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A REMARK ON CLARKE'S TANGENT CONE

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Consider a multivalued mapping F from a normed space X into another Y, and denote by $T_\Omega(z_0)$ the Clarke's tangent cone to the set $\Omega =$ graph F at $z_0 = (x_0, y_0) \in \Omega$. The aim of this paper is to describe a relation between the cone $T_\Omega(z_0)$ and the set $F(x_0)$. Namely we shall establish the following equality.

$$T_{F(x_0)}(y_0) = \{y : (0, y) \in T_\Omega(z_0)\} \quad (1)$$

Before proving this result let us recall some definitions.

DEFINITION 1. *Let Z be a normed space, let $\Omega \subset Z$ and $z_0 \in \Omega$. The Clarke's tangent cone to Ω at z_0, denoted by $T_\Omega(z_0)$, is the set of all $z \in Z$ with the following property: For every $\varepsilon > 0$ there exist $\lambda > 0$ and $\delta > 0$ such that*

$$[z' + t(z + B_Z(o, \varepsilon)] \cap \Omega \neq \phi,$$

for all $t \in (0, \lambda)$ and all $z' \in [z_0 + B_Z(0, \delta)] \cap \Omega$, where $B_Z(0,\alpha)$ denotes the closed ball in Z with radius α around $z = 0$ and ϕ stands for the empty set.

This definition is due to R. T. Rockafellar [1]. It is equivalent to the original definition of F. H. Clarke in [2].

As is well-known ([1], Theorem 1), $T_\Omega(z_0)$ is a nonempty closed convex cone. In addition, it has been shown in [3] that if Ω is convex, then $T_\Omega(z_0)$ coincides with the tangent cone in the sense of Convex Analysis, that is

$$T_\Omega(z_0) = \overline{\text{con}}\,(\Omega - z_0),$$

where $\overline{\text{con}}\,A$ indicates the closure of the cone generated by A.

In the sequel, the product space Z of the spaces X and Y will be equipped with the norm

$$\| z \| = \sqrt{\| x \|^2 + \| y \|^2} \quad (z = (x, y) \in Z).$$

Let us associate to the multivalued mapping F the sets

$$\Omega = \text{graph } F = \{(x, y) : x \in X, y \in F(x)\},$$
$$\text{dom } F = \{x \in X : F(x) \neq \phi\}.$$

DEFINITION 2. *The mapping F is lower semicontinuous at x_0, if for any $y_0 \in F(x_0)$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $F(x) \cap (y_0 + B_Y(0, \varepsilon)) \neq \phi$ whenever $x \in [x_0 + B_X(0, \delta)] \cap \text{dom } F$.*

DEFINITION 3. *The mapping F is locally Lipschitz at x_0 if there exist a neighbourhood U of x_0 and a positive real number α such that $F(x) \subset F(x') + \| x - x' \| B_Y(0, \alpha)$ for every pair x, $x' \in U$.*

Now, fix an arbitrary point $z_0 = (x_0, y_0) \in \Omega$ and consider the cone $T_\Omega(z_0)$.

PROPOSITION 1. *Let F be lower semicontinuous at x_0 and $F(x)$ be convex for all x in a neighbourhood of x_0. Then*

$$T_{F(x_0)}(y_0) \subseteq \{y : (0,y) \in T_\Omega(x_0, y_0)\} \tag{2}$$

Proof. Without loss of generality, we can assume that $x_0 = 0$ and $y_0 = 0$.

Let $y_1 \in F(0)$, $y_1 \neq 0$. We must prove that

$$(0, y_1) \in T_\Omega(z_0).$$

To this end we fix an arbitrary $\varepsilon > 0$. Since F is lower semicontinuous at x_0, there is $\delta \in (0, \varepsilon)$ such that

$$\left(y_1 + B_Y\left(0, \frac{\varepsilon}{3}\right)\right) \cap F(x') \neq \phi \tag{3}$$

whenever $x' \in B_X(0, \delta) \cap \text{dom } F$.

Let us set

$$\lambda = \frac{\varepsilon}{3 \| y_1 \|}. \tag{4}$$

Assume that z' and t satisfy the following condition:

$$z' = (x', y') \in B_Z(0, \delta) \cap \Omega,\ t \in (0, \lambda). \tag{5}$$

According to (3), there is $y'' \in F(x')$ such that

$$\| y_1 - y'' \| \leqslant \frac{\varepsilon}{3}. \tag{6}$$

If we set

$$\bar{x} = x',$$

$$\bar{y} = \frac{1}{1+t} y' + \frac{t}{1+t} y'',$$

then clearly $(\bar{x}, \bar{y}) \in \Omega$, because y', $y'' \in F(x')$ and $F(x')$ is convex.

From (3) through (6) it follows that

$$\| (x', y') + t(0, y_1) - (\bar{x}, \bar{y}) \| =$$
$$= \frac{1}{1+t} \| ty' + t^2 y_1 + t(y_1 - y'') \|$$
$$\leqslant t(\| y' \| + t \| y_1 \| + \| y_1 - y' \|)$$
$$\leqslant t(\delta + \lambda \| y_1 \| + \frac{\varepsilon}{3}) \leqslant t\varepsilon.$$

Hence,

$$(x', y') + t(0, y_1) \in \Omega + tB_Z(0, \varepsilon),$$

that is

$$[(x', y') + t((0, y_1) + B_z(0, \varepsilon))] \cap \Omega \neq \phi. \tag{7}$$

Therefore, for every $\varepsilon > 0$ we can find $\lambda > 0$ and $\delta > 0$ such that (7) holds for every pair (z', t) satisfying (3). This means that $(0, y_1) \in T_\Omega(z_0)$.

In the case where $y_1 \in F(0)$, $y_1 = 0$, the last inclusion is obvious. Since $T_{F(o)}(0) = \overline{\text{con}}\, F(0)$ and $T_\Omega(z_0)$ is a closed convex cone, the proof is thus complete.

The following proposition shows the conditions under which the converse of inclusion (2) is true.

PROPOSITION 2. *If F is locally Lipschitz at x_0 and for all x in a neighbourhood of x_0, $F(x)$ is convex, then the converse of inclusion (2) holds.*

Proof. Assume again that $x_0 = 0$, $y_0 = 0$. Because of the convexity of $F(x_0)$ we get

$$T_{F(x_0)}(y_0) = \overline{\text{con}}\, F(0).$$

To prove the desired inclusion, we have to show that $\bar{y} \notin \overline{\text{con}}\, F(0)$ implies $(0, \bar{y}) \notin T_\Omega(z_0)$. Indeed, if $\bar{y} \notin \overline{\text{con}}\, F(0)$, there is $\eta > 0$ such that, for all $t \geqslant 0$,

$$\left\{ t\bar{y} + \frac{t}{2} B_Y(0, \eta) \right\} \cap \left\{ F(0) + \frac{t}{2} B_Y(0, \eta) \right\} = \phi. \tag{8}$$

Since F is locally Lipschitz at 0, one can find a convex neighbourhood U of 0 and a positive real number α such that, for every $x \in U$,

$$F(x) \subset F(0) + \| x \| \,.\, B_Y(0, \alpha). \tag{9}$$

Let us set $W_1 = U_1 \times V_1$, where $U_1 = U \cap B_X\left(0, \frac{\eta}{2\alpha}\right)$ and $V_1 = B_Y\left(0, \frac{\eta}{2}\right)$.

To prove the condition $(0, \bar{y}) \notin T_\Omega(z_0)$, it suffices to show that, for any $\lambda > 0$ and any neighbourhood $W' = U' \times V'$ of zero, there exist $t \in (0, \lambda)$ and $(x', y') \in W' \cap \Omega$ satisfying

$$[(x', y') + t((0, \bar{y}) + W_1)] \cap \Omega = \varnothing. \tag{10}$$

Indeed, putting $t = \min\left(\frac{\lambda}{2}, 1\right)$, $x' = 0$, $y' = 0$ and taking acount of (8), one has

$$y' + t(\bar{y} + y) \notin F(0) + \frac{t}{2} B_Y(0, \eta) \tag{11}$$

for all $y \in V_1$.

On the other hand, from (9) it follows that

$$F(x' + tx) \subset F(0) + \frac{t}{2} B_Y(0, \eta) \tag{12}$$

for all $x \in U_1$

Combining (11) and (12) yields $y' + t(\bar{y} + y) \notin F(x' + tx)$, whenever $x \in U_1$ and $y \in V_1$. This means that the condition (10) holds

Q.E.D.

As a consequence of Propositions 1 and 2 we have

THEOREM. *Let X, Y be normed spaces and F a multivalued mapping from X to Y which is locally Lipschitz at x_0 and takes convex values in a neighbourhood of x_0. Then, for every $y_0 \in F(x_0)$, the inclusion (1) holds.*

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REFERENCES

[1] R.T. Rockafellar, *Clarke's tangent cone and the boundaries of closed sets in R^n*, Nonlinear Analysis 3(1979), 145—154.

[2] F.H. Clarke, *Generalized gradients and applications*, Trans. A.M.S. 205 (1975), 247—262.

[3] R.T. Rockafellar, *Generalized directional derivatives and subgradients of nonconvex functions*, Can. J. Math., Vol. XXXII, No. 2 (1980), 257—280.

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