COMPACT SUBSETS OF HOLOMORPHY OF A COMPLEX SPACE

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One of the most important properties of a Stein space X is the isonorphism of the canonical map between X and the spectrum $SO(X)$ of the algebra of holomorphic functions on X.

Given a complex space X, many conditions called «global» conditions have been known that ensure $X = SO(X)$.

In the present paper, given a compact subset K of X, we shall study « semi-global» conditions in order that $K = SO(K)$. When K is compact in a Stein manifold semi-global cohomology conditions have been established by Harvey and Wells ([7]). In the sequel we shall develop semi-global cohomology conditions for the case where K is a compact subset of holomorphy.

It is known that every compact subset of a Stein manifold is a compact subset of holomorphy ([12]). But, as will be proved later (see Remark 1), every compact subset of a cone in $\mathbf{C}^3$ is a compact subset of holomorphy. Hence a compact subset of holomorphy can have singularities. This is one of the results that follow from our study. It should also be noted that the methods of proof to be used in the sequel are quite different from those of Harvey and Wells ([7]).

1. COMPACT SET OF HOLOMORPHY.

Let X be a complex space, $SO(K)$ the spectrum of $O(X)$, where $O(X)$ denotes the Fréchet algebra of holomorphic functions on X equipped with the topology of uniform convergence on compact subsets of X.

Let $\delta_X: X \to SO(X)$ be the canonical map. A complex space X is said to have envelope of holomorphy if the spectrum $SO(X)$ of $O(X)$ has a complex structure such that

(i) δ_X is locally biholomorphic;

(ii) $O(X) \cong O(SO(X))$.

If X has envelope of holomorphy then $SO(X)$ is called the envelope of holomorphy of X.

We say that a compact set K in X is a compact set of holomorphy if K has a neighbourhood basis consisting of open sets having envelope of holomorphy.

By a Stein V-manifold we mean a complex space S which is biholomorphic to $\widetilde{S}/G$, where $\widetilde{S}$ is a Stein manifold and G is finite group of automorphisms on $\widetilde{S}$.

A complex space X is said to be a Riemann domain over S if there exists a locally biholomorphic map $\theta : X \to S$. By [12], every open subset of a Riemann domain over a Stein manifold has envelope of holomorphy. Hence every compact subset of a Riemann domain over a Stein manifold is a compact subset of holomorphy. The following theorem shows that every compact subset of a Riemann domain over a Stein V-manifold is also a compact subset of holomorphy.

1.1. THEOREM. *Let X be a Riemann domain over a Stein V-manifold S. Then X has envelope of holomorphy and $SO(X)$ is also a Riemann domain over S.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \widetilde{S}\times_S X & \xrightarrow{\ \widetilde{\pi}\ } & X \\ {\scriptstyle\widetilde{\theta}}\downarrow & & \downarrow{\scriptstyle\theta} \\ \widetilde{S} & \xrightarrow{\ \pi\ } & S \end{array}$$

where the fibered product $Z = \widetilde{S}\times_S X$ is a Riemann domain over $\widetilde{S}$. Since $\widetilde{S}$ is a Stein manifold, it follows that $SO(z)$ is a Riemann domain over $\widetilde{S}$ and $SO(Z)$ is also envelope of holomorphy of Z. Consider the map $R : SO(Z) \to SO(X)$ inducded by $\widetilde{\pi}$. Then the theorem is an immediate consequence of the following lemmas.

1.1. LEMMA. *Let θ be a finite surjective proper holomorphic map from a complex space X onto a normal space Y. Then $\theta : X \to Y$ is an analytic covering and $O(Y)$ is a continuous integral extension of finite order of X.*

Proof. As in [3] we can find analytically rare subsets $A \subset X$ and $B \subset Y$ such that

$$\theta|_{X\setminus A} : X \setminus A \to X \setminus B$$

is locally biholomorphic. Hence $\theta : X \to Y$ is an analytic covering.

Let $f \in O(X)$. We define the functions $\sigma_1, \ldots, \sigma_n$ on Y by

$$\sigma_1(y) = f(x_1) + \dots + f(x_n)$$
$$\sigma_2(y) = f(x_1)\, f(x_2) + \dots + f(x_{n-1})\, f(x_n)$$
$$\dots$$
$$\sigma_n(y) = f(x_1)\, f(x_2) \dots f(x_n)$$

for every $y \in Y$, $\theta^{-1}(y) = \{x_1, \dots, x_n\}$, where x_i are equipped with multiply. Then $\sigma_i \in \mathcal{O}(Y)$ and the polynomial

$$P(t) = t^n - \sigma_1 t^{n-1} + \dots \pm \sigma_n = (t - f(x_1)) \dots (t - f(x_n))$$

satisfies $P(f) = 0$. Hence $\mathcal{O}(Y)$ is an integral extension of $\mathcal{O}(X)$.

Let $\{f_\gamma\} \subset \mathcal{O}(X)$ be a sequence such that $f_\gamma \to f$ and $\sigma_i^\gamma \in \mathcal{O}(Y)$ be the function associated to f^γ. The problem is now reduced to proving that $\sigma_i^\gamma \to \sigma_i$ for $i = 1, \dots, n$. Take for every $y \in Y \setminus M$ a neighbourhood V of y and for every $j = 1, \dots, n$ a neighbourhood U_j of x_j such that $\theta|_{U_j} : U_j \to V$ is biholomorphic. Then

$$f_\gamma\, \theta|_{U_j} \to f\, \theta|_{U_j} \text{ for } j = 1, \dots, n.$$

Hence

$$\sigma_1^\gamma|_V = \sum_{j=1}^{n} f_\gamma \theta|_{U_j} \to \sum_{j=1}^{n} f\theta|_{U_j} = \sigma_1|_V;$$

$$\dots$$

$$\sigma_n^\gamma|_V = \prod_{j=1}^{n} f_\gamma \theta|_{U_j} \to \prod_{j=1}^{n} f\theta|_{U_j} = \sigma_n|_V.$$

Since $\mathcal{O}(Y) \subset \mathcal{O}(Y \setminus M)$ (see [9], we infer that $\sigma_i^\gamma \to \sigma_i$ for $i = 1, \dots, n$.

1.3. LEMMA. *$S\mathcal{O}(X)$ has a normal complex structure such that $\mathcal{O}(X) \cong \mathcal{O}(S\mathcal{O}(X))$.*

Proof. By a theorem of Grauert [4], it suffices to show that $R : S\mathcal{O}(Z) \to S\mathcal{O}(X)$ is finite and surjective. Since X is normal, $\tilde{\pi} : Z \to X$ is finite and surjective, by Lemma 1.2, it follows that $\mathcal{O}(Z)$ is an integral extension of $\mathcal{O}(X)$ of order

$$n = \sup \{\# \tilde{\pi}^{-1}(x) : x \in X\} < \infty.$$

Since $\mathcal{O}(Z) \cong \mathcal{O}(S\mathcal{O}(Z))$, we have $\sup \{ \# R^{-1}(\omega) : \omega \in S\mathcal{O}(X)\} < \infty$.

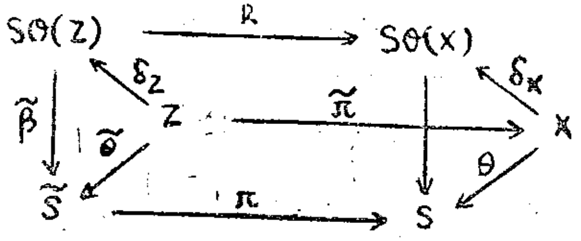
Let $\omega_o \in S\mathcal{O}(X)$ and let

$\omega' = \{ x \in \mathcal{O}(Z) : x^k + a_1 x^{k-1} + \dots + a_k = 0, \text{ for some } a_i \in \omega_o,\ k = 1, 2, \dots\}$ be the integral closure of ω_o. Since ω_o is prime, it follows that $\omega' \cap \mathcal{O}(X) = \omega_o$. Let $\tilde{\omega}$ be a maximal ideal of $\mathcal{O}(Z)$ which contains ω'. Then $\tilde{\omega} \cap \mathcal{O}(X) = \omega_o$. Since

$\mathcal{O}(X)\widetilde{/}\omega$ is integral over $\mathcal{O}(X)/\omega_o \cong \mathbf{C}$ and $\mathbf{C}$ is algebraically closed, we get $\mathcal{O}(X)/\omega_o = \mathbf{C}$. Let $\widetilde{\omega} = \operatorname{Ker} \widetilde{f}$, where $\widetilde{f}$ is a multiplicative linear functional on $\mathcal{O}(X)$. Since $\mathcal{O}(X)$ is a continuous integral extension of $\mathcal{O}(X)$ of finite order, by the continuity of $\widetilde{f}_{|\mathcal{O}(X)}$, we infer that f is continuous. Hence $\widetilde{\omega} \in S\mathcal{O}(X)$ and $R\widetilde{\omega} = \omega_o$.

1.4. LEMMA. *The canonical map $\beta: S\mathcal{O}(X) \to S$ induced by $\theta: X \to S$ is locally biholomorphic.*

Proof. Consider the commutative diagram



a) By Lemma 1.2, $R: S\mathcal{O}(Z) \to S\mathcal{O}(X)$ is an analytic covering of order $m = \sup\{\# R^{-1}(\omega) : \omega \in S\mathcal{O}(X)\} \leqslant n$. First we check that $m = n$ (n is the number in the proof of Lemma 1.3)

Since $\sup\{\# \widehat{\pi}^{-1}(x) : x \in X\} = \sup\{\# \pi^{-1}(\theta x) : x \in X\}$, we find $x_o \in X$ such that

$$\pi^{-1}(\theta x_o) = \{\widetilde{s}_1^{\,o}, \dots, \widetilde{s}_n^{\,o}\}.$$

Take $h \in \mathcal{O}(\widetilde{S})$ such that $h(s_i^o) = i$ for $i = 1, \dots, n$. Then $h\widetilde{\beta}(\delta_Z(\widetilde{s}_i^{\,o}, x_o)) = i$ for $i = 1, \dots, n$. Hence $\# R^{-1}(\delta_X(x_o)) \geqslant n$ and $m = n$.

b) We now prove that

$$\# R^{-1}(\omega) = \# \pi^{-1}(\beta\omega) \text{ for every } \omega \in S\mathcal{O}(X).$$

Given $\omega \in S\mathcal{O}(X)$, let $\pi^{-1}(\beta\omega) = \{\widetilde{s}_1, \dots, \widetilde{s}_k\}$ and $R^{-1}(\omega) = \{\widetilde{\omega}_1, \dots, \widetilde{\omega}_l\}$. Take $g \in \mathcal{O}(\widetilde{S})$ such that $g(\widetilde{s}_i) = i$ for $i = 1, \dots, k$ and $(g\widetilde{s}_i^{\,o}) = i$ for $i = 1, \dots, n$. Putting $f = g\widetilde{\beta}$, we have

$$f\delta_Z(\widetilde{s}^{\,o}, x_o) = g\widetilde{\beta}\delta_Z(\widetilde{s}_i^{\,o}, x_o) = g\widetilde{\theta}(\widetilde{s}_i^{\,o}, x_o) = g(\widetilde{s}_i^{\,o}) = i$$

for $i = 1, \dots, n$. Suppose that $P_g \in \mathcal{O}(S)[\lambda]$ and $P_f \in \mathcal{O}(X)[\lambda]$ are monic polynomials of order n such that $P_g(g) = 0$ and $P_f(f) = 0$. Since

$$P_g(f)\omega = P_g(g\widetilde{\beta})\omega = P_g\, g(\widetilde{\beta}\omega) = 0$$

for every $\omega \in SO(X)$, it follows that $P_g(f) = 0$. Then, from the uniqueness of p_f, we have $P_f = P_g$. Combining this with the relations

$$\sum_{j=1}^{e} \text{mul}\, f(\tilde{\omega}_j) = n; \quad \sum_{j=1}^{k} \text{mul}\ g(\tilde{s}_j) = n$$

yields

$$\sum_{\tilde{\beta}\tilde{\omega}_j = \hat{s}_j} \text{mul}\, g(\tilde{s}_j) = \sum_{j=1}^{k} \text{mul}\ g(\tilde{s}_j).$$

This implies $k = l$ and $\beta(\{\tilde{\omega}_1, \ldots, \tilde{\omega}_k\}) = \{\tilde{s}_1, \ldots, \tilde{s}_k\}$.

c) Let $\omega \in SO(X)$, $s = \beta\omega$. From b) we have

$$R^{-1}(\omega) = \left\{ \tilde{\omega}_1, \ldots, \omega_k \right\}; \ \pi^{-1}(s) = \left\{ \tilde{s}_1, \ldots, \tilde{s}_k \right\}$$

and $\tilde{\beta}\,\tilde{\omega}_j = s_j$ for $j = 1, \ldots, k$. Take a neighbourhood U_j of $\tilde{\omega}_j$ and a neighbourhood V_j of $\tilde{s}_j$ such that $\beta|_{U_j}: U_j \to V_j$ is biholomorphic for every j. Since the maps R and π are proper, there exist neighbourhoods $\tilde{W}$ and W of ω and s respectively such that $\beta\tilde{W} \subset W$ and $R^{-1}(\tilde{W}) \subset \bigcup_{j=1}^{n} U_j$, $\pi^{-1}(W) = \bigcup_{j=1}^{k} V_j$. Then from b) we infer that $\beta: \tilde{W} \to W$ is an imbedding. Hence, by the normality of W and since $\dim \tilde{W} = \dim W$, it follows that $\beta\tilde{W}$ is an open neighbourhood of s. Thus $\beta: \tilde{W} \to \beta\tilde{W}$ is biholomorphic.

This completes the proof of Lemma 1.4 and there by that of Theorem1.1. Before closing tbis section let us also prove the following

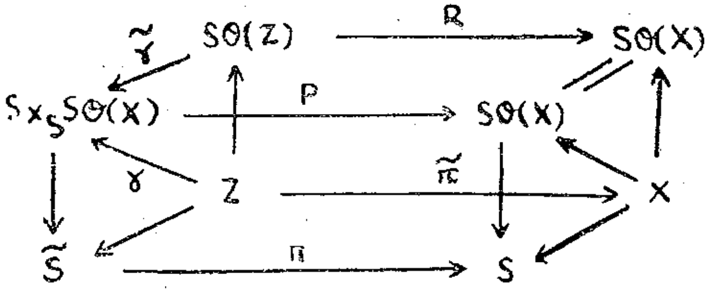
1.5. PROPOSITION. *Let X be as in Theorem 1.1. Then*

$$SO(\tilde{S} \times_S X) = \tilde{S} \times_S SO(X).$$

Proof. Consider the holomorphic map $\gamma: Z \to \tilde{S} \times_S SO(X)$ defined by the formula

$$\gamma_z = (\beta_z, R\delta_Z z) \text{ for every } z \in Z$$

and the canonical map $p: \tilde{S} \times_S SO(X) \to SO(X)$. Since p is finite and surjective, and $\tilde{S}$, $SO(X)$ are Stein, it follows that $S \times_S SO(X)$ is Stein. Thus there exists a unique holomorphic extension $\tilde{\gamma}: SO(Z) \to \tilde{S} \times_S SO(X)$ of γ. Hence we have the commutative diagram



where π, $\tilde{\pi}$, p and R are analytic coverings of order $\sup\{\#\,\pi^{-1}(s) : s \in S\}$. Then as in the proof of Theorem 1.1, it follows that $\#\,R^{-1}(\omega) = \#\,p^{-1}(\omega)$ for every $\omega \in SO(X)$ and $\tilde{\gamma}$ is biholomorphic. The proposition is proved.

2. THE CONDITION $K = S\mathcal{O}(K)$

Let K be a compact subset of a complex space X. Put

$$\mathcal{O}(K) = \varinjlim_{U \supset K} \mathcal{O}(U),$$

where U runs over open subsets of X which contain K, and suppose that $\mathcal{O}(K)$ is equipped with the inductive topology. Let $S\mathcal{O}(K)$ be the spectrum of the algebra $\mathcal{O}(K)$ and $\delta_K : K \to S\mathcal{O}(K)$ the canonical map. If δ_K is an imbedding, we shall identify K with $\delta_K K$.

2.1. THEOREM. *Let K be a compact subset of holomorphy of a complex space $(X, \mathcal{O})$. Then $K = S\mathcal{O}(K)$ if and only if $H^p(K, \mathcal{O}) = 0$ for every $p \geqslant 1$.*

For the proof of this theorem we need several lemmas.

2.2. LEMMA *Let $H^p(K, \mathcal{O}) = 0$ for every $p \geqslant 1$. Then $H^p(K, \mathcal{O}^F) = 0$ for every $p \geqslant 1$ and every Fréchet space F.*

Proof. Let $\{U_n\}$ be a neighbourhood basis of K consisting of relatively compact open subsets which have envelope of holomorphy and $U_{n+1} \ll U_n$ for every $n = 1, 2, \ldots$.

For every n, take a Stein open covering $\mathcal{U}_n$ of U_n such that

(i) $\mathcal{U}_n$ is finite;

(ii) For every $V \in \mathcal{U}_n$, there exists $U \in \mathcal{U}_{n+1}$ such that $U \subset V$;

(iii) For every $U \in \mathcal{U}_{n+1}$, there exists $V(U) \in \mathcal{U}_n$ such that $U \ll V(U)$.

Let $\theta_n : C^p(\mathcal{U}_n) \to C^p(\mathcal{U}_{n+1})$ be the map defined by

$$\theta_n f(U_0, \ldots, U_p) = g \mid_{U_0 \cap \ldots \cap U_p}$$

for every $f \in C^p(\mathcal{U}_n)$ and every $(U_0, ..., U_p) \in N(\mathcal{U}_{n+1})$, where $g : V(U_0) \cap \cdots \cap V(U_p) \to \mathbb{C}$ is a «component» of f. From (i) — (iii), it is easily seen that θ_n is injective and compact. Hence, for every $p \geqslant 1$, $F_p = \varinjlim C^p(\mathcal{U}_n)$ is a DFN-space and $\varinjlim C^p(\mathcal{U}_n)$ is regular (see [10]). Put

$$Z_p = \text{Ker } \{\delta^p : F_p \to F_{p-1}\}.$$

Then Z_p is also a DFN-space. Consider the exact sequence

$$F_{p-1} \to Z_p \to 0.$$

Since $H^p(K, \mathcal{O}^F) = \varinjlim H^p(U_n, \mathcal{O}^F) = \varinjlim H^p(\mathcal{U}_n, \mathcal{O}^F)$, it suffices to show that

$$\delta_F^{p-1} : \varinjlim (C^{p-1}(\mathcal{U}_n)_\varepsilon F) \to \varinjlim (Z^p(\mathcal{U}_n)_\varepsilon F)$$

is surjective, where $A_\varepsilon F = A \widehat{\otimes}_\varepsilon F$.

a) First consider the special case where F is a Banach space.

It is know that $Z^p(\mathcal{U}_n)_\varepsilon B \cong \mathcal{L}_\varepsilon(B'_c, Z^p(\mathcal{U}_n))$, where B'_c denotes the vector space of continuous linear fuactionals on B equipped with the topology uniform convergence on equicontinuous subsets of of B'. Using the regularity of $\varinjlim Z^p(\mathcal{U}_n)$, we shall prove that

$$\varinjlim (Z^p(\mathcal{U}_n)_\varepsilon F) = (\varinjlim Z^p(\mathcal{U}_n))_\varepsilon F.$$

Indeed, it is obvious that $\varinjlim (Z^p(\mathcal{U}_n)_\varepsilon F) \subset (\varinjlim Z^p(\mathcal{U}_n))_\varepsilon F$.

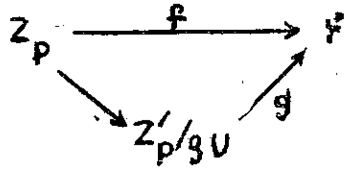
To show the converse inclusion, take $f \in (\varinjlim Z^p(\mathcal{U}_n))_\varepsilon F$ and let S be the unit ball of the Banach space F. Since $f(S)$ is bounded, we find n_0 such that $f(S) \subset Z^p(\mathcal{U}_{n_0})$, hence $f \in Z^p(\mathcal{U}_{n_0})_\varepsilon F$.

Similarly, we have

$$\varinjlim (C^p(\mathcal{U}_n)_\varepsilon F) = (\varinjlim C^p(\mathcal{U}_n))_\varepsilon F.$$

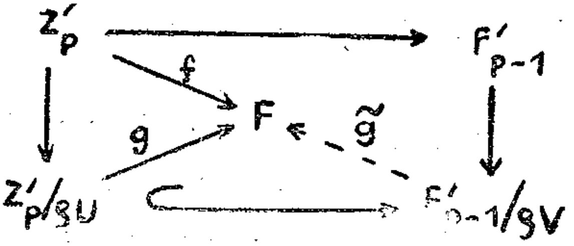
Take $f \in (\varinjlim Z^p(\mathcal{U}_n)_\varepsilon F) = \mathcal{L}_\varepsilon(Z_p, F)$. Since every DFN-space is barrelled B-complete, it follows that $F_{p-1} \to Z_p$ is open. By [13], $Z'_p \to F_{p-1}$ is an imbedding.

Consider now f as a map from a nuclear space into a Banach space. From [11], there exists a neighbourhood U of zero in Z'_p such that the following diagram is commutative



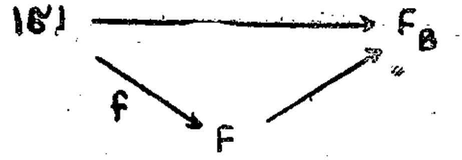
where g is nuclear, ρ is the Minkowski functional of U and $Z'_{p/\rho U}$ is the space $Z'_p \,/\, \rho^{-1}(0)$ equipped with the norm induced by ρ.

Let V be a balanced convex neighbourhood of zero in F'_{p-1} such that $V \cap Z'_p = U$. Consider the normed space $Z'_p \,/\, \rho U$ contained in the normed space $F'_{p-1} / \rho V$. By the nuclearity of g and the Hahn-Banach Theorem, we find a continuous linear map $\tilde{g} : F'_{p-1} / \rho V \to F$ such that the following diagram is commutative



Thus there exists $f : F'_{p-1} \to F$ such that $\delta_F^{p-1} f = f$.

b) Turning to the general case, let $f \in Z^p(\mathcal{U}_n)_\varepsilon F \subset C^p(\mathcal{U}_n, \mathcal{O}^F)$. Without loss of generality, we may assume that $\cup\{f(\,|\sigma|\,) ; \sigma \in N(\mathcal{U}_n)\}$ is bounded Let B be the Banach space spanned on B equipped with the norm defined by B. For every $\sigma \in N(\mathcal{U}_n)$, consider the map $f : |\sigma| \to F_B$ in the following commutative diagram



Since $f : |\sigma| \to F$ is holomorphic and $f(|\sigma|)$ is bounded in F_B, by the Cauchy's integral formula, we have that $f : |\sigma| \to F_B$ is holomorphic. Hence $f \in Z^p(\mathcal{U}_n)_\varepsilon F_B$. By a), $f = \delta_{F_B}^{p-1} \tilde{f}$ for some $\tilde{f} \in \varinjlim (C^p(\mathcal{U}_{n-1})_\varepsilon F)$. The lemma is proved.

2.3. LEMMA. *Assume that $H^p(K, \mathcal{O}) = 0$ for every $p \geqslant 1$ and is a φ coherent analytic sheaf on $\widehat{U} = SO(U)$, where U is some neighbourhood having envelope of holomorphy of K. Then $H^p(K, H^0(\widehat{U}, \varphi) \widehat{\otimes}_{\mathcal{O}(\widehat{U})} \mathcal{O}) = 0$ for every $p \geqslant 1$.*

Proof. Put $F = H^0(\widehat{U}, \varphi)$. For every open set $V \subset U$, we have $F\widehat{\otimes}_{O(\widehat{U})}O(V) = F\widehat{\otimes}_\varepsilon O(V)/\mathrm{Im}\, d$, where $d : F\widehat{\otimes}_\varepsilon O(\widehat{U})\widehat{\otimes}_\varepsilon O(V) \to F\widehat{\otimes}_\varepsilon O(V)$ is a map defined by $d(u\otimes\sigma\otimes v) = \sigma u\otimes v - u\otimes\sigma v$.

Let $F\widehat{\otimes}_{O(U)} O$ be the sheaf generated by the presheaf $\{F\widehat{\otimes}_{O(\widehat{U})}O(V)\}$. When V is small enough, by [8] $\widehat{\mathrm{Tor}}^1_{O(\widehat{U})}(F, O(V)) = 0$. Hence $\widehat{\mathrm{Tor}}^1_{O(\widehat{U})}(F, O) = 0$.

Put $P_0 = F\widehat{\otimes}_\varepsilon O(\widehat{U})$ and $P_k = P_{k-1}\widehat{\otimes}_\varepsilon O(\widehat{U})$ for every $k \geqslant 1$. By [8] we have the direct projective resolution of F

$$\ldots \to P_k \overset{d_k}{\to} P_{k-1} \to \ldots \to P_0 \to F.$$

Putting $F_k = \mathrm{Ker}\, d_k$, we obtain the exact sequences

$$0 \to F_1 \to P_0 \to F \to 0;$$
$$0 \to F_{k-1} \to P_k \to F_k \to 0 \text{ for every } k \geqslant 1.$$

Consider the homology sequences associated to these exact sequences. Since $\widehat{\mathrm{Tor}}^1_{O(\widehat{U})}(F, O) = 0$ and $\widehat{\mathrm{Tor}}^1_{O(\widehat{U})}(P_k, O) = 0$ for every $k \geqslant 1$, by the induction on k we have $\widehat{\mathrm{Tor}}^1_{O(\widehat{U})}(F_k, O) = 0$ for every $k \geqslant 1$. Hence we obtain the exact sequences

$$0 \to F_1\widehat{\otimes}_{O(\widehat{U})}O \to P_0\widehat{\otimes}_{O(\widehat{U})}O \to F\widehat{\otimes}_{O(\widehat{U})}O \to 0;$$

$$0 \to F_{k-1}\widehat{\otimes}_{O(\widehat{U})}O \to P_k\widehat{\otimes}_{O(\widehat{U})}O \to F_k\widehat{\otimes}_{O(\widehat{U})}O \to 0$$

for every $k \geqslant 2$.

Obviously

$$P_0\widehat{\otimes}_{O(\widehat{U})}O = F\widehat{\otimes}O(\widehat{U})\widehat{\otimes}_{O(\widehat{U})}O = F\widehat{\otimes}_\varepsilon O;$$

$$P_k\widehat{\otimes}_{O(\widehat{U})}O = P_{k-1}\widehat{\otimes}_\varepsilon O \text{ for every } k \geqslant 1.$$

Thus, by Lemma 2.2, $H^p(K, P_k\widehat{\otimes}_{O(\widehat{U})}O) = 0$ for every $p \geqslant 1$, $k \geqslant 0$. Consider the cohomology sequences associated to the tensor exact sequences. We have, for $p = 1, 2, \ldots$

$$H^p(K, F\widehat{\otimes}_{O(\widehat{U})}O) = H^{p+1}(K, F_1\widehat{\otimes}_{O(\widehat{U})}O);$$
$$H^{p+k}(K, F\widehat{\otimes}_{O(\widehat{U})}O) = H^{p+k+1}(K, F\widehat{\otimes}_{O(\widehat{U})}O) \text{ for every } k \geqslant 1$$

Therefore

$$H^p K, F\widehat{\otimes}_{O(\widehat{U})}O) = H^{p+k}(K, F_k\widehat{\otimes}_{O(\widehat{U})}O) \text{ for every } k \geqslant 1.$$

Since $H^{p+k}(K, F_k\widehat{\otimes}_{O(\widehat{U})}O) = 0$ if $p + k > \dim_R X$, we get $H^p(K, F\widehat{\otimes}_{O(\widehat{U})}O) = 0$ for every $p \geqslant 1$.

2.4 LEMMA. *Let K be a compact subset of holomorphy of X and $K = SO(K)$. Then there exists a neighbourhood W of K such that $\mathcal{O}(W)$ separates points of W.*

Proof. a) Let U be a neighbourhood having envelope of holomorphy of K. Then $\delta_U : U \to SO(U)$ is locally biholomorphic. Hence, for every $x \in K$ there exists a neighbourhood U_x of x such that $\mathcal{O}(U)$ separates points of U_x. Put

$$A = K \times K \setminus \bigcup \{U_x \times U_x : x \in K\}.$$

For every $(x,y) \in A$ there exists $f \in \mathcal{O}(G)$, with G some neighbourhood of K, such that $f(x) \neq f(y)$. Hence for every $(x, y) \in A$, there exist neighbourhoods $U(x, y)$ and $V(x,y)$ of (x, y) and K respectively such that $\mathcal{O}(V(x,y))$ separates points of $U(x, y)$.

We cover K by a finite number of neighbourhoods $U(x_1, y_1)$,..., $U(x_k, y_k)$ and put $V = V(x_1, y_1) \cap ... \cap V(x_k, y_k) \cap U$. Then V is a neighbourhood of K, and $\mathcal{O}(V)$ separates points of K.

b) By a) the canonical map $\delta : K \to SO(V)$ is a homeomorphism of K onto its image. Since $\delta: V \to SO(V)$ is locally biholomorphic, we may extend $\delta^{-1} : \delta K \to K$ to a holomorphic map β from a neighbourhood G of δK into V such that $\delta\beta = \mathrm{id}$. Hence $\delta|_{\beta G}$ is biholomorphic. Putting $W = \beta G$, we then have the desired neighbourhood.

2.5 PROOF OF THEOREM 2.1. Assume that $H^p(K,\mathcal{O}) = 0$ for every $p \geqslant 1$. Let U be a neighbourhood of K having envelope of holomorphy $\widehat{U}$. We shall prove that $\mathcal{O}(K)$ separates points of K. Since $\widehat{U}$ is Stein, it is sufficient to treat the case $x, y \in K$ in which $\delta x = \delta y$. Let $\mathcal{J}$ be the ideal sheaf on U associated to the set $\{x, y\}$, and $\widehat{\mathcal{J}}$ the ideal sheaf on U associated to $\{\delta x\}$. Since

$\mathcal{J}_z \cong \widehat{\mathcal{J}}_{\delta z} = (H^0(\widehat{U}, \widehat{\mathcal{J}}) \widehat{\otimes}_{\mathcal{O}(\widehat{U})} \mathcal{O}))_z$ for every $z \in U$, by Lemma 2.3 we obtain $H^1(K, \mathcal{J}) = 0$. By considering the exact cohomology sequence associated to the exact sequence

$$0 \to \mathcal{J} \to \mathcal{O} \to \mathcal{O}/\mathcal{J} \to 0,$$

we have the exact sequence

$$H^0(K, \mathcal{O}) \to H^0(K, \mathcal{O}/\mathcal{J}) \to 0.$$

Hence, there exists $f \in H^0(K, \mathcal{O})$ such that $f(x) = 0$ and $f(y) = 1$.
This shows that $\delta : K \to SO(K)$ is injective. We now prove that is also surjective. If it were not so, then we should have $y \in SO(K) \setminus K$.

For every $x \in K$, if $y(f) = 0$ implies $f(x) = 0$ for any $f \in \mathcal{O}(K)$ then $x = y$. Therefore, by the compactness of K, we can find a neighbourhood U having envelope of holomorphy $\widehat{U}$ of K and $f_1,\dots, f_k \in \mathcal{O}(U)$ such that $y(f_j) = 0$ for $j = 1,\dots, k$ and such that these functions do not vanish simultaneously at any point in K. Consider the exact sequence

$$0 \to \operatorname{Ker} \theta \mid_U \to \mathcal{O}^k \mid_U \to \mathcal{O} \mid_U \to 0$$

where θ is defined by $\theta(g_1,\dots, g_k) = \sum_{j=1}^{k} f_j g_j$. Let $\widehat{\theta} : \mathcal{O}^k \mid_{\widehat{U}} \to \mathcal{O} \mid_{\widehat{U}}$ be the homomorphism induced by $f_1,\dots, f_k$. Then

$$\operatorname{Ker} \widehat{\theta} \mid_{\widehat{U}} \cong H^0(U, \operatorname{Ker} \theta \mid_{\widehat{U}}) \widehat{\otimes}_{\mathcal{O}(\widehat{U})} \mathcal{O} \mid_{\widehat{U}}.$$

Consequently,

$$(\operatorname{Ker} \theta)_z \cong (\operatorname{Ker} \widehat{\theta} \mid_{\widehat{U}})_{\delta_U z} \cong H^0 (\widehat{U}, \operatorname{Ker} \widehat{\theta} \mid_{\widehat{U}}) \widehat{\otimes}_{\mathcal{O}(\widehat{U})} \mathcal{O} \mid_U)_z$$

for every $z \in U$. By Lemma 2.3, $H^1(K, \operatorname{Ker} \theta) = 0$. Hence, the sequence

$$(\mathcal{O}(U))^k \to \mathcal{O}(U) \to 0$$

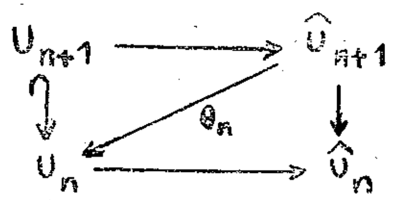
is exact, and there exist $g_1,\dots, g_k \in \mathcal{O}(U)$ such that $\sum_{j=1}^{k} f_j g_j = 0$.

This is impossible, because

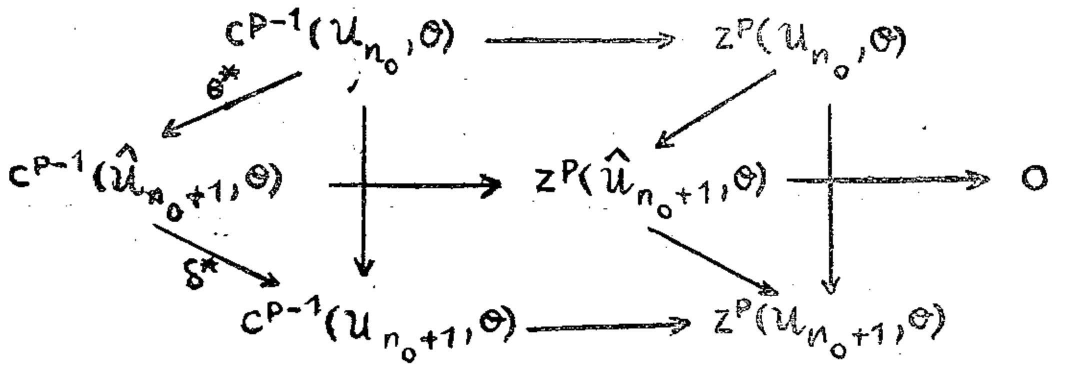
$$1 = y(1) = \Sigma\, y(f_j)\, y\, (g_j) = 0.$$

The « if » part is thus proved.

Turning to the « only if » part, by Lemma 2.4, there exists a neighbourhood basis $\{U_n\}$ of K such that each U_n having envelope of holomorphy $\widehat{U}_n$ and such that for each $n = 1, 2,\dots$ the diagram



is commutative. Let $f \in H^p(K, \mathcal{O})$. Then there exists n_0 such that $f \in H^p(U_{n_0}, \mathcal{O})$. Denote by $\mathcal{U}_{n_0}$ the cover consisting of all Stein open subsets of U_{n_0} meeting $\theta_{n_0} (\widehat{U}_{n_0+1})$. Let $\widehat{\mathcal{U}}_{n_0+1}$ and $\mathcal{U}_{n_0+1}$ be the covers consisting of all Stein open subsets of $\widehat{U}_{n_0+1}$ and U_{n_0+1} respectively. Then, for every $U \in \mathcal{U}_{n_0}$ there exists $\widehat{V}(U) \in \widehat{\mathcal{U}}_{n_0+1}$ such that $\theta \widehat{V}(U) \subset U$ and for every $\widehat{V} \in \mathcal{U}_{n_0+1}$ there exists $W(\widehat{V}) \in \mathcal{U}_{n_0+1}$ such that $\delta W(\widehat{V}) \subset \widehat{V}$. Obviously, $W(\widehat{V}(U)) \subset U$. Hence, we obtain the commutative diagram



where the middle row is exact. Hence $f = 0$ in $H^p(\mathcal{U}_{n_0+1}, \mathcal{O})$.

Thus $H^p(K, \mathcal{O}) = 0$ for every $p \geqslant 1$. The proof of Theorem 2.1 is complete.

2.6. PROPOSITION. *Let K be a compact subset of a Riemann domain over a Stein V-manifold. Then $K = S\mathcal{O}(K)$ if and only if $\dim_{\mathbf{C}} H^p(K, \mathcal{O}) < \infty$ for every $p \geqslant 1$.*

Proof. According to Theorems 1.1 and 1.2, it suffices to show that $H^p(K, \mathcal{O}) = 0$ for every $p \geqslant 1$. Denote by $L(H^p(K, \mathcal{O}))$ the space of linear maps of $H^p(K, \mathcal{O})$ into $H^p(K, \mathcal{O})$. Define a map $\theta : \mathcal{O}(X) \to L(H^p(K, \mathcal{O}))$ by

$$\theta f(\varphi) = f\varphi \text{ for every } f \in \mathcal{O}(X),\ \varphi \in H^p(K, \mathcal{O}).$$

Since $\dim_{\mathbf{C}} \mathcal{O}(X) = \infty$ and $\dim H^p(K, \mathcal{O}) < \infty$, we have $\operatorname{Ker} \theta \neq 0$. Hence, there exists $f_1 \in \mathcal{O}(X)$ such that f_1 is non-constant on each irreducible branch of X, $f_1 H^p(K, \mathcal{O}) = 0$, $\dim V(f_1) = \dim_{\mathbf{C}} X - 1$ and the sequence

$$0 \to \mathcal{O} \overset{\widehat{f_1}}{\to} \mathcal{O} \to \mathcal{O}/f_1\mathcal{O} \to 0$$

is exact, where $\widehat{f_1}$ is defined by multiplication by f_k. By considering the cohomology sequence associated to this exact sequence, we obtain the exact sequence

$$0 \to H^p(K, \mathcal{O}/f_1\mathcal{O}) \to H^{p+1}(K, \mathcal{O}).$$

Hence $\dim_{\mathbf{C}} H^p(K, \mathcal{O}/f_1\mathcal{O}) < \infty$. By induction on $\dim X$, we find functions $f_1, \ldots, f_n \in \mathcal{O}(X)$, where $n = \dim_{\mathbf{C}} X$, such that

(i) $f_k H^p(K, \mathcal{O}/\sum_{j=1}^{k} f_j\mathcal{O}) = 0$ for $k = 1, \ldots, n$, where $\sum_{j=1}^{0} f_j\mathcal{O} = 0$;

(ii) $\dim V(f_1, \ldots, f_k) = \dim_{\mathbf{C}} X - k$ for $k = 1, \ldots, n$;

(iii) for $k = 1, \ldots n$, we have a short exact sequence

$$0 \to \mathcal{O}\Big|_{\sum_{j=1}^{k-1} f_j \mathcal{O}} \xrightarrow{\widehat{f}_k} \mathcal{O}\Big|_{\sum_{j=1}^{k-1} f_j \mathcal{O}} \to \mathcal{O}\Big|_{\sum_{j=1}^{k} f_j \mathcal{O}} \to 0.$$

Obviously, $H^p(K, \mathcal{O}|_{\sum_{j=1}^{n} f_j \mathcal{O}}) = 0$. Assume that $H^p(K, \mathcal{O}|_{\sum_{j=1}^{k+1} f_j \mathcal{O}}) = 0$.

By (iii), we have the exact sequence

$$H_p(K,\mathcal{O}|_{\sum_{j=1}^{k} f_j \mathcal{O}}) \xrightarrow{\widehat{f}_k} H_p(K,\mathcal{O}|_{\sum_{j=1}^{k} f_j \mathcal{O}}) \to 0$$

Using (i), we see that $\widehat{f}_k$ is the null homomorphism. Hence,

$H^p(K,\mathcal{O}/\sum_{j=1}^{k+1} f_j \mathcal{O}) = 0$. Thus, we have $H^p(K,\mathcal{O}/\sum_{j=1}^{k} f_j \mathcal{O}) = 0$ for $k = 0, \ldots, n$.

When $k = 0$ this yields $H^p(K,\mathcal{O}) = 0$. The proof is complete.

Remarks

1. Let G be the group generated by the isomorphism $(u,v) \mapsto (-u,-v)$ of C^2. In view of the isomorphism. $(u,v) \mapsto (u^2 - v^2, 2uv, u^2 + v^2)$, we obtain

$$\mathbf{C}^2/_G \cong Y = \{(x,y,z) \in \mathbf{C}^3 : x^2 + y^2 = z^2\}.$$

Hence the cone Y is a Stein V-manifold.

2. We can prove the following: «Let K be a compact set of holomorphy. Then $K = S\mathcal{O}(K)$ if and only of $H^p(K,\mathcal{O}) = 0$ for every $p \geqslant 1$ and every coherent analytic sheaf φ on K». This formally resembles the condition for a complex space to be Stein.

3. If K is a Stein compact set, then $H^p(K,\varphi) = 0$ for every $p \geqslant 1$ and every coherent analytic sheaf φ on K. Note, however, that the converse of this statement need not necessarily hold. Indeed, it is well known [1] that there exists a compact subset $K \subset \mathbf{C}^2$ such that $K = S\mathcal{O}(K)$ and K is non-Stein.

4. For any given compact subset $K \subset \mathbf{C}^n$, and a real number m, $0 \leqslant m \leqslant n$, let $\lambda_m(A)$ denote the m-dimensional Hausdorff measure of A. From [6] it is known that if $\lambda_{2m}(K) = 0$ then $H^p(K, \mathcal{O}) = 0$ for every $p \geqslant m$. Therefore $H^p(K,\mathcal{O}) = 0$ for every $p \geqslant 1$ provided $\lambda_2(K) = 0$.

5. Proposition 2.6 was already contained in [14], but the proof presented above is new.

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