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LOCAL CONTROLLABILITY FOR LIPSCHITZIAN DISCRETE-TIME SYSTEMS

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1. INTRODUCTION

Given a subset Ω of the Euclidean space R^m and mappings $f_k : R^n \times R^n \to R^n$ $(k = 0, 1, 2, \ldots)$, let us consider the discrete-time system

$$\begin{cases} x_{k+1} = f_k(x_k, u_k), \\ x_k \in R^n, \ u_k \in \Omega \ (k = 0, 1, 2, \ldots) \end{cases} \tag{$\mathcal{D}$}$$

and suppose that

$$0 \in \operatorname{int} \Omega, \tag{1}$$

$$f_k(0, 0) = 0 \qquad (k = 0, 1, 2, \ldots). \tag{2}$$

The aim of this paper is to give some sufficient conditions for local reachability and local controllability of this system under the assumption that f_k are locally Lipschitzian.

When the mappings f_k are continuously differentiable, the controllability problem was studied by L. Weiss [1]. Our results extend Theorem 2 of that paper. The reader is referred to [2] and the references therein for the development of the theory of controllability of discrete-time systems during the last twenty years.

Our main result is Theorem 3 formulated in Section 2 and proved in Section 3. Some corollaries including the mentioned above theorem of Weiss are given in Section 4.

Throughout the paper, the following notations will be used:
$N(x)$: the collection of all neighbourhoods of x,

$B(x, \delta)$: the open ball around x with radius δ,

A^*: the conjugate operator of A (or, the transposed matrix of A if A is a matrix),

int M: the interior of M.

Consider the system ($\mathcal{D}$) and suppose that the conditions (1) and (2) are satisfied. As in [1], if $u_0, u_1, \ldots, u_{M-1}$ are some vectors (controls) in Ω, $x_0 = a$, $x_M = b$ and $x_1, x_2, \ldots, x_M$ are defined by ($\mathcal{D}$) with the chosen vectors $u_0, u_1, \ldots, u_{M-1}$, then we denote the fact by the symbol

$$a \xrightarrow[u_0, u_1, \ldots, u_{M-1}]{(\mathcal{D})} b.$$

DEFINITION 1. *System ($\mathcal{D}$) is said to be locally reachable (from the origin) after M steps if there exists a neighbourhood $U \in N(0)$ with the property that for every $x \in U$, there are M vectors $u_0, u_1, \ldots, u_{M-1} \in \Omega$ such that*

$$0 \xrightarrow[u_0, u_1, \ldots, u_{M-1}]{(\mathcal{D})} x.$$

DEFINITION 2. *System ($\mathcal{D}$) is said to be locally controllable (into the origin) after M steps if one can find a neighbourhood $U \in N(0)$ with the property that for $x \in U$, there are M vectors $u_0, u_1, \ldots, u_{M-1} \in \Omega$ such that*

$$x \xrightarrow[u_0, u_1, \ldots, u_{M-1}]{(\mathcal{D})} 0.$$

Recall that a mapping f from R^p into R^q is called locally Lipschitzian at $\bar{x} \in R^p$ if there exist a neighbourhood $U \in N(\bar{x})$ and a number $L > 0$ such that $\| f(x_1) - f(x_2) \| \leqslant L \| x_1 - x_2 \|$, for every $x_1, x_2 \in U$.

As usual, the set of linear operators from R^p into R^q is denoted by $L(R^p, R^q)$. Every element of this set can be identified with a $m \times n$ — matrix.

DEFINITION 3. *A closed convex set $\Delta \subset L(R^p, R^q)$ is called a shield for the mapping f at $\bar{x} \in R^q$ if, for any $\varepsilon > 0$, there exists $U \in N(\bar{x})$ such that: whenever $x_1, x_2 \in U$, there is an element $A \in \Delta$ satisfying*

$$\| f(x_1) - f(x_2) - A(x_1 - x_2) \| \leqslant \varepsilon \| x_1 - x_2 \|.$$

This definition of shield is equivalent to the original version given in [3].

Now let us state two known results.

THEOREM 1. *Suppose that f is a mapping from an open set $E \subset R^p$ into R^q. If $\Delta \subset L(R^p, R^q)$ is a compact shield for f at $\bar{x} \in E$ and every $A \in \Delta$ is surjective, then $f(\bar{x}) \in \text{int}\, f(U)$ for every $U \in N(\bar{x})$.*

THEOREM 2. *Suppose that f is a mapping from an open set $E \subset R^p \times R^q$ into R^p and $f(\bar{x}, \bar{y}) = 0$, where $(\bar{x}, \bar{y}) \in E$. Let Δ be a compact shield for f at $(\bar{x}, \bar{y})$ such that for every $C \in \Delta$ the operator $A = pr_x C$ is nonsingular. Then, there exist $U \in N(\bar{x})$, $V \in N(\bar{y})$ and a locally Lipschitzian mapping $\varphi: V \to U$ such that: for every $(x, y) \in U \times V$, $f(x, y) = 0$ if and only if $x = \varphi(y)$.*

Here, $(pr_x C)(h) := C(h, 0)$ for every $h \in R^p$.

Theorems 1 and 2 are due to A. D. Ioffe [4] and P.H. Dien [5]. Some extensions of these results can be found in [8] and [9].

Before going further, let us recall Clarke's concept of Generalized Jacobian of a mapping f from R^p into R^q which is locally Lipschitzian at $\bar{x}$: the Generalized Jacobian of f at $\bar{x}$, denoted $\partial f(\bar{x})$, is the convex hull of the set of all the limits of the form $A = \lim_{x_k \to \bar{x}} f'(x_k)$, where $f'(x_k)$ is the Fréchet derivative of f at x_k. It is known from [6] that $\partial f(\bar{x})$ is a compact shield for f at $\bar{x}$.

Suppose the mappings f_k are locally Lipschitzian at $(0, 0) \in R^n \times R^m$. Denote by Δ_k the Generalized Jacobian of f_k at $(0, 0)$ and define the following sets (called the Partial Generalized Jacobians of f_k at $(0, 0)$): $\Delta_{k,x} = \{pr_x C : C \in \Delta_k\}$, $\Delta_{k,u} = \{pr_u C : C \in \Delta_k\}$, where $(pr_u C)(v) := C(0, v)$ for all $v \in R^m$.

We are now in a position to state the main result of the paper.

THEOREM 3. *Assume that the conditions (1) and (2) are satisfied and f_k ($k = 0, 1, 2, \ldots$) are locally Lipschitzian at $(0, 0)$. If there exist $P_0, P_1, \ldots, P_{M-1} \in L(R^n, R^m)$ such that the convex hull of the set*

$$\Sigma = \{ A_{M-1} \ldots A_1 B_0 P_0 + A_{M-1} \ldots A_2 B_1 P_1 + \ldots + B_{M-1} P_{M-1} : A_k \in \Delta_{k,x} \ (k = 1, 2, \ldots, M-1),\ B_k \in \Delta_{k,u} (k = 0, 1, \ldots, M-1)\} \tag{3}$$

consists only of nonsingular operators, then ($\mathcal{D}$) is locally reachable and locally controllable after M steps.

3. PROOF OF THEOREM 3

Let $P_0, P_1, \ldots, P_{M-1}$ be operators such that the convex hull of Σ consists only of nonsingular operators.

Proof of reachability. For every $v \in R^n$ let us set

$$\begin{cases} x_1(v) = f_0(0, P_0, v), \\ x_2(v) = f_1(x_1(v), P_1 v), \\ \dots \\ x_M(v) = f_{M-1}(x_{M-1}(v), P_{M-1} v). \end{cases} \tag{4}$$

Since $0 \in \operatorname{int} \Omega$ and $P_k v$ $(k = 0, 1, \dots, M-1)$ are continuous mappings, we can find a real number $\bar{\sigma} > 0$ such that $P_k v \in \Omega$ for every $v \in B(0, \bar{\sigma})$ and every $k \in \{0, 1, \dots, M-1\}$. Define $\varphi(v) = x_M(v)$ $(v \in R^n)$. According to (2) and (4) we have $\varphi(0) = 0$. Hence, to prove the local reachability of system $(\mathcal{D})$, it remains to show that

$$0 \in \operatorname{int} \varphi(U), \tag{5}$$

for every $U \in N(0)$.

To this end we shall prove that, for every $\varepsilon > 0$, there is a real $\sigma > 0$ such that, if $v, v' \in B(0, \sigma)$, then there exists $R \in \Sigma$ satisfying

$$\| \varphi(v) - \varphi(v') - R(v - v') \| \leqslant \varepsilon \| v - v' \| . \tag{6}$$

Given a positive number ε, we set $x_0(v) = 0$ and pick a positive number ε_1 such that

$$\varepsilon_1 < \frac{\varepsilon}{M(M+2)\, K^{2M+1}}, \tag{7}$$

where

$$K = \max \Big\{1, \max_{k=0,1,\dots,M-1} \| P_k \|, \sup_{\substack{A_k \in \Delta_{k,x} \\ k=1,2,\dots,M-1}} \| A_k \|, \sup_{\substack{B_k \in \Delta_{k,u} \\ k=0,1,\dots,M-1}} \| B_k \| \Big\}. \tag{8}$$

Notice that $K < \infty$ because $\Delta_{k,x}$ and $\Delta_{k,u}$ are compact sets (see Section 2). By assumption, the mappings f_k $(k = 0,1,\dots, M-1)$ are continuous in a neighbourhood of $(0,0) \in R^n \times R^m$. Further, $x_k(0) = 0$. Then we can find $\sigma > 0$ such that for every pair $v, v' \in B(0, \sigma)$ and every $k \in \{0,1,\dots, M-1\}$ there is $C_k \in \Delta_k$ satisfying

$$\| f_k(x_k(v), P_k v) - f_k(x_k(v'), P_k v') - C_k(x_k(v) - x_k(v'), P_k v - P_k v') \| \leqslant \varepsilon_1 \| (x_k(v) - x_k(v'), P_k v - P_k v') \| . \tag{9_k}$$

Hence, by setting $A_k = \operatorname{pr}_x C_k$, $B_k = \operatorname{pr}_u C_k$ (see Section 2) we get

$$f_k(x_k(v), P_k v) - f_k(x_k(v'), P_k v') - A_k(x_k(v) - x_k(v')) - B_k(P_k v - P_k v') = (\| x_k(v) - x_k(v') \| + \| P_k v - P_k v' \|)\omega_k \tag{10_k}$$

for some $\omega_k \in B(0, \varepsilon_1)$.

For any $k \in \{1,2..., M\}$ combining $(10_{k-1}),..., (10_0)$ yields

$$x_k(v) - x_k(v') = A_{k-1} \dots A_1 B_o(P_o v - P_o v') + \dots + A_{k-1} B_{k-2}(P_{k-2}v -$$

$$- P_{k-2}v') + B_{k-1}(P_{k-1}v - P_{k-1}v') + \| P_o v - P_o v' \| \; A_{k-1} \dots A_1 \omega_o +$$

$$+ (\| x_1(v) - x_1(v') \| + \| P_1 v - P_1 v' \|) \; A_{k-1} \dots A_2 \omega_1 + \dots + (\| x_{k-1}(v)$$

$$- x_{k-1}(v') \| + \| P_{k-1}v - P_{k-1}v' \|) \; \omega_{k-1}. \qquad (11_k)$$

Consequently, in view of (8) we have

$$\| x_k(v) - x_k(v') \| \leqslant MK^{M+1} \| v - v' \| + M\varepsilon_1 K^M \| v - v' \|$$

$$+ \varepsilon K^M (\| x_0(v) - x_0(v') \| + \dots + \| x_{k-1}(v) - x_{k-1}(v') \|). \qquad (12_k)$$

From (12_k) ($k = 1, 2, \dots M$) and (7) it is easy to see that

$$\| x_k(v) - x_k(v') \| \leqslant (M+1) K^{M+1} \| v - v' \|, \qquad (13_k)$$

for every $k \in \{1, 2, \dots, M\}$.

Therefore, taking (11_M), (13_k) $(k = 1, 2, \dots, M)$ and (7) into account we have

$$\| x_M(v) - x_M(v') - R(v - v') \| \leqslant M\varepsilon_1 K^M \| v - v' \| +$$

$$\varepsilon_1 K^M (\| x_0(v) - x_0(v') \| + \dots + \| x_{M-1}(v) - x_{M-1}(v') \|) \leqslant \varepsilon \; \| v - v' \|,$$

where $R = A_{M-1} \dots A_1 B_0 P_0 + \dots + A_{M-1} B_{M-2} P_{M-2} + B_{M-1} P_{M-1} \in \Sigma$.

Thus, the assertion (6) holds. Consequently, co Σ (the convex hull of Σ) is a shield for φ at $0 \in R^n$. Since $\Delta_{k,x}$, $\Delta_{k,u}$ $(k = 0, 1, \dots, M-1)$ are compact sets it follows that co Σ is a compact shield. On the other hand, every operator from co Σ is nonsingular. Then, by virtue of Theorem 1 we have (5). This completes the proof of the local reachability of system ($\mathcal{D}$).

Proof of controllability. For every $(h, v) \in R^n \times R^n$ let us define

$$\begin{cases} x_1(h, v) = f_0(h, P_0 v), \\ x_2(h, v) = f_1(x_1(h, v), P_1 v) \\ \dots\dots \\ x_M(h, v) = f_{M-1}(x_{M-1}(h, v), P_{M-1} v). \end{cases}$$

and consider the mappings $\varphi(h, v) = x_M(h, v)$. By an argument similar to the previous one, we can prove that the convex hull of the set

$$\widetilde{\Sigma} = \{ [A_{M-1} \dots A_1 A_0, A_{M-1} \dots A_1 B_0 P_0 + \dots + A_{M-1} B_{M-2} P_{M-2} + B_{M-1} P_{M-1}] : A_k \in \Delta_{k,x}, B_k \in \Delta_{k,u} \ (k = 0,1 \dots, M-1)\}$$

is a compact shield for φ at $(0,0) \in R^n \times R^n$. Here, for any $A, B \in L(R^n, R^n)$ the operator $[A, B] \in L(R^n \times R^n, R^n)$ is defined by

$$[A, B]\,(h,v) = Ah + Bv$$

for every $(h, v) \in R^n \times R^n$. By virtue of Theorem 2, we can find two neighbourhoods $U, V \in N(0)$ and a locally Lipschitzian mapping Ψ from V into U such that, for every $(h, v) \in U \times V$, the equality $\varphi(h, v) = 0$ holds if and only if $v = \Psi(h)$. Since $\Psi(0) = 0$ and $0 \in \operatorname{int} \Omega$ it follows from the continuity of Ψ that there exists $U_1 \in N(0)$ such that $P_k \Psi(h) \in \Omega$ for $k \in \{0,1, \dots, M-1\}$ and $h \in U_1$. Without loss of generality we can assume that $U_1 \subset U$. Hence, for every $h \in U_1$, the vectors

$$u_0 = P_0 \Psi(h), \dots, u_{M-1} = P_{M-1} \Psi(h)$$

satisfy the condition

$$h \xrightarrow[u_0,\, u_1,\, \dots,\, u_{M-1}]{(\mathcal{D})} 0.$$

According to Definition 2, system $(\mathcal{D})$ is locally controllable after M steps.

4. COROLLARIES

In this section we shall consider system $(\mathcal{D})$ under the assumption that conditions (1) and (2) are satisfied. We also assume that the mappings f_k are locally Lipschitzian at $(0,0) \in R^n \times R^m$ and denote by $\Delta_{k,x}$ and $\Delta_{k,u}$ their Partial Generalized Jacobians (see Section 2).

Given $2M - 1$ operators $\overline{A}_k \in \Delta_{k,x}$ $(k = 1,2, \dots, M-1)$, $\overline{B}_k \in \Delta_{k,u}$ $(k = 0,1, \dots, M-1)$, we define

$$P_0 = (\overline{A}_{M-1} \dots \overline{A}_1 \overline{B}_0)^*,\ P_1 = (\overline{A}_{M-1} \dots \overline{A}_2 \overline{B}_1)^*, \dots, P_{M-1} = \overline{B}^*_{M-1}, \tag{14}$$

$$\overline{R} = \overline{A}_{M-1} \dots \overline{A}_1 \overline{B}_0 P_0 + \overline{A}_{M-1} \dots \overline{A}_2 \overline{B}_1 P_1 + \dots + \overline{B}_{M-1} P_M, \tag{15}$$

$$\Lambda = [\overline{A}_{M-1} \dots \overline{A}_1 \overline{B}_0, \dots, \overline{A}_{M-1} \overline{B}_{M-2}, \overline{B}_{M-1}], \tag{16}$$

and

$$\rho_M = \max \{ \sup_{\substack{A_k \in \Delta_{k,x} \\ k=1,2,\dots,M-1}} \| A_k - \bar{A}_k \|, \quad \sup_{\substack{B_k \in \Delta_{k,u} \\ k=0,1,\dots,M-1}} \| B_k - \bar{B}_k \| \}. \tag{17}$$

Note that every element of $L(R^n, R^m)$ can be identified with a $m \times n$ — matrix. Thus, the matrix defined by (16) contains M blocks, each of which is a $n \times m$ — matrix.

COROLLARY 1. *Assume there exist $2M-1$ operators $\bar{A}_k \in \Delta_{k,x}$ $(k = 1,2,\dots, M-1)$, $\bar{B}_k \in \Delta_{k,u}$ $(k = 0,1,\dots, M-1)$ such that the convex hull of the set (3) with $P_0, P_1,\dots, P_{M-1}$ defined by (14) consists only of nonsingular operators. Then the conclusion of Theorem 3 holds.*

COROLLARY 2. *Suppose there exist $2M-1$ operators $\bar{A}_k \in \Delta_{k,x}$ $(k=1,2,\dots M-1)$, $\bar{B}_k \in \Delta_{k,u}$ $(k = 0,1,\dots, M-1)$ such that the matrix $\wedge$ defined by (16) has maximal rank and*

$$\rho_M < \frac{1}{2M\,K^{2M-1}\,\| \bar{R}^{-1} \|}, \tag{18}$$

where K is the same as in (8). Then the conclusion of Theorem 3 holds.

Proof. According to (15) and (16) we have $\bar{R} = \wedge \cdot \wedge^*$. Since $\wedge$ has maximal rank, R is nonsingular. Using (17), (18) together with Theorem 4 in [7, Chap. IV, Section 5], we deduce that every element of the set co Σ is a nonsingular operator. Hence our result follows immediately from Corollary 1.

Now suppose that f_k are continuously differentiable in a neighbourhood of (0,0) and denote by A_k (resp., B_k) their partial derivative with respect to x (resp., u) at (0,0). As is known, such mappings are locally Lipschitzian at (0,0). Evidently $\Delta_{k,x} = \{A_k\}$ and $\Delta_{k,u} = \{B_k\}$.

The following result is due to L. Weiss ([1], Theorem 2).

COROLLARY 3. *Assume that f_k $(k = 0, 1, 2,\dots)$ are continuously differentiable in a neighbourhood of (0,0). Let A_k, B_k be defined as above. If there exists an integer M such that the matrix $\wedge$ in (16) has maximal rank, then system ($\mathcal{D}$) is locally reachable and locally controllable after M steps.*

The proof is immediate from Corollary 2, because in this case $\rho_M = 0$ and (18) holds automatically.

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