1/x - \alpha - g(x,y,z)) \text{ if } y < 0 \end{cases}$
3	$S_1(1;1,1)$, $S_1(0;1,1,1)$	$\begin{cases} \alpha + g(x,y,z) + \max(0,y,z,1/x - \alpha - g(x,y,x)) \text{ if } x \neq 0 \\ \alpha + g(0,y,z) + \max(0,y,z) \text{ if } x = 0 \end{cases}$
3	$S_1(0;2)$	$\begin{cases} 1/z & \text{if } z > 0 \\ \alpha + g(x, y, z) + \lvert Re\sqrt{x+iy}\rvert & \text{if } z \leqslant 0 \end{cases}$
3	$S_2(2)$	$\alpha + g(x,y,z) + \begin{cases} \max(\sqrt{x}, \xi(y,z) - \alpha - g(x,y,z)) \text{ if } x \geqslant 0 \\ \max(0, \xi(y,z) - \alpha - g(x,y,z)) \text{ if } x < 0 \end{cases}$
3	$S_2(1;1)$, $S_2(0;1,1)$	$\alpha + g(x, y, z) + \max(0, x, \xi(y, z) - \alpha - g(x, y, z))$
3	$S_3(1)$, $S_3(0;1)$	$\alpha + g(x, y, z) + \max(0, \eta(x, y, z) - \alpha - g(x, y, z))$
		where $h(x)$, $k(y, z)$, $g(x, y, z)$ are differentiable functions, $h(0) = k(0,0) = g(0,0,0) = 0$ $(Dh)_0 \neq 0$ $(Dk)_{(0,0)} \neq 0$, $\frac{\partial g}{\partial x}(0,0,0) \neq 0$, $\mu(y, z)$, $\nu(x, y, z)$. $\xi(y, z)$, $\eta(x, y, z)$ are the greatest real part of all the roots of the polynomials $t^3 - yt - z$, $t^4 - xt^2 - yt - z$, $yt^2 + zt + 1$, $xt^3 + yt^2 + zt + 1$, respectively.

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