ACTA MATHEMATICA VIETNAMICA
Vol. II No 2 (1986)

THE MOD 2 COHOMOLOGY ALGEBRA OF A SYLOW 2-SUBGROUP OF THE GENERAL LINEAR GROUP GL(4, Z_2)

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INTRODUCTION

In [8], Huỳnh Mùi proposed an approach to study the mod p cohomology algebra $H^*(GL_{n,p}; \mathbf{Z}_p)$ of the Sylow p-subgroup $GL_{n,p} \subset GL(n, \mathbf{Z}_p)$ consisting of all the upper triangular matrices with 1 in the diagonal. Here $\mathbf{Z}_p$ means the prime field of p elements. Since $GL_{3,2}$ is isomorphic to the dihedral group of order 8, its cohomology algebra is well known. In the present paper, we determine the algebra $H^*(GL_{4,2}, \mathbf{Z}_p)$ following the approach proposed in [8]. Applying this result, the algebra $H^*(GL_4; \mathbf{Z}_2)$ will be computed in a subsequent paper [10]. Throughout the sequel, we shall use the notation $H^*(G) = H^*(G, \mathbf{Z}_2)$. The main result is the following.

THEOREM. *The cohomology algebra $H^*(GL_{4,2})$ is the commutative algebra generated by elements:*

$$v_{12}, v_{23}, v_{34}, v_{13}, v_{24}, v_{14}, z_1, z_2$$

with $|v_{i,\, i+k}| = k$, $1 \leqslant k \leqslant 3$, $\quad 1 \leqslant i \leqslant 4-k$,

$|z_1| = 2$, $|z_2| = 3$ *and with the following structure*

1) As a module,

$$H^*(GL_{4,2}) = \mathbf{Z}_2\,[v_{12}, v_{23}, v_{34}, v_{13}, v_{24}, v_{14}]/I$$

$$\oplus\ \mathbf{Z}_2\,[v_{23}, v_{13}, v_{24}, v_{14}]\,\{z_1, z_2, z_1 z_2\},$$

where I is the ideal generated by the elements:

$$v_{12}v_{23},\ v_{23}v_{34},\ v_{12}v_{24} + v_{34}v_{13},\ v_{12}v_{24}^2 + v_{34}v_{13}^2,\ v_{12}^2v_{24} + v_{34}^2v_{13}$$

2) The multiplication is defined by the identities:

(i) $$v_{12}z_1 = v_{34}z_1 = v_{12}v_{24} = v_{34}v_{13},$$
$$v_{12}z_2 = v_{34}z_2 = 0,$$

(ii) $$z_1^2 = v_{23}z_2 + v_{23}^2 z_1 + v_{13}v_{24},$$
$$z_2^2 = v_{23}^2 v_{14} + v_{23}z_1 z_2 + (v_{13} + v_{24})z_1^2.$$

Here $|z|$ denotes the degree of an element z in a graded module and $K\{x_1, \ldots, x_n\}$ means the free module generated by $x_1, \ldots, x_n$ over an algebra K.

The paper consists of 3 sections and an appendix. In Section 1 we shall study the modular invariants for two pairs of variables which have been initiated in 1914 by W.C. Krathwohl [3]. As in [5] and [7], the modular invariant theory will be the main tool of our study. Using this theory and the Hochschild-Serre spectral sequence, we shall determine the cohomology algebras $H^*(GL_{4,2} / Z(GL_{4,2}))$ in Section 2 and $H^*(GL_{4,2})$ in Section 3. In the appendix, we shall compute the mod 2 cohomology algebra of a factor group of $GL_{n,2}$.

1. MODULAR INVARIANTS OF TWO SETS OF VARIABLES

Suppose that we are given two sets of variables $\{x_{i1}, x_{i2}\}$, $i = 1, 2$. Let $\mathbf{Z}_2[x_{11}, x_{12}, x_{21}, x_{22}]$ be the polynomial algebra generated by $x_{11}, x_{12}, x_{21}, x_{22}$ over $\mathbf{Z}_2$. Let (G_1, G_2) be a pair of groups of linear transformations, $G_i \subset GL_2 = GL(2, \mathbf{Z}_2)$, on the 2-dimensional vector space $\mathbf{Z}_2^2$. The natural action of (G_1, G_2) on $\mathbf{Z}_2[x_{11}, x_{12}, x_{21}, x_{22}]$ is defined as follows.

For every $(w_1, w_2) \in (G_1, G_2)$ and $f \in \mathbf{Z}_2[x_{11}, x_{12}, x_{21}, x_{22}]$, $(w_1, w_2) f(x_{11}, x_{12}, x_{21}, x_{22}) = f(x'_{11}, x'_{12}, x'_{21}, x'_{22})$ where x'_{ij}, $1 \leqslant i, j \leqslant 2$, are given by

$$\begin{bmatrix} x'_{21} & x'_{22} \\ x'_{11} & x'_{12} \end{bmatrix} = w_1 \begin{bmatrix} x_{21} & x_{22} \\ x_{11} & x_{12} \end{bmatrix} w_2^{-1}$$

By this action $\mathbf{Z}_2[x_{11}, x_{12}, x_{21}, x_{22}]$ is certainly a right $G_1 \times G_2$ — algebra. If $(w_1, w_2)f = f$ for all $(w_1, w_2) \in (G_1, G_2)$, f is called an invariant of (G_1, G_2)

or a (G_1, G_2) — invariant. It is easy to see that all (G_1, G_2)— invariants form an algebra. A set of generators of this algebra is called a full system of the invariants of (G_1, G_2). In addition, if the generators are algebraically independent, we say that they form a fundamental system.

Let $GL_{2,2}$ be the Sylow 2-subgroup of GL_2 consisting of all upper triangular matrices with 1 in the diagonal. For later use, we shall compute the invariants of $(1, GL_{2,2})$ and $(GL_{2,2}, GL_{2,2})$. Note that the invariants of $(1, GL_2)$ have been computed by W.C. Krathwohl [3], and those of (GL_2, GL_2) by the author in [10].

Let $V_{i1} = x_{i1}$, $V_{i2} = x_{i2}^2 + x_{i2}x_{i1}$, $1 \leqslant i \leqslant 2$.

Then we have

PROPOSITION 1.1. (L.E. Dickson [2], [5])

$$\mathbf{Z}_2[x_{i1}, x_{i2}]^{GL_{2,2}} = \mathbf{Z}_2[V_{i1}, V_{i2}],\ 1 \leqslant i \leqslant 2.$$

Define $M = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11}x_{22} + x_{12}x_{21}$.

Then, M is obviously a $(1, GL_2)$ — invariant (see W.C. Krathwohl [3]).

THEOREM 1.2. *We have*

$$\mathbf{Z}_2\,[x_{11}, x_{12}, x_{21}, x_{22}]^{(1,\, GL_{2,2})} = \mathbf{Z}_2[V_{11}, V_{12}, V_{21}, V_{22}]\,\{1, M\}.$$

Therefore, the invariants $V_{11}, V_{12}, V_{21}, V_{22}$, M form a full system of the $(1, GL_{2,2})$—invariants. Furthermore, we have

$$M^2 = V_{11}^2 V_{22} + V_{21}^2 V_{12} + V_{11}V_{21}M.$$

Proof. From Proposition 1.1, the invariants V_{ij}, $1 \leqslant i, j \leqslant 2$, are clearly algebraically independent. Furthermore, by an easy computation, we obtain the expression for M^2 as in the theorem. On the other hand, the degree of any element of $\mathbf{Z}_2\,[V_{11}, V_{12}, V_{21}, V_{22}]$ in x_{22} is congruent to zero modulo 2, but that of M is 1, then 1 and M are linearly independent over $\mathbf{Z}_2[V_{11}, V_{12}, V_{21}, V_{22}]$. Hence, it remains only to prove that the invariants $V_{11}, V_{12}, V_{21}, V_{22}$ and M form a full system of $(1, GL_{2,2})$—invariants.

Suppose f to be a homogeneous invariant. Without loss of generality, f can be assumed to be homogeneous in $\{x_{21}, x_{22}\}$. Write

$$f = x_{22}^n f_0(x_{11}, x_{12}) + \sum_{k=1}^{n} x_{22}^{n-k} x_{21}^k f_k(x_{11}, x_{12}).$$

If $n = 0$, by Proposition 1.1, f can be expressed in terms of V_{11}, V_{12}. Suppose that $n > 0$. According to W.C. Krathwohl [3, §§ 7,8], $f_0(x_{11}, x_{12})$ is an $(1, GL_{22})$ — invariant of the form

$$f_0(x_{11}, x_{12}) = V_{11}^s \, g_0(V_{11}, V_{12}),$$

where s is certain integer such that $s \equiv n \pmod 2$, $0 \leqslant s \leqslant 1$. This shows that f is of the form

$$f = M^s V_{22}^t \, g_0(V_{11}, V_{12}) + f',$$

where $n = 2t + s$ and f' is an $(1, GL_{2,2})$ — invariant with $f'_0 = 0$, i.e.

$$f' = \sum_{k=1}^{n} x_{22}^{n-k} x_{21}^{k} f'_k(x_{11}, x_{12}).$$

Consequently, we need only to consider the invariant f with $f_0 = 0$. In this case, we have $f = V_{21} g$. By induction on the degree of f, the theorem then follows

Now we consider the invariants of $(GL_{2,2}, GL_{2,2})$.

Let f be a $(GL_{2,2}, GL_{2,2})$ — invariant. Clearly, f is then an $(1, GL_{2,2})$—invariant. According to Theorem 1.2, f can be expressed in terms of V_{ij} and M. Since $M = (V_{12} + M) + V_{12}$, we can write

$$f = \sum_{r,s} V_{11}^r \, V_{21}^s \, f_{rs}(V_{12}, V_{22}, V_{12} + M).$$

Further, we have

$$wV_{11} = V_{11}, \qquad wV_{21} = V_{21} + V_{11},$$

$$wV_{12} = V_{12}, \qquad wV_{22} = V_{22} + (V_{12} + M),$$

$$wM = M, \text{ for } w = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in (GL_{2,2}, GL_{2,2}). \tag{1}$$

From these relations, we see that f can be decomposed as a sum of $(GL_{2,2}, GL_{2,2})$ — invariants which are homogeneous in $\{V_{11}, V_{12}\}$. Hence, we need only to consider the invariants of the form

$$f = \sum_{k=0}^{n} V_{21}^{n-k} \, V_{11}^{k} \, f_k\,(V_{12}, V_{22}, V_{12} + M). \tag{2}$$

LEMMA 1.3. *Let f be as in (2). Then $f_0(V_{12}, V_{22}, V_{12} + M)$ is an invariant of $(GL_{2,2}, GL_{2,2})$. Further, if n is odd, f_0 has the factor $V_{12} + M$.*

Proof. Following the argument used in [3, §§ 7, 8], we write

$$\begin{aligned} f &= \sum_{k=0}^{n} V_{21}^{n-k} V_{11}^{k} f_k(V_{12}, V_{22}, V_{12} + M) \\ &= V_{21}^{n} f_0(V_{12}, V_{22}, V_{12} + M) + V_{21}^{n-1} V_{11} f_1(V_{12}, V_{22}, V_{12} + M) \\ &\quad + \cdots \end{aligned}$$

Let w be as in (1). We have

$$wf = (V_{21}+V_{11})^n (wf_0) + (V_{21}+V_{11})^{n-1} V_{11}(wf_1) + \dots$$
$$= V_{21}^n (wf_0) + V_{21}^{n-1} V_{11} (n\,(wf_0) + wf_1) + \dots$$

Since $wf = f$, we have $wf_0 = f_0$ and $n(wf_0) + wf_1 = f_1$. The first relation shows that f_0 is a $(GL_{2,2}\, GL_{2,2})$ – invariant. If n is odd, $f_1 = wf_1 + wf_0 = wf_1 + f_0$.

Writing f_1 in the form

$$f_1 = \sum_{r,s,t} a_{rst}\, V_{12}^r V_{22}^s (V_{12} + M)^t\,,\ a_{rst} \in \mathbf{Z}_2,$$

we have

$$f_0 = f_1 - wf_1 = \sum_{r,s,t} a_{rst}\ V_{12}^r\ (V_{22}^s - (V_{22} + (V_{12} + M))^s)\,(V_{12} + M)^t\,.$$

Thus, f has the factor $V_{12} + M$. The lemma follows.

By a direct verification we have the following invariants of $(GL_{2,2},\, GL_{2,2})$:

$$W_{11} = x_{11},\ W_{21} = x_{21}^2 + x_{21}x_{11},\ W_{12} = x_{12}^2 + x_{12}x_{11},$$
$$W_{22} = x_{22}\,(x_{22} + x_{12})\,(x_{22} + x_{21})\,(x_{22} + x_{12} + x_{21} + x_{11}),$$
$$M = x_{11}x_{22} + x_{12}x_{21},$$
$$K = x_{11}x_{22}^2 + x_{11}^2 x_{22} + x_{12}x_{21}^2 + x_{12}^2 x_{21}.$$

THEOREM 1.4.

$$\mathbf{Z}_2\,[x_{11}, x_{12}, x_{21}, x_{22}]^{(\mathrm{GL}_{2,2}\mathrm{GL}_{2,2})} = \mathbf{Z}_2\,[W_{11}, W_{12}, W_{21}, W_{22}]\ \{1,\ M,\ K,\ MK\}.$$

Therefore, the invariants W_{11}, W_{12}, W_{21}, W_{22}, M, K form a full system of invariants of $(GL_{2,2},\, GL_{2,2})$ and

$$M^2 = W_{11}K + W_{11}^2\, M + W_{12}W_{21},$$
$$K^2 = W_{11}^2\, W_{22} + W_{11}MK + (W_{12} + W_{21})M^2.$$

Proof. The proof of this theorem is similar to that of Theorem 1.2. Obviously, the invariants W_{ij} are algebraically independent. By a direct computation we obtain the expression for M^2 and K^2 as in the theorem. Furthermore, the degree of any element of $\mathbf{Z}\,[W_{11}, W_{12}, W_{21}, W_{22}]$ in x_{22} is congruent to zero modulo 4, but those of M, K, MK are 1, 2, 3 respectively. Then 1, M, K, MK are linear independent over $\mathbf{Z}_2\,[W_{11}, W_{12}, W_{21}, W_{22}]$. It remains only to prove that invariants W_{ij}, M, K form a full system of $(GL_{2,2},\, GL_{2,2})$–invariants.

Let f be a homogeneous invariant of the form (2). Let $n=2t+s$, $0 \leqslant s \leqslant 1$. According to Lemma 1.3, f_0 is a $(GL_{2,2},\ GL_{2,2})$ – invariant of the form $f_0 = (V_{12} + M)^s\, g_0$. Hence,

$f = K^s W_{21}^t g + f'$, where f' is a $(GL_{2,2}, GL_{2,2})$ — invariant with $f'_0=0$. Thus, it is sufficient to consider the invariant f with $f_0 = 0$. In this case, $f = W_{11}g$ and the theorem can be proved by induction on the degree of f.

Remark 1.5. Consider $\mathbf{Z}_2[x_{11}, x_{12}, x_{21}]$ as a $(GL_{2,2}, GL_{2,2})$ — subalgebra of $\mathbf{Z}_2[x_{11}, x_{12}, x_{21}, x_{22}]$. As a consequence of Theorem 1.4, we have

$$\mathbf{Z}_2[x_{11}, x_{12}, x_{21}]^{(GL_{2,2},\ GL_{2,2})} = \mathbf{Z}_2[W_{11}, W_{12}, W_{21}].$$

2. COHOMOLOGY ALGEBRA $H^*(D(3,2))$

a) Let us consider the Hochschild-Serre spectral sequence

$$E_2 \cong H^*(G/Z) \otimes H^*(Z) \Rightarrow H^*(G)$$

for the central group extension

$$1 \to Z \to G \to G/Z \to 1, \qquad (E)$$

where $Z \cong Z_2 \times Z_2$. Let a_1, a_2 be two generators of Z and $x_i : Z \to \mathbf{Z}_2$ the duals of a_i, $i = 1,2$. Then $H^*(Z) = \mathbf{Z}_2[x_1, x_2]$. Let us denote by $\tau : H^*(Z) \to H^{*+1}(G/Z)$ the trangression as usual. Then, from [4, ch. 2] we have

$$\tau x = x(z_E) \in H^2(G/Z) \text{ for } x \in H^1(Z) = \mathrm{Hom}(Z, \mathbf{Z}_2)$$

where $z_E \in H^2(G/Z,Z)$ is the cohomology class corresponding to the extension (E).

PROPOSITION 2.1. *In the Hochschild — Serre spectral sequence for the central group extension (E), we have*

$$E_3 \cong H^*(G/Z)/(\tau x_1, \tau x_2) \otimes \mathbf{Z}_2 [x_1^2, x_2^2]$$

$$\oplus Ann_{H^*(G/Z)}(\tau x_1, \tau x_2) \otimes \mathbf{Z}_2 [x_1^2, x_2^2] x_1 x_2$$

$$\oplus \frac{A \,.\, \mathbf{Z}_2 [x_1^2, x_2^2]}{(\tau x_2 \otimes x_1 + \tau x_1 \otimes x_2)(H^*(G/Z) \otimes \mathbf{Z}_2 [x_1^2, x_2^2])},$$

where $A = \{ y_1 \otimes x_1 + y_2 \otimes x_2 / y_i \in H^*(G/Z),\ y_1 \tau x_1 + y_2 \tau x_2 = 0 \}$.

Proof. Identify E_2 with $H^*(G/Z) \otimes \mathbf{Z}_2[x_1, x_2]$. Since

$y \otimes x_1^n x_2^m = (y \otimes 1)(1 \otimes x_1)^n (1 \otimes x_2)^m$ lies in E_2 for $y \in H^*(G/Z)$, we have

$$d_2(y \otimes x_1^n x_2^m) = n(y\tau x_1 \otimes x_1^{n-1} x_2^m) + m(y \tau x_2 \otimes x_1^n x_2^{m-1}).$$

Then, computing $E_3 = \ker d_2/\mathrm{im}\ d_2$, we obtain easily the proposition.

b) *Computation of* $E_0(H^*(D(3,2)))$

Let $C(n, k)$ be the subgroup of $GL(n+1, \mathbf{Z}_2)$ consisting of all matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & * & * & \cdots & * & * \\ & 1 & 0 & \cdots & 0 & 0 & * & \cdots & * & * \\ & & & & \cdots & & & & & \\ & & & & & 1 & 0 & \cdots & * & * \\ & & & & & & 1 & 0 \cdots & 0 & * \\ & & & & & & & 1 \cdots & 0 & 0 \\ & 0 & & & & & & & \cdots & \\ & & & & & & & & 1 & 0 \\ & & & & & & & & & 1 \end{pmatrix} \Bigg\} k \text{ lines}$$

and $D(n, k) = GL_{n+1,2}/C(n, k+1)$. Then there is the central group extension

$$1 \to Z(D(n, k)) \to D(n, k) \to D(n, k-1) \to 1$$

with $\quad Z(D(n, k)) = C(n, k) \,/\, C(n, k+1) \cong \mathbf{Z}_2^{n-k-1}.$

PROPOSITION 2.2. (Huỳnh Mùi [8, Theorem A])

In the Hochschild — Serre spectral sequence

$$H^*(D(n, k-1)) \otimes H^*(Z(D(n, k))) \Rightarrow H^*(D(n,k)),\ k \geqslant 1,$$

for the above central group extension, we have $E_0 = E_{2^{k-1}+1}$.

We shall consider only the case $n = 3$, $k = 2$.

Let e_{ij} be the elementary matrix i.e. the matrix with 1 in position (i, j) and zero elsewhere. Set

$$c_{i,i+2} = (I + e_{i,i+2})C(3,3) \in D(3,2), \qquad 1 \leqslant i \leqslant 2,$$

$$c_{i,\,i+1} = (I + e_{i,\,i+1})\ C(3,2) \in D(3,1), \quad 1 \leqslant i \leqslant 3.$$

Let $x_{ij} : D(3, 2) \to \mathbf{Z}_2$ or $x_{ij}: D(3, 1) \to \mathbf{Z}_2$ be the duals of c_{ij}. Then

$$H^*(\mathbf{Z}\,(D\,(3,\,2))) = \mathbf{Z}_2[x_{13},\ x_{24}],$$

$$H^*(D\,(3,\,1)) = \mathbf{Z}_2\,[x_{12},\ x_{23},\ x_{34}].$$

LEMMA 2.3. *In the Hochschild-Serre spectral sequence for the central group extension*

$$1 \to Z(D(3, 2)) \to D(3, 2) \to D(3, 1) \to 1, \tag{3}$$

we have

$$E_0H^*\,(D(3,\,2)) = \mathbf{Z}_2\,[x_{12},\ x_{23},\ x_{34}] \,/\, (x_{12}x_{23},\ x_{23}x_{34}) \otimes \mathbf{Z}_2\,[x_{13}^2,\ x_{24}^2]$$

$$\oplus \frac{(x_{12} \otimes x_{24} + x_{34} \otimes x_{13})\ (\mathbf{Z}_2\,[x_{12},\ x_{23},\ x_{34}] \otimes \mathbf{Z}_2\,[x_{13}^2,\ x_{24}^2])}{(x_{12}x_{23} \otimes x_{24} + x_{23}x_{34} \otimes x_{13})\ (\mathbf{Z}_2\,[x_{12},\ x_{23},\ x_{34}] \otimes \mathbf{Z}_2\,[x_{13}^2,\ x_{24}^2])}.$$

Proof. According to Proposition 2.2, we need only to compute $E_3 = E_0H^*(D(3,2))$. We observe that the projection $D\,(3,2) \to D\,(3,1)$ has an inverse map $t : D(3,1) \to D(3,2)$ given by

$$t(c_{i,\,i+1}\,c_{j,\,j+1}) = c_{i,\,i+1}\,c_{j,\,j+1}\,[c_{i,\,i+1},\ c_{j,\,j+1}],\quad 1 \leqslant i,\ j \leqslant 3.$$

The factor set f corresponding to the extension (3) with $f(x, y) = t(x)t(y)t(xy)^{-1}$ satisfies the following relation

$$f(c_{i,\,i+1},\, c_{j,\,j+1}) = \begin{cases} c_{i,\,i+2} & \text{if } j = i+1, \\ 1 & \text{if } j \neq i+1, \\ & 1 \leqslant i,\, j \leqslant 3. \end{cases}$$

By a direct verification, we have

$$d_2(1 \otimes x_{i,\,i+2}) = [x_{i,\,i+2}\, f] = x_{i,\,i+1}\, x_{i+1,\,i+2}, \quad i = 1,2.$$

The Lemma follows easily from Proposition 2.1.

c) *The restrictions.* Let A, B be the subgroups of $D(3, 2)$ given by $A = \langle c_{12}, c_{13}, c_{24}, c_{34}\rangle$, $B = \langle c_{23}, c_{13}, c_{24}\rangle$ with $c_{ij} = (I + e_{ij})\ C(3, 3) \in D(3, 2)$. Obviously, A and B are maximal elementary abelian 2-subgroups of $D(3, 2)$. Let us again denote by $x_{ij}\colon A \to \mathbf{Z}_2$ or $x_{ij}\colon B \to \mathbf{Z}_2$ the homomorphisms given by $x_{ij}(c_{kl}) = \delta_{ij,\,kl}$ (Kronecker symbol). Then

$$H^*(A) = \mathbf{Z}_2\,[x_{12}, x_{13}, x_{34}, x_{24}],$$
$$H^*(B) = \mathbf{Z}_2\,[x_{23}, x_{13}, x_{24}].$$

By a well known result, we have

$$\operatorname{Im} \operatorname{res}\,(A, D(3, 2)) \subset H^*(A)^{W_{D(3,2)}(A)},$$

$$\operatorname{Im} \operatorname{res}\,(B, D(3, 2)) \subset H^*(B)^{W_{D(3,2)}(B)}.$$

Here $W_G(S) = N_G(S)/C_G(S)$ denotes the weyl group of a subgroup S in a group G and $H^*(S)$ is considered as a $W_G(S)$-module via the adjoint isomorphism (cf. Huỳnh Mùi [5].

$$\text{Let } V_{13,\,A} = x_{13}^2 + x_{13}x_{12},$$

$$V_{24,\,A} = x_{24}^2 + x_{24}x_{34},$$

$$M_A = x_{12}\,x_{24} + x_{34}\,x_{13}.$$

We have

LEMMA 2.4. *$W_{D(3,2)}(A) \cong GL_{2,2}$ and under this isomorphism $H^*(A)$ becomes an $1 \times GL_{2,2}$-module, that is*

$$H^*(A)^{W_{D(3,2)}(A)} = \mathbf{Z}_2[x_{12}, x_{34}, V_{13,\,A}\, V_{34,\,A}]\,\{1, M_A\}.$$

Further, $M_A^2 = x_{12}^2\, V_{24,\,A} + x_{34}^2\, V_{13,\,A} + x_{12}\, x_{34}\, M_A$.

Proof. Since $W_{D(3,2)}(A) = N_{D(3,2)}(A)/C_{D(3,2)}(A) = D(3,2)/A \cong GL_{2,2}$; the lemma follows from Theorem 1.2.

LEMMA 2.5 *Let* $V_{13,B} = x_{13}^2 + x_{13}x_{23}$, $V_{24,B} = x_{24}^2 + x_{24}x_{23}$.

Then $$H^*(B)^{W_{D(3,2)}(B)} = \mathbf{Z}_2[x_{23}, V_{13,B}, V_{24,B}].$$

Proof. Since $W_{D(3,2)}(B) = D(3,2)/B \cong GL_{2,2} \times GL_{2,2}$, the lemma follows from Remark 1. 3.

By an argument analogous to that used in [7], we prove

LEMMA 2.6. *(i) There exists the unique cohomology class* $z \in H^2(D(3,2)$ *such that* $z|_A = M_A$, $z|_B = 0$.

(ii) There exist the unique cohomology classes $v_{13} v_{24} \in H^2(D(3,2))$ *such that*
$$v_{i,i+2|A} = v_{i,i+2,A},$$
$$v_{i,i+2|B} = v_{i,i+2,B}.$$

Proof. We prove only the existence. The uniqueness will follow from Lemma 2.9.

(i) From Lemma 2.3, there exists a cohomology class $\overline{z} \in H^2(D(3,2))$ such that
$$\overline{z} \in F^1 H^2(D(3,2)) \to x_{12} \otimes x_{24} + x_{34} \otimes x_{13} \in E_0^{1,1}.$$
Here $\{F^i H^*(D(3,2))\}$ means the Hochschild-Serre filtration on $H^*D(3,2))$ with respect to (3). According to Lemma 2.4, $\overline{z}|A$ is of the form
$$\overline{z}|_A = \alpha_1 x_{12}^2 + \alpha_2 x_{12} x_{34} + \alpha_3 x_{34}^2 + \beta_1 V_{13,A} + \beta_2 V_{24,A} + \beta_3 M_A \quad \text{with}$$
$\alpha_i, \beta_j \in \mathbf{Z}_2$, $1 \leqslant i, j \leqslant 3$.

On the other hand, we have the commutative diagram
$$\begin{array}{ccc} F^1 H^2(D(3,2)) & \to & E_0^{1,1}(D(3,2), Z(D(3,2))) \\ \downarrow & & \downarrow \\ F^1 H^2(A) & \longrightarrow & E_0^{1,1}(A, Z(D(3,2))) \end{array}$$
induced by the inclusion $(A, Z(D(3,2))) \subset (D(3,2), Z(D(3,2)))$.
From this diagram we observe that $\beta_1 = \beta_2 = 0$, $\beta_3 = 1$.
This means that $\overline{z}|_A = \alpha_1 x_{12}^2 + \alpha_2 x_{12}x_{34} + \alpha_3 x_{34}^2 + M_A$ with $\alpha_i \in \mathbf{Z}_2$.

Similarly, $\overline{z}|_B = \beta x_{23}^2$ with $\beta \in \mathbf{Z}_2$. Set
$$z = \overline{z} + \alpha_1 v_{12}^2 + \alpha_2 v_{12} v_{34} + \alpha_3 v_{34}^2 + \beta v_{23}^2,$$
where $v_{i,i+1} = \inf(D(3,2), D(3,1))\, x_{i,i+1}$, $1 \leqslant i \leqslant 3$.
It is easy to see that z satisfies the condition (i).

(ii) From Lemma 2.3, there exists a cohomology class $\overline{v}_{13} \in H^2(D(3,2))$ such that $\overline{v}_{13} \mid_{Z(D(3,2))} = x_{13}^2$.

Consider the commutative diagram

$$\begin{array}{ccc} H^*(D(3,2)) & \xrightarrow{res} & H^*(A) \\ \downarrow id & & \downarrow res \\ H^*(D(3,2)) & \xrightarrow{res} & H^*(Z(D(3,2))). \end{array}$$

induced by the inclusions $Z(D(3,2)) \subset A \subset D(3,2)$.

By Lemma 2.4, $\overline{v}_{13} \mid_A = \alpha_1 x_{12}^2 + \alpha_2 x_{12} x_{34} + V_{13,A} + \beta M_A + \alpha_3 x_{34}^2$ with $\alpha_i, \beta \in Z_2$, $1 \leqslant i \leqslant 3$. Similarly,

$\overline{v}_{13} \mid_B = \gamma x_{23}^2 + V_{13,B}$, $\gamma \in \mathbf{Z}_2$. Set

$$v_{13} = \overline{v}_{13} + \alpha_1 v_{12}^2 + \alpha_2 v_{12} v_{34} + \alpha_3 v_{34}^2 + \beta z + \gamma v_{23}^2.$$

Then v_{13} satisfies the condition (ii). The element v_{24} is obtained in a similar way. The lemma is proved.

From Lemmas 2. 4, 2.5 and 2. 6 we have

COROLLARY 2.7.

$$\text{(i) Im res } (A, D(3,2)) = H^*(A)^{W_{D(3,2)}(A)},$$

$$\text{(ii) Im res } (B, D(3,2)) = H^*(B)^{W_{D(3,2)}(B)}.$$

Remark 2. 8. From the proof of Lemma 2. 6, we have that

$$z \in F^1H^2(D(3,2)) \mapsto x_{12} \otimes x_{24} + x_{34} \otimes x_{13} \in E_0^{1,1},$$

$$v_{i,i+2} \in F^0H^2(D(3,2)) \mapsto 1 \otimes x_{i,i+2} - E_0^{0,2},$$
$$1 \leqslant i \leqslant 2.$$

Now we determine the algebra $H^*(D(3,2))$. According to Lemmas 2. 3, 2. 6 and Remark 2. 8, this algebra is generated by the elements:

$$v_{12}, v_{23}, v_{34}, v_{13}, v_{24}, z,$$

where $v_{i,i+1} = \inf(D(3,2), D(3,1))\, x_{i,i+1}$, $1 \leqslant i \leqslant 3$ and $H^*(D(3,2))$ can be decomposed as the direct sum of modules

$$H^*(D(3,2)) = Z_2[v_{12}, v_{23}, v_{34}, v_{13}, v_{24}] / (v_{12}v_{23}, v_{23}v_{34}) \oplus Z_2[v_{12}, v_{34}, v_{13}, v_{24}]\{z\}. \quad (4)$$

It remains to compute z^2 and $v_{23}z$ to determine the algebra structure of $H^*(D(3,2))$. To this end we prove

LEMMA 2.9. *The homomorphism*

$$\text{Res}: H^*(D(3,2)) \to H^*(A) \times H^*(B),$$

given by the restriction homomorphisms, is injective.

Proof. We shall denote by $\left\{ F^i H^*(D(3,2)) \right\}$ the Hochschild–Serre filtration with respect to (3). We consider the element $x \in H^n(D(3,2))$ such that $\operatorname{Res}(x) = 0$, i.e. $x / A = x / B = 0$.

Suppose that $x \in F^{n-k} H^n(D(3,2))$. We shall prove $x = 0$ by induction on k. If $k = 0$, i.e. $x \in F^n H^n(D(3,2)) = E_0^{n,0}$, by Lemma 2.3, it is easy to see that $x = 0$.

Suppose that $k > 0$. Write $k = 2l + r$, $0 \leqslant r \leqslant 1$. We consider only the case $r = 0$, since the case $r = 1$ can be treated in a similar way. According to Remark 2.8 and (4), x is of the form

$$x = \sum_{i+j=l} \alpha_{ij} v_{13}^i v_{24}^j v_{23}^{n-2(i+j)}$$

$$+ \sum_{i+j=l} \beta_{ijs} v_{13}^i v_{24}^j v_{12}^s v_{34}^{n-s-2(i+j)} + X$$

with $X \in F^{n-k+1} H^n(D(3,2))$.

On the other hand, consider the commutative diagrams

$$\begin{array}{ccc} F^{n-k} H^n(D(3,2)) & \longrightarrow & E_0^{n-k,k}(D(3,2), Z(D(3,2))) \\ \downarrow & & \downarrow \\ F^{n-k} H^n(A) & \longrightarrow & E_0^{n-k,k}(A, Z(D(3,2))) \end{array} \qquad (5)$$

$$\begin{array}{ccc} F^{n-k} H^n(D(3,2)) & \longrightarrow & E_0^{n-k,k}(D(3,2), Z(D(3,2))) \\ \downarrow & & \downarrow \\ F^{n-k} H^n(B) & \longrightarrow & E_0^{n-k,k}(B, Z(D(3,2))) \end{array} \qquad (6)$$

induced by the inclutions $(A, Z(D(3,2))) \subset (D(3,2), Z(D(3,2)))$ and $(B, Z(D(3,2))) \subset (D(3,2), Z(D(3,2)))$, respectively. By Remark 2.8 and (5), we have

$$0 = x \mid A \in F^{n-k} H^n(A) \longrightarrow \sum_{i+j=l} \beta_{ijs} (x_{12}^s x_{34}^{n-s-2(i+j)} \otimes x_{13}^{2i} x_{24}^{2j}).$$

Since A is an elementary abelian 2-group, we have $\beta_{ijs} = 0$. Similarly, by Remark 2.8 and (6) $\alpha_{ij} = 0$. Hence $x = X \in F^{n-(k-1)} H^n(D(3,2))$. The lemma follows.

For the restrictions of the elements $v_{23} z$, z^2 on $H^*(A)$ and $H^*(B)$, we have

COROLLARY 2. 10.

(*i*) $v_{23} z = 0$

(*ii*) $z^2 = v_{12}^2 v_{24} + v_{34}^2 v_{13} + v_{12} v_{34} z.$

Combining the above results, we obtain

THEOREM 2.11.

$H^*(D(3,2)) = \mathbf{Z}_2 [v_{12}, v_{23}, v_{34}, v_{13}, v_{24}, z] / I$

with $I = (v_{12}v_{23},\ v_{23}v_{34},\ v_{23}z,\ z^2 + v_{12}^2 v_{24} + v_{34}^2 v_{13} + v_{12}v_{34}z)$.

3. COHOMOLOGY ALGEBRA $H^*(GL_{4,2})$

a) Consider the central group extension

$$1 \to Z(GL_{4,2}) \to GL_{4,2} \to D(3,2) \to 1 \qquad \text{(E)}$$

with $Z(GL_{4,2}) \cong \mathbf{Z}_2$.

Let $z_E \in H^2(D(3,2))$ be the cohomology class corresponding to extension (E). From Lemma 2.6, by considering the restrictions of the elements z_E, $Sq^1 z_E$, $Sq^2 Sq^1 z_E$ on $H^*(A)$ and $H^*(B)$, we obtain

LEMMA 3.1.

(i) $z_E = z$,

(ii) $Sq^1 z_E = v_{12}v_{24} + v_{34}v_{13} + (v_{12} + v_{34}) z_E$,

(iii) $Sq^2 Sq^1 z_E = v_{12}v_{24}^2 + v_{34}v_{13}^2 + (v_{12}^2 + v_{34}^2 + v_{12}v_{34}) Sq^1 z_E + (v_{12} + v_{34})(v_{12}v_{34} + z) z_E$.

Let $c = I + e_{14} \in GL_{4,2}$ be a generator of $Z(GL_{4,2})$ and $x: Z(GL_{4,2}) \to \mathbf{Z}_2$ the dual of c. Then, as is well known, $H^*(Z(GL_{4,2})) = \mathbf{Z}_2[x]$.

LEMMA 3.2. *In the Hochschild-Serre spectral sequence for the central group extension (E), we have*

$$E_4 \cong \mathbf{Z}_2 [v_{12}, v_{23}, v_{34}, v_{13}, v_{24}] / I_1 \otimes \mathbf{Z}_2[x^4]$$
$$\mathbf{Z}_2 [v_{23}, v_{13}, v_{24}] v_{23} / I_1 \otimes \mathbf{Z}_2[x^4]\, x$$
$$\mathbf{Z}_2 [v_{23}, v_{13}, v_{24}] v_{23} / I_2 \otimes \mathbf{Z}_2[x^4]\, x^2$$
$$\mathbf{Z}_2 [v_{23}, v_{13}, v_{24}] v_{23} / I_2 \otimes \mathbf{Z}_2[x^4]\, x^3.$$

Here $I_1 = (v_{12}v_{23},\ v_{23}v_{34},\ v_{12}^2 v_{24} + v_{34}^2 v_{13})$,

$I_2 = (I_1,\ v_{12}v_{24} + v_{34}v_{13})$.

Proof. From [6,2.1], we have

$$E_4 \cong H^*(D(3,2)) / (z_E, Sq_1 z_E) \otimes \mathbf{Z}_2[x^4]$$
$$\oplus\ Ann_{H^*(D(3,2))}(z_E) / (Sq^1 z_E) \otimes \mathbf{Z}_2[x^4]\, x$$
$$\oplus\ Ann_{H^*(D(3,2)) / (z_E)}(Sq^1 z_E) \otimes \mathbf{Z}_2[x^4]\, x^2$$
$$\oplus\ Ann_{H^*}(D(3,2))\ (z_E\, Sq^1 z_E) \otimes \mathbf{Z}_2[x^4]\, x^3.$$

The lemma now follows from Theorem 2.11 and Lemma 3.1.

LEMMA 3.3.

$$d_4(v_{23} \otimes x^3) = (v_{12}v_{24}^2 + v_{34}v_{13}^2) \otimes 1 \text{ in } E_4^{5,0}.$$

Proof. Suppose that $d_4(v_{23} \otimes x^3) \neq (v_{12}v_{24}^2 + v_{34}v_{13}^2) \otimes 1$.

Then, it is easy to see that $(v_{12}v_{24}^2 + v_{34}v_{13}^2) \otimes 1$ is not zero in $E_5^{5,0}$. On the other hand, according to Proposition 2.2, $E_5 = E_0H^*(GL_{4,2})$. Hence, we have

$$0 = d_5(x^4) = \tau(x^4) = \tau(Sq^2Sq^1x) = Sq^2Sq^1(\tau x) =$$

$$= Sq^2Sq^1z_E = (v_{12}v_{24}^2 + v_{34}v_{13}^2) \otimes 1 \text{ in } E_0^{5,0}$$

(by Lemma 3.1).

This contradiction proves the Lemma.

Computing $E_0 = E_5 = \text{Ker } d_4/\text{im } d_4$, we obtain

PROPOSITION 3.4.

$$\begin{aligned} E_0H^*(GL_{4,2}) = {} & \mathbf{Z}_2[v_{12}, v_{23}, v_{34}, v_{13}, v_{24}]/I \otimes \mathbf{Z}_2[x^4] \\ & \oplus \mathbf{Z}_2[v_{23}, v_{13}, v_{24}]v_{23}/I_1 \otimes \mathbf{Z}_2[x^4]x \\ & \oplus \mathbf{Z}_2[v_{23}, v_{13}, v_{24}]v_{23}/I_2 \otimes \mathbf{Z}_2[x^4]x^2 \\ & \oplus \mathbf{Z}_2[v_{23}, v_{13}, v_{24}]v_{23}^2/I_2 \otimes \mathbf{Z}_2[x^4]x^3. \end{aligned}$$

Here I_1, I_2 are as in Lemma 3.2 and $I = (I_1, v_{12}v_{24}^2 + v_{34}v_{13}^2)$,

b) The restrictions

We use the technique developed in Section 2,c) to compute the images of the restrictions of $H^*(GL_{4,2})$ on $H^*(A)$ for each maximal elementary abelian 2-subgroup A of $GL_{4,2}$.

Let e_{ij} be the emementary matrix (i.e as defined previously, the matrix with 1 in position (i, j) and zero elsewhere). Set $c_{ij} = I + e_{ij}$ and

$$A_1 = \langle c_{12}, c_{13}, c_{14} \rangle, \quad A_2 = \langle c_{13}, c_{14}, c_{23}, c_{24} \rangle,$$

$$A_3 = \langle c_{14}, c_{24}, c_{34} \rangle, \quad A_4 = \langle c_{12}, c_{34}, c_{14} \rangle,$$

$$A_5 = \langle c_1, c_2, c_3 \rangle$$

with $c_1 = c_{12}c_{34}$, $c_2 = c_{13}c_{24}$, $c_3 = c_{14}$.

Let $x_{ij}: A_k \to \mathbf{Z}_2$ be the duals of c_{ij} with suitable indices (i, j) for $1 \leqslant k \leqslant 4$ and let $x_i: A_5 \to \mathbf{Z}_2$ be the duals of c_i. Then

$$H^*(A_k) = \begin{cases} \mathbf{Z}_2[x_{ij}, \text{ with suitable indices } (i, j)], & k \leqslant 4, \\ \mathbf{Z}_2[x_1, x_2, x_3], & k = 5. \end{cases}$$

By indentifying $v_{i,i+k} = \inf(GL_{4,2}, D(3,2))\, v_{i,i+k}$, $1 \leqslant k \leqslant 2$, $1 \leqslant i \leqslant 4-k$, it is easy to see that

$$v_{13} \mid A_i = x_{13}^2 + x_{13}\, x_{i,i+1}\,,\ i = 1,2,$$

$$v_{24} \mid A_i = x_{24}^2 + x_{24}\, x_{i,i+1}\,,\ i = 2,3,$$

$$v_{13} \mid A_5 = v_{24} \mid A_5 = x_2^2 + x_2 x_1.$$

Let us set

$$V_{14,i} = \begin{cases} x_{14}(x_{14} + x_{13})(x_{14} + x_{12})(x_{14} + x_{13} + x_{12}), & i = 1, \\ x_{14}(x_{14} + x_{13})(x_{14} + x_{24})(x_{14} + x_{13} + x_{24} + x_{23}), & i = 2, \\ x_{14}(x_{14} + x_{24})(x_{14} + x_{34})(x_{14} + x_{24} + x_{34}), & i = 3, \\ x_{14}(x_{14} + x_{12})(x_{14} + x_{34})(x_{14} + x_{12} + x_{34}), & i = 4, \\ x_3(x_3 + x_2)(x_3 + x_1)(x_3 + x_2 + x_1), & i = 5, \end{cases}$$

$$M = x_{23}x_{14} + x_{13}x_{24},$$

$$K = x_{23}^2 x_{14} + x_{13}x_{24}^2 + x_{13}^2 x_{24} + x_{23}x_{14}^2.$$

We have

LEMMA 3.5. *$W_{GL_{4,2}(A_2)} \cong GL_{2,2} \times GL_{2,2}$ and under this isomorphism $H^*(A_2) = Z_2[x_{23}, x_{13}, x_{24}, x_{14}]$ becomes a $GL_{2,2} \times GL_{2,2}$ – module, that is*

$$H^*(A_2)^{W_{GL_{4,2}(A_2)}} = Z_2[x_{23}, v_{13}|_{A_2}, v_{24}|_{A_2}, V_{14,2}, M, K]/I$$

where I is the ideal generated by the elements

$$M^2 = x_{23}K + x_{23}^2 M + (v_{13}\big|_{A_2})(v_{24}\big|_{A_2}),$$

$$K^2 = x_{23}^2 V_{14,2} + x_{23}MK + (v_{13}\big|_{A_2} + v_{24}\big|_{A_2})M^2.$$

Proof. We have $W_{GL_{4,2}}(A_2) = N_{GL_{4,2}}(A_2)/C_{GL_{4,2}}(A_2) =$

$= GL_{4,2}/A_2 \cong GL_{2,2} \times GL_{2,2}$. The Lemma now follows from Theorem 1.4.

LEMMA 3.6.

$$H^*(A_4)^{W_{GL_{4,2}}(A_4)} = \mathbf{Z}_2\,[x_{12}, x_{34}, V_{14,4}].$$

Proof. It is easy to see that $W_{GL_{4,2}}(A_4) \cong \langle \bar{c}_{13}, \bar{c}_{24} \rangle$.

Here $\bar{c}_{i,i+2} = c_{i,i+2}\, A_4$. Further, the action of $W_{GL_{4,2}}(A_4)$ on $H^*(A_4) = Z_2[x_{12}, x_{34}, x_{14}]$ is as follows

$$\bar{c}_{i,i+2}\, x_{j,j+1} = x_{j,j+1},\ i = 1,2,\ j = 1 \text{ or } 3,$$
$$\bar{c}_{12} x_{14} = x_{14} + x_{34},\ \bar{c}_{34}\, x_{14} = x_{14} + x_{12},$$

proving the Lemma.

Since $W_{GL_{4,2}}(A_i) \cong GL_{3,2}$ for $i = 1$ or 3 or 5 we have proved (see[5]).

LEMMA 3.7.

$$H^*(A_i)^{W_{GL_{4,2}}(A_i)} = \begin{cases} \mathbf{Z}_2\,[x_{12},\ v_{13}|_{A_1},\ V_{14,1}],\ i = 1, \\ \mathbf{Z}_2\,[x_{34},\ v_{24}|_{A_3},\ V_{14,3}],\ i = 3, \\ \mathbf{Z}_2\,[x_1,\ v_{13}|_{A_5},\ V_{14,5}],\ i = 5. \end{cases}$$

By the argument used in the proof of Lemma 2.6, we obtain easily

LEMMA 3.8. (i) There exist unique cohomology classes

$z_1 \in H^2(GL_{4,2})$, $z_2 \in H^3(GL_{4,2})$ such that

$z_i|_{A_1} = z_i|_{A_3} = z_i|_{A_4} = 0,\ i = 1,2,$

$z_1|_{A_2} = M,\ z_1|_{A_5} = v_{13}|_{A_5},$

$z_2|_{A_2} = K,\ z_2|_{A_5} = 0.$

(ii) There exists a unique cohomology class $v_{14} \in H^4(GL_{4,2})$ such that $v_{14}|_{A_i} = V_{14,i}$ $1 \leqslant i \leqslant 5$.

(iii) $z_i \in F^1H^{1+i}(GL_{4,2}) \mapsto v_{23} \otimes x^i \in E_0^{1,i},\ i = 1,2,$

$v_{14} \in F^0H^4(GL_{4,2}) \mapsto 1 \otimes x^4 \in E_0^{0,4}.$

COROLLARY 3.9. $\operatorname{Im} \operatorname{res}(A_i, GL_{4,2}) = H^*(A_i)^{W_{GL_{4,2}}(A_i)},\ 1 \leqslant i \leqslant 5.$

PROPOSITION 3.10. *The homomorphism*

$$\operatorname{Res} : H^*(GL_{4,2}) \longrightarrow \prod_{i=1}^{5} H^*(A_i)$$

given by the restriction homomorphisms, is injective.

Proof. The proof is similar to that of Proposition 2.9.

COROLLARY 3.11.

(i) $z_1^2 = v_{23} z_2 + v_{23}^2 z_1 + v_{13} v_{24},$

(ii) $z_2^2 = v_{23}^2 v_{14} + v_{23} z_1 z_2 + (v_{13} + v_{24})\, z_1^2,$

(iii) $v_{12} z_1 = v_{34} z_1 = v_{12} v_{24},$

(iv) $v_{12} z_2 = v_{34} z_2 = 0.$

Proof. Considering the restrictions of the elements z_1^2, z_2^2, $v_{12} z_1$, $v_{34} z_1$, $v_{12} z_2$, $v_{34} z_2$ on $H^*(A_i)$, $1 \leqslant i \leqslant 5$, we obtain the required formulas.

Finally, we derive the main result of the paper :

THEOREM 3.12.

$H^*(GL_{4,2}) = \mathbf{Z}_2[v_{12}, v_{23}, v_{34}, v_{13}, v_{24}, v_{14}, z_1, z_2]/I$, *where* I *is the ideal generated by the elements:*

$$v_{12}v_{23},\ v_{23}v_{34},\ v_{12}v_{24}+v_{34}v_{13},\ v_{12}v_{24}^2+v_{34}v_{13}^2,\ v_{12}^2v_{24}+v_{34}^2v_{13},$$

$$z_1+v_{23}z_2+v_{23}^2z_1+v_{13}v_{24},\ z_2^2+v_{23}^2v_{14}+v_{23}z_1z_2+(v_{13}+v_{24})z_1^2,$$

$$(v_{12}+v_{34})z_1,\ v_{12}(z_1+v_{24}),\ v_{12}z_2,\ v_{34}z_2.$$

APPENDIX. THE COHOMOLOGY ALGEBRA H* (D(n, 2))

In this appendix, we compute the cohomology algebra of the group $D(n, 2)$ defined in Section 2. b) The result is stated as follows.

THEOREM A. 1. *$H^*(D(n, 2))$ is the commutative algebra generated by the elements: u_i, v_j, z_k, $1 \leqslant i \leqslant n$.*

$1 \leqslant j \leqslant n-1$, $1 \leqslant k \leqslant n-2$ with $|u_i| = 1$, $|v_j| = 2$, $|z_k| = 2$; the algebraic relations between these elements are the following identifies:

(i) $u_i u_{i+1} = z_i u_{i+1} = 0, \quad u_i z_{i+1} + u_{i+3} z_i = 0,$

(ii) $z_i^2 = u_i^2 v_{i+1} + u_{i+2}^2 v_i + u_i u_{i+2} z_i,$

(iii) $z_i z_{i+1} = u_i u_{i+3} v_{i+1}.$

a) Consider first the Hochschild-Serre spectral sequence

$$E_2 \cong H^*(G/Z) \otimes H^*(Z) \Rightarrow H^*(G)$$

for the central group extension

$$1 \to Z \to G \to G/Z \to 1 \text{ with } Z = \mathbf{Z}_2^n. \qquad (E)$$

Let $a_1, a_2, \ldots, a_n$ be the generators of Z and $x_i : Z \to \mathbf{Z}_2$ be the duals of a_i. We know that $H^*(Z) = \mathbf{Z}_2[x_1, x_2, \ldots x_n]$. Proposition 2.1 can easily be generalized as follows.

PROPOSITION A.2. *For the central group extension (E), we have*

$$E_3 \cong H^*(G/Z) \setminus (\tau x_1, \ldots, \tau x_n) \otimes \mathbf{Z}_2[x_1^2, \ldots, x_n^2]$$

$$\oplus \frac{\bigoplus_{I \in P_n} A_I \otimes \mathbf{Z}_2[x_1^2, \ldots, x_n^2]}{\left(\sum_{i \in I} \tau(x_i) \otimes X_{I/\{i\}},\ I \in \underline{P_n}\right)}.$$

Here $P_{\underline{n}}$ is the family of all subsets of $\underline{n} = \{1, 2, \ldots, n\}$,

$$A_I = \{ \sum_{i \notin I} y_i \otimes x_i X_I, \ \ y_i \in H^*(G/Z), \ \ \sum_{i \notin I} y_i \tau x_i = 0,$$
$$y_i \in A_{nn_{H^*(G/Z)}}(\tau x_j), \ \ i \notin I, \ \ j \in I\}, \ \ X_I = \prod_{i \in I} x_i .$$

Now we apply Proposition A.2 to the central group extension

$$1 \to Z(D(n, 2) \to D(n, 2) \to D(n, 1) \to 1. \qquad (A)$$

Suppose that $c_{i,i+2}$ (resp. $c_{j,j+1}$) are elements of $Z(D(n,2))$ (resp. $D(n,1)$) represented by $I + e_{i,i+2}$ (resp. $I + e_{j,j+1}$) $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$). Let x_i: $Z(D(n,2) \to \mathbb{Z}_2$ (resp. y_j: $D(n, 1) \to \mathbb{Z}_2$) be the duals of $c_{i,i+2}$(resp. $c_{j,j+1}$), $1 \leqslant$ $\leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$.

Then $H^*(D(n,2)) = \mathbb{Z}_2[x_1, x_2, \ldots, x_{n-1}]$,

$H^*(D(n,1)) = \mathbb{Z}_2(y_1 y_2, \ldots, y_n]$.

From Propositions 2.2 and A,2 we can derive

LEMMA A.3. *For the central group extension (A), we have*

$$E_0 H^*(D(n,2) = \mathbb{Z}_2[y_1, \ldots, y_n]/(y_1 y_2, \ldots, y_{n-1} y_n) \otimes \mathbb{Z}_2[x_1^2, \ldots, x_n^2]$$

$$\oplus \frac{\bigoplus_{i=1}^{n-2} (y_i \otimes x_{i+1} + y_{i+2} \otimes x_i)(\mathbb{Z}_2[y_1, \ldots y_n] \otimes \mathbb{Z}_2[x_1^2, \ldots, x_{n-1}^2])}{(\sum_{i \in I} \tau(x_i) \otimes X_{I/\{i\}}, I \in P_{\underline{n}})}.$$

b) *The restrictions.* Consider a subset $I = \{ i_1, i_2, \ldots, i_s \}$ with $i_1 < i_2 < \ldots < i_s$ of $P_{\underline{n}}$. We say that I is admissible if

$i_1 \leqslant 2$, $i_s \geqslant n-1$, $2 \leqslant i_{j+1} - i_j \leqslant 3, 1 \leqslant j \leqslant s-1$.

Set $A_I = < c_{i,i+1}, c_{j,j+2}, i \in I, \ 1 \leqslant j \leqslant n-1 > \subset D(n,2)$, where c_{ij} is represented by $I + e_{ij}$. Clearly, A_I is a normal elementary abelian 2-subgroup of $D(n,2)$ and A_I is self centralized; so it is also a maximal elementary abelian 2-subgroup of $D(n,2)$. Let us denote by $y_i : A_I \to \mathbb{Z}_2$ (resp. x_j: $A_I \to \mathbb{Z}_2$) the duals of $c_{i,i+1}$ (resp. $c_{j,j+2}$). We have

$$H^*(A_I) = \mathbb{Z}_2[x_i, y_j, 1 \leqslant i \leqslant n-1, \ j \in I].$$

We set
$$V_{1i} = y_j, \ i \in I,$$

$$V_{2j} = \begin{cases} x_j^2 + x_j y_{j+k} & \text{if } j+k \in I, \ k = 0 \text{ or } 1, \\ x_j & \text{otherwise, } 1 \leqslant j \leqslant n-1, \end{cases}$$

$$M_k = y_k x_{k+1} + y_{k+2} x_k \ \text{ if } k, k+2 \in I.$$

LEMMA A. 4. *Denote by* $W(I) = N_{D(n,2)}(A_I)/C_{D(n,2)}(A_I)$ *the weyl group of the group* A_I *in the group* $D(n, 2)$. *Then*

$$H^*(A_I)^{W(I)} = Z_2[V_{1i}, V_{2j}, M_k, \ i \in I, \ 1 \leqslant j \leqslant n-1, \ k, k+2 \in I]/J,$$

where J *is the ideal generated by the elements*:

$$M_i^2 + V_{1i}^2 V_{2,i+1} + V_{1,i+2}^2 V_{2i} + V_{1i} V_{1,i+2} M_i, \ i, i+2 \in I.$$

Proof: Let $\{j_1, j_2, \ldots, j_{n-s}\}$ $(1 \leqslant j_1 < j_2 < \ldots < j_{n-s} \leqslant n)$ be the complement of I in $\underline{n}$. It is easy to see that $W(I) = D(n,2)/A_I \cong \langle a_k, 1 \leqslant k \leqslant n-s \rangle$ where a_k is represented by $I + e_{j_k, j_k+1}$. By identifying $W(I)$ with $< a_k$, $1 \leqslant k \leqslant n-s >$, the action of $W(I)$ on $H^*(A_I)$ is defined by

$$a_k y_i = y_i, \ i \in I,$$

$$a_k x_t = \begin{cases} x_t + y_{t+1} \text{ if } t+1 \in I, \ t = j_k, \\ x_t + y_t \quad \text{if } t \in I, \ t = j_k - 1, \\ x_t \text{ etherwise.} \end{cases}$$

It suffices to consider the case $i_s = n$. Let $f \in H^*(A_I)^{W(I)}$. Without loss of generality, we can suppose that f is homogeneous in $\{x_{n-1}, y_n\}$ (see Section 1). This means that f is of the form

$$f = \sum_{k=0}^{m} x_{n-1}^{n-k} y_n^k f_k(y_i, x_j, i \in I, i \neq n, \ 1 \leqslant j \leqslant n-2).$$

As in the proof of Theorem 1.2, the Lemma follows by induction on the degree of f.

As in the case $n = 3$ (see Section 2), we have

LEMMA A. 5.

(i) There exist unique cohomology classes $z_i \in H^2(D(n, 2))$, $1 \leqslant i \leqslant n-2$ *such that*

$$z_{i/A_I} = \begin{cases} M_i & \text{if } i, i-2 \in I, \\ V_{1i} V_{2,i+1} & \text{if } i \in I, i+2 \notin I, \\ V_{1,i+2} V_{2i} & \text{if } i \notin I, i+2 \notin I, \\ O & \text{if } i \notin I, i+2 \notin I. \end{cases}$$

(ii) *There exist unique cohomology classes* $v_i \in H^2(D(n,2))$, $1 \leqslant i \leqslant n-1$ *such that*

$$v_i|_{A_I} = \begin{cases} V_{2i}^2 \text{ if } i, i+1 \notin I, \\ V_{2i} \text{ otherwise.} \end{cases}$$

Further, $z_i \in F^1H^2(D(n,2)) \longrightarrow y_i \otimes x_{i+1} + y_{i+2} \otimes x_i \in E_0^{1,1}$,

$$v_i \in F^0H^2(D(n,2)) \longrightarrow 1 \otimes x_i^2 \in E_0^{0,2}.$$

Now let us denote $u_i = \inf(D(n,2), D(n,1))y_i$, $1 \leqslant i \leqslant n$.

By Lemmas A.3 and A.5, $H^*(D(n,2))$ is the algebra generated by the elements: u_i, v_j, z_k. We have the following

LEMMA A.6. *The homomorphism* Res: $H^*(D(n,2) \to \prod_I H^*(A_I)$ *given by the restriction homomorphisms, is injective.*

Here the direct product runs over the set of all admissible subsets of $\underline{n}$.

Proof. Suppose that $x \in F^{m-k}H^m(D(n,2))$, $x|A_I = 0$ for each I. If $k = 0$, $x \in F^mH^m(D(n,2) = H^m(D(n,1))$. Clearly $x = 0$. Suppose that $k > 0$. By using the following commutative diagrams

$$\begin{array}{ccc} F^{m-k}H^m(D(n,2)) & \longrightarrow & E^{m-k,k}(D(n,2), Z(D(n,2))) \\ \downarrow & & \\ F^{m-k}H^m(A_I) & \longrightarrow & E_0^{m-k,k}(A_I, Z(D(n,2))) \end{array}$$

induced by the inclutions $(A_I, Z(D(n,2))) \subset (D(n,2), Z(D(n,2)))$ for each I, we have $x \in F^{m-(k-1)}H^m(D(n,2))$. The Lemma follows by induction on k as in the proof of Lemma 2.9.

By considering the restrictions of the elements u_i, v_j, z_k on A_I, we obtain

LEMMA A.7. $z_i u_{i+1} = 0$, $u_i z_{i+1} + u_{i+3} z_i = 0$,

$$z_i^2 = u_i^2 v_{i+1} + u_{i+2}^2 v_i + u_i u_{i+2} z_i,$$

$$z_i z_{i+1} = u_i u_{i+3} v_{i+1}.$$

This completes the proof of Theorem A.1.

Acknowledgement. I wish to acknowledge my deep gratitude to my Teacher Huỳnh Mùi for his generous and inspiring guidance. My thanks are addressed to all members of the Algebraic topology seminar at the University of Hanoi for many valuable suggestions.

REFERENCE

[1] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, 1966.

[2] I.E. Dickson, *On linear modular group and their invariants*, Bull. A.M. Soc. 20 (1920), 132 — 134.

[3] W.C. Krathwohl, *Modular invariants of two pairs of congredient variables*, Amer. Jour. of Math., 34 (1914), 449 — 460.

[4] G.P. Hochschild and J.P. Serre, *Cohomology of group extension*, Trans. A.M. Soc., 74 (1953), 110 — 135.

[5] Huỳnh Mùi, *Modular invariant theory and cohomology algebras of symmetric groups*, Jour. Fac. Science, Univ. Tokyo, Sec. IA, Vol. 22, No.3, (1975), 319 — 369.

[6] Huỳnh Mùi, *The cohomology algebras of the groups of order* 2^4, Preprint series, Institute of Mathematics, Hanoi, No. 3, 1982.

[7] Huỳnh Mùi, *The cohomology algebras of the extra-special p-group* $E(p^3)$, preprint.

[8] Huỳnh Mùi, *An approach to study the mod p cohomology algebras of the general linear group over a prime field of p elements*, to appear in Acta Math. Vietnamica.

[9] G. Lewis, *The integral cohomology rings of group of order* p^3, Trans. A.M. Soc., 12 (1968), 501 — 529.

[10] Phạm Việt Hùng, *The algebra* $\dot{H}^*(GL(4,Z_2), Z_2)$, to appear in Acta Math. Vietnamica.

[11] M. Tezuka and N. Yagita, *The mod p cohomology ring of* $GL_3(Fp)$, Journal of Algebra, 2 (1983), 295 — 303.

[12] M. Tezuka and N. Yagita, *The cohomology of subgroups of* $GL_n(F_q)$, Proceeding of the Northwerstern Homotopy Theory Conference, Contemporary Mathematics, 19 (1983), 379 — 396.

Received June 15, 1984

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