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THE DISC CONDITION AND THE LOCAL MAXIMUM PRINCIPLE

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INTRODUCTION

The continuous principle or the disc condition is one of the important geometrical properties of the complex spaces. An interesting problem in the complex analysis is to study Steiness of the complex spaces by considering the continuous principle.

In the case of the Riemann domains over the Stein manifold this problem has been solved by Oka [11], Norguet [10] and Bremmermann [2] by considering the plurisubharmonicity of the boundary distant function. Recently, Fornaess [5] has shown that there exist open sets in a Stein space satisfying the continuous principle which are however not Stein.

On the other hand, as is well-known, one can say that the continuous principle is sufficiently good for the study of certain problems in the complex analysis, for example, for the extension problem of the holomorphic maps [13]. In this paper, we study the realization of this principle for a family of the Riemann domains over $\mathbf{C}^n$. In some special cases these results have been obtained in [15].

In section 1, we prove that for an 1-dimensional parametric family of the Riemann domains over $\mathbf{C}^n$, the disc condition and the plurisubharmonicity of the boundary distant function are equivalent. In particular, when the family is an open set in $\mathbf{C}^n$, the result has been established by Slodkowski [15]. We also consider the realization of the disc condition for open sets constructed from the spectrum of holomorphic operator functions.

According to Shiffman's results [13] we establish in section 2 the equivalence beetwen the disc condition and the extension of the holomorphic maps for the open sets of the complex manifolds satisfying the disc condition.

Finally, in section 3 we prove the equivalence of the Steiness and the maximum local principle for the Riemann domains over $\mathbf{C}^n$. When $n=2$ the result has been proved by Slodkowski [15] and Wermer [16].

DISC CONDITION FOR A FAMILY OF STEIN SPACES

We shall use the following notations:

$$\Delta = \{ z \in C : |z| < 1 \}$$

$$A_{r1} = \{ z \in C \; r < |z| < 1 \} \qquad \text{with some } r < 1.$$

For a complex space X, $\mathcal{O}(\Delta, X)$, denotes the space of all holomorphic maps from Δ into X equipped with the compact open topology.

1.1. DEFINITION

We say that the complex space X satisfies the disc condition if for all sequence $\{f_n\}$ *in* $\mathcal{O}(\Delta, X)$, $\{f_n\}$ *converges to a map* $f \in \mathcal{O}(\Delta, X)$ *whenever the sequence* $\{f_{n} |_{A_{r1}}\}$ *converges in* $\mathcal{O}(A_{r1}, X)$ *for some* $r < 1$. *(cf. [13]).*

For this definition we first give some examples.

1.2. By maximum modulus principle, $\mathbf{C}^n$ satisfies the disc condition and hence any Stein space does.

1.3. Let G be a complex Lie group. Then by [8] there exists a biholomorphic map from G onto $\Gamma \times S$, where Γ is a commutative complex Lie group and S is a Stein space. Let L be the tangent space of Γ at unit element. Since Γ is commutative the map $\exp: L \to \Gamma$ is a holomorphic convering map and hence by [13] Γ satisfies the disc condition. Thus G satisfies also the disc condition.

1.4. Considering the sequence $f_n : \Delta \to CP$ defined by $f_n(z) = (2^n z^n \; : \; 1)$ it follows that CP does not satisfy the disc condition.

Let $p : Y \to \mathbf{C}^n$ be a Riemann domain over $\mathbf{C}^n$. Denote by $d_Y(x)$ the boundary distance function on Y.

$$d_Y(x) = \sup \left\{ \varepsilon > 0 : \begin{array}{l} \text{there exists a connected neighbourhood } U \\ \text{containing } x \text{ such that } p_U : U \cong \Delta(p(x), \varepsilon) \end{array} \right\}$$

Where $\Delta(p(x), \varepsilon)$ denotes the polydisc with center $p(x)$ and radius ε. Let X be a complex space, Y be a Riemann domain over $\mathbf{C}^n$. For any open set $\Omega \subset X \times Y$, $\Omega_x = (x \times Y) \cap \Omega$ and $d(z, \partial\Omega_x) = d_{\Omega_x}(z)$ for any $z \in \Omega_x$.

1.5. DEFINITION

Let Ω *be as above. Then* Ω *is said to satisfy the boundary condition if* $\partial\Omega_x = \partial\Omega \cap (x \times Y)$ *for every* $x \in X$.

1.6. THEOREM: *Let* Ω, X, Y *be as above. Then:*

(i) If Ω *is relatively compact,* Ω *satisfies boundary condition the projection* $pr_1 \Omega$ *of* Ω *onto the first component satisfies the disc condition and the map* $\varphi(x, z) := -\log d(z, \partial\Omega_x)$ *is plurisubharmonic on* Ω, *then* Ω *satisfies the disc condition.*

(ii) If Ω *satisfies the disc condition, then* $\varphi(x, z)$ *is plurisubharmonic on* Ω.

Proof. (i) Let $\{f_n\}$ be a sequence in $O(\Delta, \Omega)$ which converges to f in $O(A_{r_1}, \Omega)$ for some $r < 1$, and $r_1 = \inf\{0 < s < 1 : f$ is extended on $A_{s_1}\}$. First we show that $r_1 = 0$. Assume the contrary that $r_1 > 0$. Consider the holomorphic envelop $\widetilde{Y}$ of Y. Then $\widetilde{Y}$ satisfies the disc condition. Hence $Pr_1 \Omega \times Y$ satisfies the disc condition, too. For the canonical map $g : \Omega \to \to Pr_1 \Omega \times \widetilde{Y}$, let $\widetilde{f}_n = gf_n$. Then $\{\widetilde{f}_n\}$ converges to $\widetilde{f} = g.f$ uniformly on every compact set of A_{r_1}. Since $Pr_1 \Omega \times Y$ satisfies the disc condition $\{\widetilde{f}_n\}$converges then to $\widetilde{f}$ in $O(\Delta, Pr_1 \Omega \times \widetilde{Y})$. Let t_0 be arbitrary point of Δ with $|t_0| = = r_1$. Write $\widetilde{f}(t_0) = (x_0, \widetilde{z}_0) \in Pr_1 \Omega \times \widetilde{Y}$. Let $f_n(t_0) = (x_n, z_n) \in \Omega$, $\widetilde{f}_n(t_0) = = gf_n(t_0) = (x_n, \widetilde{z}_n)$. Since $\widetilde{f}_n(t_0) \to \widetilde{f}(t_0)$, we have $x_n \to x_0$, $\widetilde{z}_n \to \widetilde{z}_0$. On the other hand, since Ω is relatively compact we can assume that (x_n, z_n) converges to $(x_0, z_0) \in \overline{\Omega}$. First we suppose that $(x_0, z_0) \in \Omega$. By assumption $f(t_0) = = (x_0, z_0)$. Then $gf(t_0) = (x_0, \widetilde{z}_0) = \widetilde{f}(t_0)$. Take a compact neighbourhood W_{t_0} of t_0 such that $\widetilde{f}(W_{t_0})$ is contained in $\widetilde{V}(x_0, \widetilde{z}_0)$ where $\widetilde{V}(x_0, \widetilde{z}_0)$ is a neighbourhood of $(x_0, \widetilde{z}_0)$ such that $g^{-1}\big|_{\widetilde{V}(x_0, \widetilde{z}_0)}$ is biholomorphic onto a neighbourhood $V(x_0, z_0)$ of (x_0, z_0). Then $\widetilde{f}_n \to \widetilde{f}$ on W_{t_0}. Thus $g^{-1}\widetilde{f}_n \to g^{-1}\widetilde{f}$ on W_{t_0}. Obviously, $g^{-1}\widetilde{f}_n = f_n$ and $g^{-1}f\big|_{W_{t_0} \cap A_{r_1}} = f$. This shows that f can be extended holomorphically on W_{t_0}. Covering the circle $|z| = r_1$ by such neighbourhoods W_{t_0} with $t_0 \in \Delta$, $|t_0| = r_1$, we infer that f is extended holomorphically on $A_{r'_1}$ with $r' < r_1$. This contradicts the choice of r_1. Hence $r_1 = 0$ and f is extended holomorphically on $\Delta \setminus \{0\}$. Obviously the sequence $\{f_n\}$ uniformly converges to f on every compact set of $\Delta \setminus \{0\}$. As above, for $t_0 = 0$ we can prove that f is extended holomorphically on Δ and the sequence $\{f_n\}$ converges to f in $O(\Delta, \Omega)$.

Thus it suffices to show that $(x_0, z_0) \in \Omega$. Indeed, assume that $(x_0, z_0) \overline{\in} \Omega$. Then $(x_0, z_0) \in \partial\Omega$. Since Ω satisfies the boundary condition, it follows that

$z_0 \in \partial\Omega_{x_0}$. By plurisubharmonicity of $\varphi . f_n$ we have $\sup_{\substack{n\in N\\ |\lambda - t_0| \leqslant \varepsilon}} \varphi . f_n(\lambda)$
$= \sup_{\substack{n\in N\\ |\lambda - t_0| = \varepsilon}} \varphi . f_n(\lambda) = c < +\infty$ for $\varepsilon > 0$ sufficiently small.

Thus $\varphi . f_n(t_0) \leqslant c$. Hence $d(z_n, \partial\Omega x_n) \geqslant e^{-c} > 0$.

Take $0 < 2\delta < e^{-c}$. Then $\Delta(z_n, 2\delta) = \{z \in \Omega x_n : d(z, z_n) < 2\delta\}$ is relatively compact in Ωx_n for every n. For every $z \in \Delta(z_0, \delta)$ we have $d(z, z_n) \leqslant$ $\leqslant d(z, z_0) + d(z_0, z_n) < \delta + d(z_0, z_n)$. Thus $d(z, z_n) < 2\delta$ for sufficiently large n. Since $\overline{\Delta}(z_n, 2\delta) \subset \Omega x_n$ and $x_n \to x_0$, it follows that $\overline{\Delta}(z_0, \delta) \subset \overline{\Omega}_{x_0}$. This is impossible, since $z_0 \in \partial\Omega_{x_0}$. Hence (i) is proved.

(ii) By [9] without loss of generality we may assume that X is a complex manifold. Suppose φ being not plurisubharmonic at $u_0 = (x_0, z_0) \in \Omega$. Then there exists a disc $\Delta = \{u_0 + \lambda u_1 : |\lambda| \leqslant 1\}$, $\Delta \subset \Omega$ and a holomorphic function f over $\mathbf{C}$ such that $\varphi(u_0 + \lambda u_1) \leqslant \operatorname{Re} f(\lambda)$, $|\lambda| = 1$. We may assume that $\varphi(u_0) = \max \varphi(u_0 + \lambda u_1)$ and $\varphi(u_0) > \operatorname{Re} f(0)$. Write $u_0 + \lambda u_1 = (x_0 + \lambda x_1, z_0 + \lambda z_1)$. Then $-\log d(z_0 + \lambda z_1, \partial\Omega_{x_0 + \lambda x_1}) \leqslant \operatorname{Re} f(\lambda) : |\lambda| = 1$.

$-\log d(z_0, \partial\Omega_{x_0}) > \operatorname{Re} f(0)$

and $-\log d(z_0, \partial\Omega_{x_0}) \geqslant -\log d(z_0 + \lambda z_1, \partial\Omega_{x_0 + \lambda x_1})$ with $|\lambda| \leqslant 1$

Since $-\log d(z_0, \partial\Omega_{x_0}) > \operatorname{Re} f(0)$, then $d(z_0, \partial\Omega_{x_0}) < e^{-Ref(o)} =$ $= |e^{-f(o)}|$. Put $\rho_1 = \dfrac{d(z_0, \partial\Omega_{x_0})}{|e^{-f(o)}|}$. Note that $\rho_1 < 1$.

Take a unit vector $a \in \mathbf{C}^n$ such that:

$$d(z_0, \partial\Omega_{x_0}) = d(z_0, z_0 + \rho_1 e^{i\alpha} . e^{-f(o)} a).$$

Since $u_0 + \lambda u_1 \in \Omega$,

$$d(u_0 + \lambda u_1, u_0 + \lambda u_1 + \rho e^{i\alpha} . e^{-f(o)} a) = \rho |e^{-f(o)}| < \rho_1 |e^{-f(o)}|$$
$= d(z_0, \partial\Omega_{x_0}) \leqslant d(z_0 + \lambda z_1, \partial\Omega_{x_0 + \lambda x_1})$ for every $|\lambda| \leqslant 1$ and $0 < \rho < \rho_1$.

Therefore $u_0 + \lambda u_1 + \rho . e^{i\alpha} . e^{-f(o)} a \in \Omega$. Thus for $0 < \rho < \rho_1$ we may define the maps $f_\rho : \Delta \to \Omega$ and $f_{\rho_1} : \Delta \setminus \{0\} \to \Omega$ by

$$f_\rho(\lambda) = u_0 + \lambda u_1 + \rho . e^{i\alpha} . e^{-f(o)} a,$$
$$f_{\rho_1}(\lambda) = u_0 + \lambda u_1 + \rho_1 . e^{i\alpha} . e^{-f(o)} a.$$

Obviously, $f_\rho(\lambda)$ converges to $f_{\rho_1}(\lambda)$ as $\rho \to \rho_1$ in $\mathcal{O}(\Delta \setminus \{0\}, \Omega)$. Since Ω satisfies the disc condition, $f_{\rho_1}(0) \in \Omega$. This contradicts the condition that $f_{\rho_1}(0) =$ $= u_0 + \rho_1 e^{i\alpha} . e^{-f(o)} a \in \partial\Omega$. The proof of the theorem is complete.

In the case where X is an one-dimensional complex space, we have the following result.

1.7. THEOREM. *Let X be an 1-dimensional complex space, Y and Ω satisfy the condition of Theorem 1.6 such that $Pr_1\,\Omega$ is not compact. Then Ω satisfies the disc condition if and only if the function $\varphi(x, z) = -\log d(z, \partial\Omega_x)$ is plurisubharmonic.*

Proof. By Theorem 1.6 it is enough to prove that if $\{f_n\} \subset \mathcal{O}(\Delta, \Omega)$ and $\{f_n\}$ converges in $\mathcal{O}(A_{r_1}, \Omega)$ to f for some $r_1 > 0$, then $\{f_n\}$ converges to f in $\mathcal{O}(\Delta, \Omega)$. As in Theorem 1.6, it suffices to show that if $t_0 \in \Delta$ for $|t_0| = r$, then f can be extended holomorphically on a neighbourhood of t_0. Since $\mathrm{Pr}_1\,\Omega$ is not compact and $\dim Pr_1\Omega = 1$, $Pr_1\Omega$ is Stein by [7]. Hence $Pr_1\,\Omega$ satisfies the disc condition. Using the notations of Theorem 1.6, it follows that $\{\widetilde{f_n}\}$ converges to $\widetilde{f}$ uniformly on every compact set of Δ. Put $f_n(t_0) = (x_n, z_n)$. Since Ω is relatively compact, (x_n, z_n) has a limit point (x_0, z_0). As in the proof of Theorem 1.6, $g(x_0, z_0) = (x_0, \widetilde{z_0})$. We may assume that $(x_0, z_0) \in \partial\Omega$. Put $f_n = (f_n^1, f_n^2)$. Then $f_n^1(t_0) \to x_0$. Take a neighbourhood U of x_0 in $\mathrm{Pr}_1\,\Omega$ such that (U, θ, Δ) is an analytic covering, where Δ is an open disc in $\mathbf{C}$ with the center at zero and $\theta^{-1}(0) = x_0$. Note that $\{\theta f_n^1\}$ converges to θf^1 uniformly on every compact set of Δ and $\theta f_n^1(t_0) \to \theta f^1(0) = 0$. Applying Hurwitz's theorem, without loss of generality we may assume that for every n, there exists an $\lambda_n \in \Delta$ such that $\lambda_n \to t_0$ and $\theta f_n^1(\lambda_n) = 0$. Thus $f_n(\lambda_n) = (x_0, f_n^2(\lambda_n))$ and $f_n^2(\lambda_n) \to z_0$, $z_0 \in \partial\Omega_{x_0}$. This contradicts the following fact:

$$\infty > \sup_{\substack{n\in N \\ |\lambda - t_0| = \varepsilon}} \varphi f_n(\lambda) \geqslant \lim_{n>N} \varphi f_n(\lambda_n) = \lim_{n>N} \varphi(x_0, f_n^2(\lambda_n)) =$$

$$- \lim \log d(f_n^2(\lambda_n), \partial\Omega_{x_0}) = \infty.$$

The proof of the Theorem is complete.

Remark: When $X = Y = \mathbf{C}$, Theorem 1.7 has been proved by Slodkowski [15]. In Theorems 1.6 and 1.7 the disc condition for open sets $\Omega \subset X \times Y$ are considered. Concerning the special case where sets Ω are constructed from the spectrum of holomorphic operator, we have the following theorem.

1.8. THEOREM: *Let X be a complex space satisfying the disc condition, B a Banach algebra and $T: X \to B$ a holomorphic map. Then the open set:*

$$\Omega_T = \{ (x,z) \in X \times \mathbf{C} : z \in \sigma T(x) \}$$

satisfies the disc condition.

Proof. For each $b \in B$ as ussal, $\sigma(b)$ denotes the spectrum of b. Suppose that $f_n \in \mathcal{O}(\Delta, \Omega_T)$ and $\{f_n\}$ converges in $\mathcal{O}(A_{r1}, \Omega_T)$ to $f \in \mathcal{O}(A_{r1}, \Omega_T)$. Since $X \times \mathbf{C}$ satisfies the disc condition, it follows that $f \in \mathcal{O}(\Delta, X \times \mathbf{C})$ and $\{f_n\}$ converges to f in $\mathcal{O}(\Delta, X \times \mathbf{C})$. Thus it suffices to show that $f(\Delta) \subset \Omega_T$. As in Theorem 1.6, it remains to prove that for every $t_o \in \Delta$, $|t_o| = r$ there exists a neighbourhood W_{t_o} of t_o such that $f(W_t) \subset W_T$. Write f_n in the form $f_n = (f_n^1, f_n^2)$ and $f = (f^1, f^2)$. Then, since X satisfies the disc condition, f_n^1 converges to f^1 in $\mathcal{O}(\Delta, X)$. Put $x_o = \lim f_n^1(t_o)$ and take a Stein neighbourhood U of x_o. We may suppose that $U \subset \Delta^m$, where Δ^m is an open polydisc in $\mathbf{C}^m$ for some m. By Cartan-Bungart's theorem [3], T can be extended to a holomorphic map $\widetilde{T}: \Delta^m \to B$. Consider $\Omega_{\widetilde{T}}^m = \{(x, z) \in \Delta^m \times B : z \in \sigma\widetilde{T}(x)\}$. Since U is an analytic set in Δ^m, it is easy to see that Ω_T is an analytic set in $\Omega_{\widetilde{T}}^m$. Considering a sufficiently small neighbourhood V_{t_o} of t_o, we infer that f_n converges to f in $\mathcal{O}(V_{t_o} \cap A_{r1}, \Omega_T^U) \subset \mathcal{O}(V_{t_o} \cap A_{r1}, \Omega_{\widetilde{T}}^m)$. Let $\varphi(x,z) = -\log d(z, \partial\Omega_{\widetilde{T}}^m)$. By the results of Soldkowski [15], $\varphi(x,z)$ is plurisubharmonic on $\Omega_{\widetilde{T}}^m$. Hence for a sufficiently small positive number δ such that $\{\lambda : |\lambda - t_o| \leqslant \delta\} \subset V_{t_o}$, we have

$$C = \sup_{|\lambda - t_o| < \delta} \varphi f_n(\lambda) < \infty.$$

Put $W = \left\{(x, z) \in \Omega_{\widetilde{T}}^m : \varphi(x, z) < C\right\}$ and $\Delta^{(t_o)} = \{\lambda : |\lambda - t_o| \leqslant \delta\}$.

Then $\varphi f_n(\lambda) < C$ for every $\lambda \in \Delta^{(t_o)}$. Hence $f_n(\Delta^{(t_o)}) \subset W$ for every $n \geqslant 1$; implying $\bigcup_{n=1}^{\infty} f_n(\Delta^{(t_o)}) \subset W$. If $(x, z) \in W$ then $\varphi(x, z) = -\log d(z, \partial\Omega_{\widetilde{T}(x)}^m) < C$. Hence $d(z, \partial\Omega_{\widetilde{T}(x)}^m) > e^{-C} > 0$. Take $0 < \delta < e^{-C}$, then $\overline{\Delta}(z, \delta) = \{z' \in C : |z' - z| \leqslant \delta\} \subset \Omega_{\widetilde{T}(x)}^m$. Let $x_o = \lim_{n\to\infty} f_n^2(t_o) \in \mathbf{C}$. Then $|z - z_o| < \frac{\delta}{2}$, $\Delta\left(z_o \frac{\delta}{2}\right) \subset \Omega_{\widetilde{T}(x)}^m$ for every $(x, z) \in W$. Put:

$$W_1 = \left\{(x, z \in W : |z - z_o| < \frac{\delta}{2}\right\}.$$

Since $f_n^2(t_o) \to z_o$ and $f_n^1(t_o) \to x_o$, $(f_n^1(t_o), f_n^2(t_o)) \in W$, it follows that

$x_o \in Pr_1 W$. We may assume that $f_n(\Delta^{(t_o)}) \subset W_1$ and $f(\Delta^{(t_o)} \setminus \{0\}) \subset W_1$. Let $x \in Pr_1 W_1$ and $z_x \in \mathbf{C}$ such that $(x, z_x) \in \Omega^m_{\tilde{T}}$ and $|z_x - z_o| < \frac{\delta}{2}$. Then $\Delta\left(z_o, \frac{\delta}{2}\right) \subset \Delta(z_x; \delta) \subset \Omega^m_{\tilde{T}(x)}$. Thus we may define holomorphic maps H and G on $Pr_1 W$ with values in $GL(A(\overline{\Delta}\left(z_o, \frac{\delta}{2}\right), B)$, where $A(\overline{\Delta}, B)$ denotes the Banach algebra of all continuous maps from $\overline{\Delta}$ into B, holomorphic in Δ by the formulas:

$$H(x)(z) = \tilde{T}(x) - z,$$

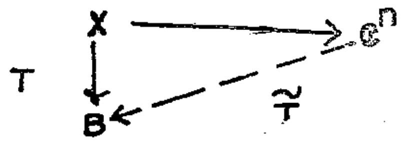
$$G(x)(z) = (T(x) - z)^{-1}.$$

By Benke-Zomer's Theorem, H and G can be extended to holomorphic maps $\widetilde{H}$ and $\widetilde{G}$ respectively on a neighbourhood of $f^1(\overline{\Delta})$ containing x_o. Since $HG = GH =$ $=$ id on $Pr_1 W_1$, we have $\widetilde{H}\widetilde{G} = \widetilde{G}\widetilde{H} =$ id. This implies that $(x_o, z_o) \in \Omega_T$ and hence $f(W_{t_0}) \subset \Omega_T$ for some neighbourhood W_{t_0} of t_0. The proof of the theorem is complete.

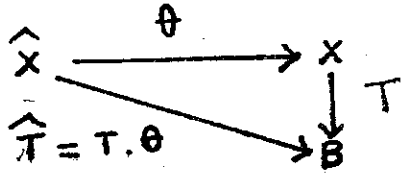
When X is Stein, we have the following

1.9. COROLLARY. *Let X be a Stein space and $T: X \to B$ be a holomorphic map. Then $\Omega_T = \{(x, z) \in X \times \mathbf{C} : z \overline{\in} 6\ T(x)\}$ is Stein.*

Proof. First, suppose that X is embedded in $\mathbf{C}^n$ with some n. Consider the diagram:



By Cartan-Bungart Theorem [3] T can be extended to a holomorphic map $\tilde{T}: \mathbf{C}^n \to B$. By Theorem 1.8, $\Omega_{\tilde{T}}$ satisfies the disc condition. Hence $\Omega_{\tilde{T}}$ is Stein. Since Ω_T is an analytic subset of $\Omega_{\tilde{T}}$, Ω_T is also Stein. In general, consider the commutative diagram:



where $\theta : \widehat{X} \to X$ is the normalization of X. Let $\widetilde{\theta} : \Omega_{\widetilde{T}} \to \Omega_T$, $(\widetilde{x}, z) \to (\theta\widetilde{x}, z) = (x, z)$ be the map induced by θ. Since θ is finite, proper and surjective, by [9] it suffices to prove that $\Omega_{\widetilde{T}}$ is Stein. Let $\widehat{X} = \bigsqcup_{j \in I} W_j$, where W_j are irreducible branches of $\widehat{X}$. Then $\Omega_{\widetilde{T}} = \bigsqcup_{j \in I} \Omega_{\widetilde{T}_j}$, where $\widetilde{T}_j = \widetilde{T}|_{W_j}$.

Since W_j is normal and irreducible, it can be embedded into C^{m_j} for some m_j by [7]. Hence $\Omega_{\widetilde{T}_j}$ is Stein for every $j \geqslant 1$. Thus $\Omega_{\widetilde{T}} = \bigsqcup_{j \in I} \Omega_{\widetilde{T}_j}$ is also Stein.

By Theorem 1.8, the following example shows that the disc condition and the boundary condition are independent:

1.10. *Example:* we consider the Banach algebra $B = l_\infty = \{(\lambda_n) : \lambda_n \in \mathbf{C}, \sup |\lambda_n| < \infty\}$.

Obviously, $\sigma(f) = \overline{\{\lambda_n\}}$ for every $f = (\lambda_n) \in B$. Now take two sequences (α_n) and (β_n) such that:

$$\overline{\{\alpha_n\}} = \{z : \frac{1}{3} \leqslant |z| \leqslant 1\}$$

$$\overline{\{\beta_n\}} = \{z : |z| \leqslant \frac{2}{3}\}.$$

Put $\Delta = \{z : |z| < 1\}$. Consider the map $f : \Delta \to B$ given by $f(\lambda) = (\alpha_0, \lambda\beta_0, \alpha_1, \lambda\beta_1, \ldots, \alpha_n, \lambda\beta_n, \ldots)$. Then $\sigma(f) = \overline{\mathcal{A}} \cup \overline{\mathfrak{B}} = \overline{\Delta}_{\frac{1}{3}, 1} \cup \overline{\mathfrak{B}}$, where

$\mathcal{A} = \{\alpha_n\}$ and $\mathfrak{B} = \{\lambda\beta_n\}$. Since $|\lambda\beta_n| \leqslant \frac{2}{3}|\lambda|$ it follows that $\mathfrak{B} \ll \Delta_{|\lambda|}$.

Take an increasing sequence of the positive number $\{\lambda_k\}$, $\lambda_k \to \frac{1}{2}$ as $k \to \infty$.

Consider $\{\lambda_k . \beta_n\}$. We have $|\lambda_k . \beta_n| \leqslant \frac{2}{3} |\lambda_k| < \frac{1}{3}$. For each k, take z_k such that $\frac{2}{3}\lambda_k < |z_k| < \frac{1}{3}$ and $z_k \to \frac{1}{3}$. Then $\sigma f(\lambda_k) = \overline{\Delta}_{\frac{1}{3}, 1} \cup \overline{\{\lambda_k . \beta_n\}}$.

Since $z_k \bar{\in} \overline{\Delta}_{\frac{1}{3}, 1}$ and $z_k \bar{\in} \overline{\{\lambda_k . \beta_k\}}$ we have $z_k \bar{\in} \sigma f(\lambda_k)$. Obviously $(\lambda_k, z_k) \to \left(\frac{1}{2}, \frac{1}{3}\right)$ and $\frac{1}{3} \bar{\in} \partial\Omega_f^{\frac{1}{2}}$. Thus Ω_f does not satisfy the boundary condition.

In the Example 1.10, $B = l_\infty$ is not an algebra of holomorphic functions. For such algebra we have the following theorem.

1.11. THEOREM. *Let D be an open, relatively compact set in $\mathbf{C}$, $B = \bar{A}(D)$ be the algebra of the holomorphic in D and continuous on $\bar{D}$ functions, Z be a complex space and $T : Z \to B$ be a holomorphic map. Then the set*

$$\Omega_T = \{(x,z) \in X \times \mathbf{C} : z \,\bar{\ni}\, \sigma\, T(x)\}$$

satisfies the boundary condition.

Proof. Suppose that Ω_T does not satisfy the boundary condition. Then there exists $(x_n, z_n) \in \Omega_T$ such that (x_n, z_n) converges to $(x_0, z_0) \in \partial\Omega_T$, but $z_0 \,\bar{\in}\, \partial\Omega_T^{x_0}$. Thus $z_0 \in \operatorname{Int} \sigma\, T(x_0)$. Put $a_n = Tx_n \in B$, $a = T(x_0) \in B$, $a = T(x_0) \in B$. Note that $\sigma(a) = \widehat{a}(M_B)$. Obviously, $\widehat{a}(M_B) = \widehat{a}(\bar{D}) = a(D)$. Since $z_0 \in \operatorname{Int} \sigma(a) = \operatorname{Int} a(\bar{D})$, there exists $\omega \in \bar{D}$: $a(\omega) = z_0$. We may assume $z_0 = 0$ therefore $a(\omega) = 0$. Since $0 \in \operatorname{Int} \sigma(a)$ and $\dim D = 1$, we have $\omega \in D$. Take a neighbourhood U of ω such that $\bar{U} \subset D$ and consider the analytic set $V(a) = \{a^{-1}(0)\}$ in U. Since $a \neq 0$, we can assume $V(a) \cap U = \{\omega\}$.
Consider the sequence $\{f_n\} = \{z_n - a_n\} \subset A(\bar{U})$. Since $\{f_n\}$ converges to f uniformly on all compact sets in U, where $f = -a$, and $a(\omega) = 0$, by Hurwit's Theorem there exists a number N such that $n > N$, $V(f_n) \cap \bar{U} \neq \phi$. Thus $V(f_n) \cap D \neq \phi$. This contradicts the condition $z_n \in \sigma(a_n)$. The theorem is proved.

2. DISC CONDITION AND HOLOMORPHIC EXTENSION

First we recall some definitions.

2.1. DEFINITION. *Let X be a complex space. We say that X satisfies the Hartogs extension condition if every holomorphic map $f : H_k(r) \to X$, where $H_k(r) = \{ z \in \Delta^k : | z_j | < r, 1 \leqslant j \leqslant k-1$ or $| z_k | > 1-r\}$ can be extended holomorphically on Δ^k* (cf. [13]).

2.2. DEFINITION. *Let X be a complex space, Ω an open set in X. We say that Ω* is *a local Stein if for every $p \in \partial\Omega$ there exists a neighbourhood U of p such* that *$\Omega \cap U$ is Stein.* (*cf.* [5]).

In this section we prove the following theorem.

2.3. THEOREM. *Let X be a complex manifold satisfying the disc condition, Ω an open set in X. Then the following conditions are equivalent:*

(i) *Every sequence $\{f_n\} \subset \mathcal{O}(\Delta, \Omega)$ converging in $\mathcal{O}(\Delta \setminus 0, \Omega)$ converges in* $\mathcal{O}(\Delta, \Omega)$.

(ii) *Ω satisfies Hartogs extension condition.*

(iii) Ω *is locally Stein.*

(iv) *The restriction map* $\mathcal{O}(M, \Omega) \to \mathcal{O}(M, \Omega)$ *is surjective for every Riemann domain M over a Stein manifold.*

Proof: We prove the theorem in the following scheme:

(i) $\to$ (ii) $\to$ (iii) $\to$(i) and (ii) $\longleftrightarrow$ (iv).

(i) $\to$ (ii). It is known by [13] that if Ω satisfies the hypothesis (i) then Ω satisfies Levi extension condition and hence Ω satisfies Hartogs extension condition.

(ii) $\to$ (iii). Let $p \in \partial\Omega$. Take a Stein neighbourhood V of p. By Docquier-Grauert's Theorem, it suffices to prove that $\Omega \cap V$ is Hartogs convex. Consider a holomorphic embedding $\varphi: H_k(r) \to \Omega \cap V$.

Since V is Stein, φ can be extended holomorphically to $\varphi_1 : \Delta^k \to V$. On the other hand, since Ω satisfies Hartogs extension condition, φ can be extended holomorphically to φ_2: $\Delta^k \to \Omega$. By uniqueness we have $\varphi_1 \equiv \varphi_2$. Thus φ is extended holomorphically to $\psi: \Delta^k \to V \cap \Omega$ and hence $V \cap \Omega$ is Hartogs couvex.

(iii) $\to$ (i). Let $\{f_n\} \in \mathcal{O}(\Delta, \Omega)$ such that $\{f_n\}$ converges to some f in $\mathcal{O}(\Delta \setminus \{0\}, \Omega)$. Since X satisfies the disc condition $f \in \mathcal{O}(\Delta, X)$ and $\{f_n\}$ converges to f in $\mathcal{O}(\Delta, X)$. We shall prove that $f(0) \in \Omega$. Obviously, $f(0) \in \overline{\Omega}$. Since Ω is locally Stein there exists a neighbourhood V of $f(0)$ such that $V \cap \Omega$ is Stein. Take a sufficiently small neighbourhood Δ_ε of 0 such that for $n > N$, $f_n(\overline{\Delta_\varepsilon}) \subset V \cap \Omega$. Since $V \cap \Omega$ satisfies the disc condition, $f \in \mathcal{O}(\Delta_\varepsilon, V \cap \Omega)$. Hence $f(0) \in \Omega$.

Finally, (ii) and (iv) are equivalent by [13].

2.5. *Remark.* J. E. Fornaess and R. Narasimhan [5] have constructed a Stein space X with dim $X \geqslant 2$ such that $U \setminus \{x_o\}$ satisfies the disc condition for every Stein neighbourhood U of x_o in X. Since dim $X \geqslant 2$ it follows that $X \setminus \{x_o\}$ is not locally Stein at x_o.

2.6. THEOREM. *There exists a complex manifold X which is an increasing union of Stein open sets and does not satisfy, the disc condition.*

Proof. As in [6] for each natural n we put:

$$M_n = \left\{ (z, w, \eta) \in \mathbf{C}^3 : w\eta = p_n(z),\ p_n(z) = \prod_{k=1}^{n} \left(z - \frac{1}{k} \right) \right\}.$$

Obviously, M_n are closed submanifold of $\mathbf{C}^3$. Hence M_n are Stein. For each n, consider the map: $\gamma_n : M_n \to M_{n+1}$

$$(z, w, \eta) \to \left(z, w, \eta\left(z - \frac{1}{n+1}\right)\right).$$

Clearly, γ_n is biholomorphic from M_n onto $M_{n+1} \setminus \left\{\frac{1}{n+1} \times \mathbf{C}^2\right\}$. Thus we can define $M = \underrightarrow{\lim}(M_n, \gamma_n)$. We shall prove that M does not satisfy the disc condition. Let $f_n \in \mathcal{O}(\Delta, M)$ be a function defined by

$$f_n(\lambda) = \left(\lambda, \lambda - \frac{1}{n+1}, p_n(\lambda)\right).$$

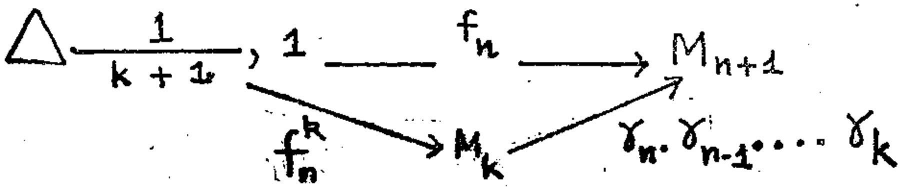
Then $f_n(\Delta) \subset M_{n+1}$. We prove that $\{f_n\}$ converges $\mathcal{O}(\Delta \setminus \{0\}, M)$. For each k, consider $f_n^k \in \mathcal{O}(\Delta_{\frac{1}{k+1}, 1}, M)$ defined by:

$$f_n^k(\lambda) = \left(\lambda, \lambda - \frac{1}{n+1}, \frac{p_k(\lambda)}{\lambda - \frac{1}{n+1}}\right), \lambda \in \Delta_{\frac{1}{k+1}, 1}$$

where $\Delta_{\frac{1}{k+1}, 1} = \left\{z \in \Delta : \frac{1}{k+1} < |z| < 1\right\}$.

Note that $f_n^k \in \mathcal{O}\left(\Delta_{\frac{1}{k+1}, 1}, M_k\right)$. For every $n > k$, $\{f_n^k\}$ converges in $\mathcal{O}\left(\Delta_{\frac{1}{k+1}, 1}, M\right)$ to the function f^k given by $f^k(\lambda) = \left(\lambda, \lambda, \frac{p_k(\lambda)}{\lambda}\right)$.

On the other hand, since the diagram



is commutative, we have $\gamma_q \cdots \gamma_p \cdot f^p = f^q$, where p, q are natural numbers with $p < q$. Thus we can define a map $f : \Delta \setminus \{0\} \to M$ by setting $f(z) = f^k(z)$ for $z \in \Delta_{\frac{1}{k+1}, 1}$. Since $\frac{1}{k+1} \to 0$, the sequence $\{f_n\}$ converges in $\mathcal{O}(\Delta \setminus \{0\}, M)$ to f. Now suppose that M satisfies the disc condition. Then $\{f_n\}$ converges to $f \in \mathcal{O}(\Delta, M)$ in $\mathcal{O}(\Delta, M)$. Consider $\overline{\Delta}_\varepsilon = \{z : |z| \leqslant \varepsilon\}$, $0 < \varepsilon < 1$.

Since $\{f_n\}$ uniformly converges on $\overline{\Delta}_\varepsilon$, it follows that $\bigcup_{n=1}^{\infty} f_n(\overline{\Delta}_\varepsilon)$ is compact;

Since $M = \bigcup_{k=1}^{\infty} M_k$, $M_k \subset M_{k+1}$ and M_k is open in M for every $k \geqslant 1$, there exists an k_0 such that $\bigcup_{n=1}^{\infty} f_n(\overline{\Delta}_\varepsilon) \subset M_{k_0}$. Hence $f^{(k_0)}(\lambda) = \left(\lambda, \lambda, \frac{p_{k_0}(\lambda)}{\lambda}\right)$ for all $\lambda \in \Delta_\varepsilon \setminus \{0\}$. Thus, $f^{k_0}(\lambda)$ can be extended holomorphically into Δ_ε. This is impossible, because $p_{k_0}(0) \neq 0$. Hence M does not satisfy the disc condition. The theorem is proved.

2.7. *Remark*: Let M_i be open sets satisfying the Hartogs extension condition and $M_i \subset M_{i+1}$ for $i = 1, 2, \ldots$ Then $M = \bigcup_{i=1}^{+\infty} M_i$ satisfies the Hartogs extension condition.

Proof: Assume that $f: H_k(r) \to M$ is a holomorphic map. Let us set ${}^{(1-\varepsilon)}\Delta^k = \{z \in \Delta^k : |z_i| < 1-\varepsilon, 1 \leqslant i \leqslant k\}$, For any $s < 1$, we denote:

$$H_{k,s}(r) = \left\{ z \in \Delta^k : |z_i| < r < 1, 1 \leqslant i \leqslant k-1, |z_k| < s \right\} \cup$$

$$\cup \left\{ z \in \Delta^k : |z_k| > s - r \right\}.$$

Take $r' < r$ and $\varepsilon > 0$ sufficiently small such that: $\varepsilon + r' < r$. Then

$$\overline{{}^{(1-\varepsilon)}H_k(r')} \subset {}^{(1)}H_k(r).$$

By compactness of $f\left[\overline{{}^{(1-\varepsilon)}H_k(r')}\right]$ there exists n_0 satisfying

$$f\left[\overline{{}^{(1-\varepsilon)}H_k(r')}\right] \subset M_{n_0}.$$

By hypothesis, f can be extended uniquely to a holomorphic map f_ε : ${}^{(1-\varepsilon)}\Delta^k \to M$. Thus $\{f_\varepsilon\}$ defines a holomorphic map $\tilde{f}: \Delta^k \to M$ which is an extension of f.

3. DISC CONDITION AND LOCAL MAXIMUM PRINCIPLE

3.1. DEFINITION [15]: *We say that an algebra A of continuous functions on a locally compact space X is a maximum modulus algebra if for every $f \in A$, and for every relatively compact open set $N \subset X$, we have:*

$$\max |f|_{\overline{N}} = \max |f|_{\overline{N} \setminus N}.$$

A connection between the Steinness of the open sets in $\mathbf{C}^n$ and the local maximum principle is established in the following theorem.

3.2. THEOREM. *Let Ω be an open set in $\mathbf{C}^n$. Then the following three conditions are equivalent:*

(i) Ω is Stein.

(ii) For every analytic set $Z \subset \mathbf{C}^n$ with dim $Z \geqslant 2$, the algebra of all polynomials restricted to $Z \setminus Z \cap \Omega$ is a maximum modulus algebra.

(iii) For every analytic set $Z \subset \mathbf{C}^n$ with dim $Z \geqslant 2$ and for every plurisubharmonic function φ on a neighbourhood of V, which is relatively open and relatively compact in $Z \setminus Z \cap \Omega$, the following relation holds:

$$\max \varphi|_{\overline{V}} = \max \varphi|_{\overline{V} \setminus V}. \quad (\star)$$

Proof: (ii) $\to$ (iii). Since (iii) has local property we may assume that V is contained in an open polydisc U of $\mathbf{C}^n$ and φ is plurisubharmonic on U. Let $\widetilde{\mathcal{P}}(U)$ denote the set of plurisubharmonic functions on U satisfying ($\star$). Then it is easy to check that:

a) For every $f \in \mathcal{O}(U)$, $|f| \in \widetilde{\mathcal{P}}(U)$.

b) If $\{f_k\} \subset \mathcal{O}(U)$ and c_k are constant, then $\sup (c_k \log f_k) \in \widetilde{\mathcal{P}}(U)$.

c) If $\{\psi_j\} \subset \widetilde{\mathcal{P}}(U)$ and $\psi_j \downarrow \psi$, then $\psi \in \widetilde{\mathcal{P}}^k(U)$.

Since every plurisubharmonic function on U is the limit of a decreasing sequence of the continuous plurisubharmonic functions (see [12]) and every continuous plurisubharmonic function on U can be represented as in b) (see [14]), it follows that ($\star$) holds for φ.

(iii) $\longrightarrow$ (i). We shall prove this implication by induction on n. If $n = 2$, this has been proved by Slodkowski [15]. Assume now that the implication has been proved for $\mathbf{C}^m$ with $m < n$ and $n > 2$.

We shall prove the statement (i) for $\mathbf{C}^n$. For $z \in \mathbf{C}^n$, we write $z = (x, \omega) \in \mathbf{C} \times \mathbf{C}^{n-1}$, and put $\Omega_x = \Omega \cap (x \times \mathbf{C}^{n-1})$, $x \in Pr_1 \Omega$.

By Theorem 1.6, it suffices to prove that the function $\varphi(x, z) = -\log d(z, \partial\Omega_x)$ is plurisubharmonic. By [5] it remains to show that the restriction of φ on every complex line is subharmonic.

First consider the case $L = \{x\} \times \mathbf{C}^{n-1}$. Then by inductive hypothesis $-\log d(z, \partial \Omega_x) \mid_{\Omega_x \cap L}$ is subharmonic.

Now consider $L = \{(\lambda, a\lambda + b) : \lambda \in \mathbf{C}, a, b \in \mathbf{C}^{n-1}\}$.

$\varphi(\lambda, a\lambda + b) = -\log d(a\lambda + b, \partial\Omega_\lambda) = -\log \min \{\| \omega - a\lambda - b \| :$

$\omega \in \partial\Omega_\lambda\} = \log \max \{\| \omega - a\lambda - b \|^{-1} : \omega \in \partial\Omega_\lambda\}$.

Then $L \cap \Omega = \{(\lambda, a\lambda + b) : a\lambda + b \in \Omega_\lambda\}$. Put $U = \{\lambda \in \mathbf{C} :$ $a\lambda + b \in \Omega_\lambda\}$ and $\beta(\lambda) = \max \| \omega - a\lambda - b \|^{-1}$. For each $a \in \mathbf{C}$ and each polynomial $p(\lambda)$, consider the function Ψ on U defined by:

$$\Psi(\lambda) = |\exp p(\lambda)| \exp |e^{a\lambda}| \beta(\lambda), \quad \lambda \in U.$$

By Cole's Lemma [1] and Rado's Theorem [1], it is enough to prove that ψ has local maximum property. Take $\lambda^* \in U$ and consider an open polydisc D in $\mathbf{C}$ containing λ^* and $\overline{D} \subset U$. Put $V = C^n \setminus \Omega \cap (D \times \mathbf{C}^{n-1})$. Then V is relatively open in $\mathbf{C}^n \setminus \Omega$. Obviously $\overline{V} \setminus V \subset (\partial D \times \mathbf{C}^{n-1}) \cap (\mathbf{C}^n \setminus \Omega)$. Then the function $\theta(\lambda, \omega) = |\exp p(\lambda)| \exp |e^{a\lambda}| \gamma(\lambda,\omega)$ (where $\gamma(\lambda, \omega) = \| \omega - \lambda a - b \|^{-1}$) is plurisubharmonic on a neighbourhood of $\overline{V}$. By (iii) we have

$$\psi(\lambda^*) = \max_{\omega \in \partial\Omega_{\lambda^*}} \theta(\lambda^*, \omega) \leqslant \max_{(\lambda,\omega) \in \overline{V}} \theta(\lambda, \omega) = \max \theta|_{\overline{V} \setminus V} \leqslant$$

$\max \theta \mid (\partial D \times \mathbf{C}^{n-1}) \cap (\mathbf{C}^n \setminus \Omega) = \max \Psi \mid \partial D$. Thus Ψ has the local maximum property on U. Hence $\varphi/_{L\cap\Omega}$ is subharmonic. By Theorem 1.6, Ω is Stein.
(iii) $\to$ (ii) is trivial.

(i) $\to$ (ii). Let Z be an analytic set in $\mathbf{C}^n$ with $\dim Z \geqslant 2$. Put $\Omega_Z = \Omega \cap Z$. Then Ω_Z is Stein. Hence there exist a strongly plurisubharmonic exhaution function θ on Ω_Z by [5]. For every $c > 0$, put $\Omega_Z^c = \{x \,(\in \Omega_Z : \theta(x) < c\}$. Then Ω_Z^c is relatively compact in Ω_Z. First we prove $(*)$ for Ω_Z^c. Let V be a relatively open set in $Z \setminus Z \cap \Omega_Z^c$ and φ be a plurisubharmonic function on a neighbourhood of $\overline{V}$. Then $\varphi + \frac{\theta}{n} \to \varphi$ as $n \to \infty$ and $\varphi + \frac{\theta}{n}$ is strongly plurisubharmonic on a neighbourhood of $\overline{V}$. Obviously, if $(\star)$ holds for every $\varphi + \frac{\theta}{n}$, then $(\star)$ also holds for φ. Thus we may suppose φ is strongly plurisubhamonic. Let $\tilde{\theta}$ be a strongly plurisubharmonic extension of θ on a neighbourhood $\tilde{\Omega}$ of Ω_Z in $\mathbf{C}^n$. By [12] such an extension exists. Put

$$\tilde{\Omega}_c = \{x \in \tilde{\Omega} : \tilde{\theta}(x) < c\}.$$

Then $\partial\Omega_c \cap Z = \partial\Omega_Z^c$. Let $\max \varphi|_{\overline{V}} = \varphi(z_0)$. By maximum principle for plurisubharmonic functions $z_0 \in \partial V$. But $\partial V \subset (\overline{V} \setminus V) \cup (V \cap \partial\Omega_Z^c)$. If $z_0 \in \overline{V} \setminus V$ the theorem is proved. Consider the case $z_0 \in V \cap \partial\Omega_Z^c$. Then $z_0 \in \partial\tilde{\Omega}_c$. Since $\tilde{\Omega}_c$ is strongly pseudo-convex, there exists a polynomial $P(z_1, z_2, \ldots, z_n)$ and a ball $B(z_0, r)$ such that

$$B(z_0, r) \cap \tilde{\tilde{\Omega}}_c \cap V(P) = \{z_0\}$$

where $V(P)$ denotes a zero-set of the polynomial P (Cf. [7]. Take a neighbourhood W of z_0 sufficiently small such that $W \subset B(z_0, r) \cap V(P)$. Then $W \cap \tilde{\tilde{\Omega}}_c = \phi$. Hence $W \subset Z \setminus \Omega_Z^c$. Since $V \subset Z \setminus \Omega_Z^c$ and V is also a neighbourhood of z_0, we may take W such that $W \subset V$, Then $\max \varphi_{/W \cap V(P) \cap Z} = \varphi(z_0)$. Since $\dim (W \cap V(P) \cap Z) \geqslant 1$, φ is constant on $W \cap V(P) \cap Z$. This contradicts the strong plurisubharmonicity of φ. In general case, let V be a relatively open set in $Z \setminus Z \cap \Omega$, φ be a plurisubharmonic function on a neighbourhood of $\overline{V}$. For each n take V_n a relatively open set V_n in $Z \setminus \Omega_Z^n$

such that $V_n \supset V_{n+1}$ and $V = \bigcap_{n=1}^{+\infty} V_n$. We may assume that φ is plurisubharmonic on V_n for every $n \geqslant 1$. Then $\max \varphi \mid_{\overline{V}} = \lim \max \varphi \mid_{\overline{V}_n} =$

$= \lim \max \varphi \mid_{\overline{V}_n \setminus V_n} = \max \varphi \mid_{\overline{V} \setminus V}$. The proof of the theorem is complete.

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