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FRACTIONAL Q-INTEGRATION AND INTEGRAL REPRESENTATIONS OF «BIBASIC» DOUBLE HYPERGEOMETRIC SERIES OF HIGHER ORDER

MUMTAZ AHMAD KHAN and ABDUL HAKIM KHAN

1. INTRODUCTION

W.A. Al-Salam [2] and R.P. Agarwal [1] have defined certain fractional q-integral operators and given their fundamental properties. The operation of fractional q-differentiation is defined by means of the correspondence

$$D_q^{\alpha} f(x) = (1-q)^{\alpha-1} \Pi_p \begin{bmatrix} -\alpha ; q \\ 1 \end{bmatrix} \int_0^x [x-tq]-\alpha-1\, f(t)d(t;q), \quad (1)$$

where the basic integral is defined through the relation (W. Hahn [31])

$$\int_0^x f(t)\, d(t;\, q) = x(t-q) \sum_{k=0}^{\infty} q^k f(xq^k).$$

Using the series definition for the basic integral, (1.1) may be written as

$$D_p^{\alpha} f(x) = (1-q)^{-\alpha} x^{-\alpha} \sum_{j=0}^{\infty} \frac{[q^{-\alpha}]_j\, q^j}{[q]_j} f(xq^j), \quad (2)$$

so that

$$D_p^{\alpha} x^{\mu-1} = (1-q)^{-\alpha} \begin{bmatrix} \mu-\alpha ; q \\ \mu \end{bmatrix} x^{\mu-\alpha-1} \quad (3)$$

In 1972, Manjari Upadhyay [7] obtained integral representations of basic double hypergeometric series of higher order with the help of (1.3).

The object of this paper is to deduce some integral representations for «bibasic» double hypergeometric series and to evaluate ceratain integrals involving basic hypergeometric series. A number of interesting special cases of the main results derived in the paper have been given.

The following usual notation is used throughout the paper;
Let for $|q| < 1$,

$$[a]_r = (1-a)(1-aq)\dots(1-aq^{r-1})\ ;\ [a]_0 = 1\ ;$$

$$[a]_{-r} = \frac{(-)^r\, q^{r(r+1)/2}\, a^{-r}}{[q/a]_r}.$$

The generalized 'bibasic' hypergeometric function is then defined as

$$\underset{A+B}{\Phi}{}_{C+D}\left[\begin{matrix} q^{(a)} : q_1{}^{(b)};\ x \\ q^{(c)} : q_1{}^{(d)};\ q^{\lambda} q_1^{\lambda_1} \end{matrix}\right] \equiv \underset{A+B}{\Phi}{}_{C+D}\left[\begin{matrix} (a) : (b);\ x \\ (c) : (d);\ q^{\lambda} q_1^{\lambda_1} \end{matrix}\right]$$

$$= \sum_{n=o}^{\infty} \frac{[q^{(a)}]_n [q_1{}^{(b)}]_n}{[q^{(c)}]_n [q_1{}^{(d)}]_n}\, x^n\, q^{(1/2)\lambda n(n+1)}\, q_1{}^{(1/2)\lambda_1 n(n+1)},$$

where $\lambda, \lambda_1 > 0$, $|q|<1$, $|q_1|<1$ and for $\lambda = 0 = \lambda_1$, $|x|<1$.

In the numerator and the denominator the terms before the colon are on the base q and those after it are on the base q_1. As usual (a_N) stands for the sequence of N parameters $a_1, a_2, \dots, a_N$. When $N = A$, we shall simply write (a) instead of (a_A). A 'bibasic' double hypergeometric series is defined as

$$\Phi\left[\begin{matrix} x \\ \\ y \end{matrix}\left|\begin{matrix} (a) \qquad (g) \\ (b);\ (c) : (h);\ (i) \\ (d) \qquad (j) \\ (e);\ (f) : (k);\ (1) \end{matrix}\right.\right]$$

$$= \sum_{m=o}^{\infty}\sum_{n=o}^{\infty} \frac{[q^{(a)}]_{m+n}[q^{(b)}]_m [q^{(c)}]_n [q_1{}^{(g)}]_{m+n} [q_1{}^{(h)}]_m [q_1{}^{(i)}]_n x^m y^n}{[q^{(d)}]_{m+n}[q^{(e)}]_m [q^{(f)}]_n [q_1{}^{(j)}]_{m+n} [q_1{}^{(k)}]_m [q_1{}^{(l)}]_n}.$$

The following basic functions shall aslo be used:

$$e_q(x) = \left\{\prod_{n=o}^{\infty}(1-xq^n)\right\}^{-1} = \sum_{r=o}^{\infty}\frac{x^r}{[q]_r}\ ;\ |x|<1,$$

$$[x-y]_\alpha = x^\alpha \prod_{n=o}^{\infty}\left[\frac{(1-yq^n/x)}{(1-yq^{n+\alpha}/x)}\right].$$

We further denote by

$$\Pi\left[\begin{matrix} x^{(a_r)} \\ x^{(b_s)} \end{matrix};\ x^y\right] \text{ or simply } \Pi_x\left[\begin{matrix}(q_r);\ x^y \\ (b_s)\end{matrix}\right]$$

the product $\displaystyle\prod_{u=o}^{\infty}\left[\frac{(1-x^{(a_r)+yu})}{(1-x^{(b_s)+yu})}\right]$

Lastly, we shall use the notation

$$D_q^{a,b}[f(x)] \text{ to denote } D_q^{b}[x^{a-1} f(x)].$$

3. In this section we shall give three basic integral representations for a 'bibasic' double hypergeometric series in the following forms:

$$\Phi\left[\begin{array}{c|ccc} x & \begin{array}{c}(a)\\(b);\ (c)\end{array} & : & \begin{array}{c}(g)\\(h);\ (i)\end{array} \\ y & \begin{array}{c}(d)\\(e);\ (f)\end{array} & : & \begin{array}{c}(j)\\(k);\ (L)\end{array}\end{array}\right]$$

$$= \frac{1}{(1-q)(1-q_1)} \Pi_q \left[\begin{array}{c} b'-e',\ e';\ q \\ b',\ 1\end{array}\right] \Pi_{q_1}\left[\begin{array}{c} i'-L',\ L';\ q_1 \\ i',\ 1\end{array}\right] \times$$

$$\times \int_0^1\int_0^1 u^{e'-1} v^{L'-1} [1-uq]_{b'-e'-1} [1-vq_1]_{i'-L'-1} \times$$

$$\times \Phi\left(\begin{array}{c|c} xu & \begin{array}{c}(a)\qquad\qquad (g)\\(b),\ b';\ (c)\ :(h);\ (i),\ i'\end{array} \\ yv & \begin{array}{c}(d)\qquad\qquad (j)\\(e),\ e';\ (f)\ :(k):(L),\ L'\end{array}\end{array}\right] d(u;q)\, d(v;q_1) \qquad (1)$$

provided that $Rl\,(e') > ,0\ (Rl)\,(L') > 0$, $|x| < 1$, $|y| < 1$.

$$\Phi\left[\begin{array}{c|ccc} xy & \begin{array}{c}(a)\\(b);\ (c)\end{array} & : & \begin{array}{c}(g)\\(h);\ (i)\end{array} \\ x & \begin{array}{c}(d)\\(e);\ (f)\end{array} & ; & \begin{array}{c}(j)\\(k);\ (L)\end{array}\end{array}\right]$$

$$= \frac{1}{(1-q)(1-q_1)} \Pi_q \left[\begin{array}{c} d',\ a'-d';\ q \\ a',\ 1\end{array}\right] \Pi_{q_1}\left[\begin{array}{c} k',\ h'-k';\ q_1 \\ h',\ 1\end{array}\right] \times$$

$$\times \int_0^1\int_0^1 u^{d'-1} v^{k'-1} [1-uq]_{a'-d'-1} [1-vq_1]_{h'-k'-1} \times$$

$$\times \Phi\left[\begin{array}{c|c} uvxy & \begin{array}{l}(a),\ a'\ :\ (g)\\(b);\ (c)\ :\ (h),\ h';\ (i)\end{array} \\ ux & \begin{array}{l}(d),\ d'\ :\qquad (j)\\(e);\ (f):\ (k),\ k\ ;\ (L)\end{array}\end{array}\right] d(u;q) d(v;q_1), \qquad (2)$$

provided $Rl\,(d') > 0$, $Rl\,(k') > 0$ $|x| < 1$, $|y| < 1$,

$$\Phi\left[\begin{array}{c|c} xuv & \begin{array}{l}(a),\ a':(g),\ g'\\(b);(c):(h);(i)\end{array} \\ yuv & \begin{array}{l}(d),\ d':(j),\ j'\\(e);(f):(k),(l)\end{array}\end{array}\right]$$

$$= \frac{1}{(1-q)(1-q_1)} \Pi_q \left[\begin{array}{c} a'\ d'-a';\ q \\ d',\ 1\end{array}\right] \Pi_{q_1}\left[\begin{array}{c} g',\ j'-g';\ q_1 \\ j',\ 1\end{array}\right] \times$$

$$\times \int_0^1\int_0^1 t^{a'-1} Z^{g'-1} [1-tq]_{d'-a'-1} [1-Zq_1]_{j'-g'-1} \times$$

$$\times \Phi \left[\begin{array}{c|c} xuvlz & \begin{array}{c} (a) \quad : \quad (g) \\ (b);(c):(h);(i) \end{array} \\ yuvlz & \begin{array}{c} (d) \quad : \quad (j) \\ (e);(f):(k);(l) \end{array} \end{array} \right] d(l;\, q)\; d\,(z;\, q_1). \tag{3}$$

provided $Rl\,(d') > 0$, $Rl(j') > 0$, $|\, xuv \,| < 1$, $|\, yuv \,| < 1$.

It may be remarked that (3.1 — 3) can be regarded as extensions of Manjari Upadhyay's results ([7], (3.1), (3.2) and (3.3)).

To prove (3.1), we have, on using (1.3)

$$D_{q,\ x}^{e'-b',\ e'}\ D_{q_1,\ y}^{L'-i',\ L'}\ \Phi \left[\begin{array}{c|c} x & \begin{array}{c} (a) \qquad (g) \\ (b),\, b';(c):(h);(i),\, i' \end{array} \\ y & \begin{array}{c} (d) \qquad (j) \\ (e),\, e',\ (f):(k);(e),\ L' \end{array} \end{array} \right]$$

$$= D_{q,x}^{e'-b'}\ D_{q_1,y}^{L'-i'} \left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{[q^{(a)}]_{m+n}[q^{(b)}]_m[q^{b'}]_m[q^{(c)}]_n}{[q^{(d)}]_{m+n}[q^{(e)}]_m[q^{e'}]_m[q^{(f)}]_n} \times \right.$$

$$\left. \times \frac{[q_1^{(g)}]_{m+n}[q_1^{(h)}]_m[q_1^{(i)}]_n[q_1^{i'}]_n\, x^{m+e-1}\ y^{n+L'-1}}{[q_1^{(j)}]_{m+n}[q_1^{(k)}]_m[q_1^{(L)}]_n[q_1^{L}]_n} \right]$$

$$= \frac{x^{b'-1}}{(1-q)^{e'-b'}}\ \frac{y^{i'-1}}{(1-q)^{L'-i'}}\, \Pi_q \left[\begin{array}{c} b';\, q \\ e' \end{array} \right] \Pi_{q_1} \left[\begin{array}{c} i';\, q_1 \\ L' \end{array} \right] \Phi \left[\begin{array}{c|c} x & \begin{array}{c} (a) \qquad (g) \\ (b);(c);(h);(i) \end{array} \\ y & \begin{array}{c} (d) \qquad (j) \\ (e);(f):(k);(l) \end{array} \end{array} \right]$$

Hence by the definition (1. 1) of the operators $D_{q,\ x}^{e'-b'}\ D_{q_1,\ y}^{L'-i'}$,

$$x^{b'-1}\ y^{i'-1}\ \Phi \left\{ \begin{array}{c|c} x & \begin{array}{c} (a) \qquad (g) \\ (b)\ ;\ (c):(h)\ ;\ (i) \end{array} \\ y & \begin{array}{c} (d) \quad : \quad (j) \\ (e)\ ;\ (f):(k):(L) \end{array} \end{array} \right\}$$

$$= \frac{1}{(1-q)\,(1-q_1)}\ \Pi_q \left[\begin{array}{c} b'-e',\ e';\, q \\ b',\quad l \end{array} \right] \Pi_{q_1} \left[\begin{array}{c} i'-L',\ L';\ q_1 \\ i',\quad l \end{array} \right] \times$$

$$\times \int_0^x \int_0^y u^{e'-1}\, v^{L'-1}\, [x-uq]_{b'-e'-1}\ [y-vq_1]_{i'-L'-1} \times$$

$$\times\ \Phi \left[\begin{array}{c|c} x & \begin{array}{c} (a) \quad : \quad (g) \\ (b),\ b';\ (c):(h);\ (i),\ i' \end{array} \\ y & \begin{array}{c} (d) \quad : \quad (j) \\ (e),\ (e')\ (f):(k);(L),\ L' \end{array} \end{array} \right] d(u;\, q)\ d(v;\, q_1)$$

for $Rl(e') > 0$, $Rl(L') > 0$. Transforming the variables through $u \to ux$ and $v \to vy$, (3. 1) follows.

To prove (3. 2 and 3), we start with

$$D_{q,x}^{d'-a',d'} D_{q_1,y}^{k'-h',k'} \Phi\left[\begin{array}{c|c} xy & (a), a' : (g) \\ & (b); (c) : (h), h'; (i) \\ x & (d); d' : (j) \\ & (e); (f) : (k), k'; (l) \end{array}\right]$$

and $$D_{q,u}^{a'-d',a'} D_{q_1,v}^{g'-j',g'} \Phi\left[\begin{array}{c|c} xyv & (a) : (g) \\ & (b); (c) : (h); (i) \\ yuv & (d) : (j) \\ & (e); (f) : (k); (L) \end{array}\right]$$

respectively and follow the same procedure of (2. 1).

4. In this section some of the interesting special cases of (3. 1 — 3) are deduced. A number of integral representations for basic Appell functions $\Phi^{(1)}$, $\Phi^{(2)}$, $\Phi^{(3)}$, $\Phi^{(4)}$ can be easily obtained as special cases of the main integrals in section 3. We give here only some typical results.

$$\int_0^1\int_0^1 u^{e'-1} v^{f'-1} [1-uq]_{b'-e'-1} [1-vq]_{c'-f'-1} \times$$

$$\times \Phi \left[\begin{array}{c|c} uq^{d-a-b-c} & b, b'^{a}; c, c' \\ vq^{d-a-c} & e'^{d}; f' \end{array}\right] d(u; q)\ d(v; q)$$

$$= (1-q)^2 \Pi_q \left[\begin{array}{l} d-a, d-b-c, b', c', 1, 1; q \\ d, d-a-b-c, b'-e', c'-f', e', f' \end{array}\right], \qquad (1)$$

(in (3.1) take $q_1 = q$, $A = B = C = D = E = F = 1$, $e = f = 1$,

$G = H = I = J = K = L = 0$, $i' = c'$, $L' = f'$; $x = q^{d-a-b-c}$, $y = q^{d-a-c}$).

$$\int_0^1\int_0^1 u^{d'-1} v^{e'-1} [1-uq]_{a'-d'-1} [1-vq]_{b'-e'-1} \times$$

$$\times \Phi \left(\begin{array}{c|c} uvq^{d-a-b-c} & a, a' \\ & b, b; c \\ vq^{d-a-c} & d, d \\ & e'; \end{array}\right) d(u; q)\ d(v; q)$$

$$= (1-q)^2 \Pi_q \left[\begin{array}{l} d-a, d-b-c, a', b', 1, 1; q \\ d, d-a-b-c, a'-d', b'-e', e', q' \end{array}\right], \qquad (2)$$

(in (3.2) take $q_1 = q$, $A = B = C = D = E = F = 1$, $e = f = 1$

$G = H = I = J = K = L = 0$, $h' = b'$, $k' = e$; $x = q^{d-a-c}$, $y = q^{-b}$)

$$\Phi^{(1)}[a'; b, c; b'; xu, yu] = \frac{1}{(1-q)} \Pi_q \left[\begin{array}{c} b'-a', a'; q \\ b', 1 \end{array}\right] \times$$

$$\times \int_0^1 t^{a'-1} [1-t-p]_{b'-a'-1} [1-xutq^b]_{-b} [1-yutq^c]_{-c} d(t; q), \qquad (3)$$

(in (3.3) take $v = 1$, $q_1 = 0$, $A = D = 0$, $E = F = B = C = 1$, $e = f = 1$, $d' = b'$). ((4.3) was given earlier by F. H. Jackson [5] also),

$$\int_0^1 t^{a'-1} [1-tq]_{b'-a'-1} \, {}_2\Phi_1\left[\begin{matrix} c, -N; tq \\ 1+a'+c-b'-N \end{matrix}\right]$$

$$\times {}_B\Phi_E\left[\begin{matrix} (b); xut \\ (e) \end{matrix}\right] d(t;q) = (1-q) \frac{[q^{b'-c}]_N}{[q^{b'-a'-c}]_N} \times$$

$$\times \Pi_q \left[\begin{matrix} b'+N, 1; q \\ b'-a'+N, a' \end{matrix}\right] {}_{B+2}\Phi_{E+2}\left[\begin{matrix} (b), a', b'-c+N; xu \\ (e), b'-c', b'+N \end{matrix}\right], \tag{4}$$

(in (3.3) take $v = 1$, $q_1 = 0$, $(e) = (e_{E+1})$ where $e_{E+1} = 1$, $A = D = 0$, $C = 2$, $F = 2$;, $C_1 = C$, $C_2 = -N$, $f_1 = 1$, $f_2 = 1 + a' + c - b' - N$, $yu = q$, N, a positive integer and use basic analogue of Saalschutz theorem (cf. [6; p. 97])).

$$\int_0^1 t^{a'-1} [1-tq]_{b'-a'-1} \, {}_2\Phi_1\left[\begin{matrix} c, -N; tq \\ 1+a'+c-b'-N \end{matrix}\right] \times$$

$$\times {}_2\Phi_1\left[\begin{matrix} b, b'-c; tq^{c-b} \\ a' \end{matrix}\right] d(t;q)$$

$$= (1-q) \Pi_q \left[\begin{matrix} b'-b+N, c, 1; q \\ b'-a'+N, a', c-b \end{matrix}\right] \frac{[q^{b'-c}]_N}{[q^{b'-a'-c}]_N}, \tag{5}$$

(in (4.4) take $B = 2$, $E = 1$; $b_1 = b' - c$, $b_2 = b$, $e_1 = a'$, $xu = q^{c-b}$).

$$\int_0^1 t^{a'-1} [1-tq]_{b'-a'-1} [1+tq]_{b'-a'-1} \, {}_B\Phi_E\left[\begin{matrix} (b); xut \\ (e) \end{matrix}\right] d(t;q)$$

$$= (1-q) \Pi_q \left[\begin{matrix} 2; q^2 \\ 2(b'-a') \end{matrix}\right] \Big\{ \Pi_q \left[\begin{matrix} 2b'-a'; q^2 \\ a' \end{matrix}\right] \times$$

$$\times \Phi^{(q^2)}\left[\begin{matrix} a', \frac{(b)}{2}, \frac{(b)+1}{2}; (xu)^2 \\ 2b'-a', \frac{(e)}{2}, \frac{(e)+1}{2}, \frac{1}{2} \end{matrix}\right] + \frac{\prod_{r=1}^{B} (1-q^{b_r}) \, xu}{\prod_{r=1}^{E} (1-q^{e_r})(1-q)} \times$$

$$\times \Pi_q \left[\begin{matrix} 1+2b'-a'; q^2 \\ 1+a' \end{matrix}\right] \times \Phi^{(q^2)}\left[\begin{matrix} a'+1, \frac{1+(b)}{2}+1, (xu)^2 \\ \frac{(e)+1}{2}, \frac{(e)}{2}+1, \frac{3}{2}, 1+2b'-a' \end{matrix}\right]\Big\} \tag{6}$$

(in (3.3) take $v = 1$, $q_1 = 0$, $A = D = 0$, $F = C = 1$, $f = 1$, $(e) = (e_{E+1})$; $e_{E+1} = 1$, $C_1 = 1 + a' - b'$, $yu = -q^{b'-a'}$ and use q-analogue of Kumer's theorem [6]).

In the above cases $\Phi^{(1)}$, $\Phi^{(2)}$, $\Phi^{(3)}$ and $\Phi^{(4)}$ are the basic analogues of Appell's hypergeometric functions defined by F.H. Jackson [4].

REFERENCES

[1] R.P. Agarwal, *Certain Fractional q-integrals and q-derivatives*. Proc. Cambridge Philos. Soc. 66 (1969), 365–70.

[2] W.A. Al-Salam, *Some Fractional q-integrals and q-derivatives*, Proc. Edinburgh Math. Soc., 15 (1966), 135–40.

[3] W. Hahn, *Uber die hoheren Heineschen Reihen undeine einheitliche Theorie der Sogenannten speziellen Funktionen*, Math. Nachr., 3 (1950), 25–94.

[4] F.H. Jackson, *On basic double hypergeometric functions*, Quart. J. Math. (Oxford), 13, (1942), 69–82.

[5] L.J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, 1969.

[6] D.B. Sears, *Transformations of basic hypergeometric functions of special type*, Proc. London Math. Soc. 52 (1951), 467–483.

[7] M. Upadhyay, *Fractional q-integration and integral representation of basic double hypergeometric series of higher order*, Ganita, 23 (1972), 19–29.

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MATHEMATICS SECTION Z.H. COLLEGE OF ENGG. AND TECH.,
A.M.U, ALIGARH 202 001, U.P. INDIA.