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ON A GRÄTZER'S PROBLEM

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1. INTRODUCTION.

It is obvious that a lattice L determines Sub (L) (the lattice of all sublattice of a lattice L), that is, if L is isomorphic to L' then Sub (L) is also isomorphic to Sub (L'). The converse is not true. L.N. Sevrin [1] and N.D. Filippov [2] (1964 and 1966) give necessary and sufficient conditions for Sub (L) is isomorphic to Sub (L'). George Grätzer in his book « General Lattice Theory », 1978 has given the following problem: « Find conditions under which Sub (L) determines L up to isomorphism » ([3], Problem I. 4). This problem is one of the basic and important topics in studying Sub (L).

Observe that for an arbitrary lattice L, if L^* is the dual lattice of L then Sub (L^*) is isomorphic to Sub (L). Therefore, this work will give conditions under which Sub (L) determines L up to isomorphism and dual isomorphism.

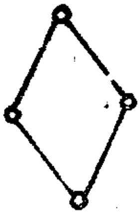
In this paper, among others we shall prove the important result: « Let L be a modular lattice of locally finite length which has no linear decompositions. Then Sub (L) determines L up to isomorphism and dual isomorphism ».

2. CONCEPTS AND RESULTS.

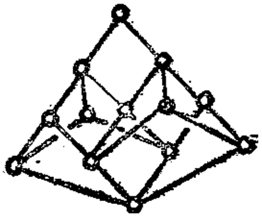
Let a, b, c, d be elements of a lattice L. A symbol $\langle a, b; c, d\rangle$ is called a square in L iff $a//b$ and $\{a\vee b, a\wedge b\} = \{c, d\}$. Thus, $\{a, b, c, d\}$ is a sublattice of L, $\{a, b, c, d\} \cong D_4$ and Sub $(\{a, b, c, d\}) \cong N_{13}$. (« $\cong$ » = « isomorphic to »)

If $a \wedge b \prec a,b \prec a \vee b$ in L, then the square $\langle a,b;\ a\wedge b, a\vee b\rangle$ is called unit. We often say that a,b,c,d are elements of $\langle a,b;\ c,d\rangle$.

For the professional symbols and definitions of Lattice Theory, see G. Grätzer [3].



D_4



N_{13}

THEOREM 1 (see also L.N. Sevrin [1], N.D. Filippov [2]).

Let L and L' be arbitrary lattices. Sub (L) is isomorphic to sub(L') iff there is a one-to-one and onto map $\varphi: L \to L'$, such that, $\langle a,b;\ c,d\rangle$ is a square in L iff $\langle \varphi(a), \varphi(b);\ \varphi(c), \varphi(d)\rangle$ is a square L'.

Proof. *Suffciiency.* We define a map $f: P(L), \to P(L')$ $P(L)$ is the set of all subsets of a set L), as follows: $f: A \in P(L) \to \varphi(A) \in P(L')$. It is obvious that $f: P(L) \longrightarrow P(L')$ is one-to-one and onto. Now we show that $A \in \mathrm{Sub}(L)$ iff $f(A) \in \mathrm{Sub}(L')$. Suppose $A \in \mathrm{Sub}(L)$. If $p,q \in f(A)$ (we can assume that $p//q$), then $\langle p,q;\ p \wedge q,\ p \vee q\rangle$ is a square in L', so $\langle \varphi^{-1}(p), \varphi^{-1}(q);\ \varphi^{-1}(p \wedge q), \varphi^{-1}(p \vee q)\rangle$ is also a square in L, that is, $\{\varphi^{-1}(p) \wedge \varphi^{-1}(q), \varphi^{-1}(p) \vee \varphi^{-1}(q)\} = \{\varphi^{-1}(p \wedge q), \varphi^{-1}(p \vee q)\}$. Hence $\varphi^{-1}(p \wedge q), \varphi^{-1}(p \vee q) \in A$, and so, $p \wedge q$, $p \vee q \in \varphi(A) = f(A)$. This means that $f(A) \in \mathrm{Sub}(L')$. Likewise, we verify that $f(A) \in \mathrm{sub}(L')$ implies that $A \in \mathrm{Sub}(L)$. Thus, $f_{\mathrm{Sub}(L)}: \mathrm{Sub}(L) \to \mathrm{Sub}(L')$ is one-to-one and onto. Moreover, for $A,B \in \mathrm{Sub}(L)$, $A \subseteq B$ iff $f_{\mathrm{Sub}(L)}(A) \subseteq f_{\mathrm{Sub}(L)}(B)$ This means that $f_{\mathrm{Sub}(L)}$ is an isomorphism, that is, $\mathrm{Sub}(L) \cong \mathrm{Sub}(L')$.

Necessity. Let f be an isomorphism from Sub(L) onto Sub(L'). We define a map $\varphi: L \to L'$ as follows: $a \in L \to a' \in L'$ iff $f(\{a\}) = \{a'\}$. It is easy to see that φ is one-to-one and onto. Suppose $f(\{a_i\}) = \{a'_i\}$, $i = 1,2,\ldots,n$. Then

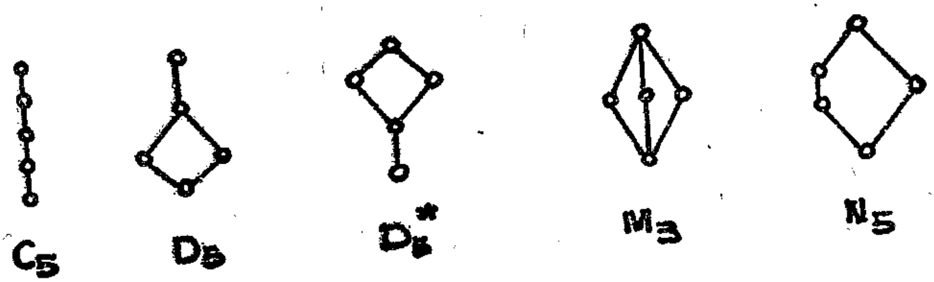
$$f([a_1,\ldots,a_n]) = f(\{a_1\} \vee \ldots \vee \{a_n\}) = f(\{a_1\}) \vee \ldots \vee f(\{a_n\}) = \{a'_1\} \vee \ldots \vee \{a'_n\} =$$

$[a'_1,\ldots,a'_n] = [\varphi(a_1),\ldots,\varphi(a_n)]$. Because the height function h is finite length-preserving for the isomorphism, so $h([a,b,c,d]) = h(f([a,b,c,d])) = h([\varphi(a), \varphi(b), \varphi(c), \varphi(d)])$ for $a,b,c,d \in L$. Therefore, $\langle a,b;\ c,d\rangle$ is a square in L iff $4 = h([a,b,c,d]) = h([a,b])$ iff $4 = h([\varphi(a), \varphi(b), \varphi(c), \varphi(d)]) = h([\varphi(a), \varphi(b)])$ iff $\langle \varphi(a), \varphi(b);\ \varphi(c), \varphi(d)\rangle$ is a square in L'. This completes the proof of the Theorem.

For getting at the main result we have to prove some lemmas.

LEMMA 1. *Sub(L) is isomorphic to Sub(N_5) iff L is isomorphic to N_5.*

Proof. It is easy to see that there are only five non-isomorphic lattices of five elements, as shown in the following figures:



It is clear that in C_5 there is no, in D_5 and D_5^* there is one, in M_3 there are three squares, while in N_5 there are two squares. Hence by Theorem 1 we obtain at once that Sub$(L) \cong$ Sub(N_5) implies that $L \cong N_5$. The converse is trivial.

LEMMA 2. *A lattice L is modular iff Sub(L) does not contain a principal ideal which is isomorphic to Sub(N_5). (A lattice L is called modular, if it satisfies the identity $(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee (x \wedge z))$ for $x, y, z \in L$).*

Proof. A lattice L is not modular iff L contains its sublattice N_5. Thus, by Lemma 1, L contains its sublattice N_5 iff sub(L) contains a principal ideal which is isomorphic to Sub(N_5).

A linear decomposition L_i, $i \in I$, of a lattice L consists of a chain I and sublattices L_i of L $(i \in I)$, such that, if $i, j \in I$, $i < j$, then $a < b$ in L for all $a \in L_i$ and $b \in L_j$ and $L = U\ (L_i \mid i \in I)$. If $|I| \geqslant 2$, we say that L has a proper linear decomposition. If for any linear decomposition L_i, $i \in I$ of the lattice L, we have $|I| = 1$, then we say that L has no linear decompositions.

LEMMA 3. *Let L be a lattice and let L_i, $i \in I$ be a linear decomposition of L. Then Sub(L) is isomorphic to Π (Sub(L_i) | $i \in I$).*

Proof. If D is a sublattice of L, then $D \cap L_i$ is a sublattice of L_i for each $i \in I$ Moreover, it is obvious that if D_i is a sublattice of L_i, $i \in I$, then $\cup\ (D_i \mid i \in I)$ is a sublattice of L. Define a map φ : Sub$(L) \rightarrow \Pi($Sub$(L_i) \mid i \in I)$ as follows: for $D \in$ Sub (L), φ: $D \rightarrow f^D \in \Pi$ (Sub $(L_i) \mid i \in I)$, where $f^D(i) = D \cap L_i$ for every

$I \in I$. Because $L_i \cap L_j = \phi$, $i \neq j$, $i, j \in I$ and $D = D \cap L = D \cap (\cup(L_i \mid i \in I)) = \cup(D \cap L_i \mid i \in I)$, we see that if $D, E \in \mathrm{Sub}(L)$ and $D \cap L_i = E \cap L_i$ for every $i \in I$, then $D = E$. This means that φ isone-to-one. If $f \in \Pi(\mathrm{Sub}(L_i) \mid i \in I)$, where $f(i) = D_i$, $D_i \in L_i$ for every $i \in I$, then $D = \cup(D_i \mid i \in I) \in \mathrm{Sub}(L)$ and $D \cap L_i = D_i$ for every $i \in I$, and hence $\varphi(D) \in \Pi(\mathrm{Sub}(L_i) \mid i \in I)$ From this it follows that φ is onto. Besides, for $D, E \in \mathrm{Sub}(L)$, $D \subseteq E$ iff $D \cap L_i \subseteq E \cap L_i$ for every $i \in I$ iff $\varphi(D) \subseteq \varphi(E)$. We have thus proved the isomorphism of φ. Hence, $\mathrm{Sub}(L) \cong \Pi(\mathrm{Sub}(L_i) \mid i \in I)$.

Lemma 3 shows that to solve the problem we have to consider lattices which have no linear decompositions.

Now we introduce some concepts which we shall need in the sequel. An element a ($a \in L$) is called linear iff a is comparable with b for every $b \in L$. An atom element $a \in L$ is called isolated iff $a \vee b \succ a, b$ for every atom $b \in L$. It is easy to see that $a \in L$ is linear iff $\{a\} \in \mathrm{Sub}(L)$ is isolated. A lattice L is called to be of locally finite length, if every interval $[p,q]$, $p,q \in L$, is of finite length.

LEMMA 4. *Let L be a lattice. If $|L| > 2$, then the following conditions are equivalent:*

(i) L is of locally finite length and has no linear decompositions.

(ii) L is of locally finite length and if c is a linear element of L and $c \neq 0,1$, then there are two squares $\langle a_1,b_1; c,d_1\rangle$ and $\langle a_2,b_2; c,d_2\rangle$, such that, a_1,b_1 are comparable with a_2, b_2.

(iii) Every Boolean principal ideal of Sub($[p, q]$), $p, q \in L$, is finite and if $\{c\}$ is an isolated element of Sub(L) and $\{c\} \neq \{0\}, \{1\}$, then there are two N_{13} principal ideals containing c, such that, if $\{a_1\}$, $\{b_1\}$ and $\{a_2\}$, $\{b_2\}$ are their atoms, respectively, and $h(\{a_1\} \vee \{b_1\}) = h(\{a_2\} \vee \{b_2\}) = 4$, then $h(\{a_1\} \vee \{a_2\}) = h(\{a_1\} \vee \{b_2\}) = h(\{b_1\} \vee \{a_2\}) = h(\{b_1\} \vee \{b_2\}) = 2$.

Proof. K **is a chain of** L **iff Sub(K) is a Boolean (principal) ideal of Sub(L), and if** K **is a finite chain, then** $|\mathrm{Sub}(K)| = 2^{|K|}$**. Hence,** L **is of locally finite length iff every Boolean principal ideal of Sub ($[p, q]$),** $p, q \in L$**, is finite.**

(i) ⇒ *(ii)*: We observe that if c is a linear element of L and if $\langle a_1, b_1; c, d_1\rangle$ and $\langle a_2, b_2; c, d_2\rangle$ are squares in L, then only one of the following conditions hold:

Condition 1. $a_1, b_1 > c > a_2, b_2$ (or $a_1, b_1 < c < a_2, b_2$). This means that a_1, b_1 are comparable with a_2, b_2.

Condition 2. $c > a_1, b_1, a_2, b_2$ (or $c < a_1, b_1, a_2, b_2$).

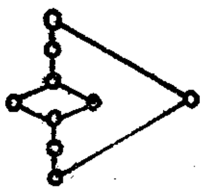
It is obvious that 0 and 1 satisfy Condition 2.

Let us assume that $c \neq 0,1$ and c satisfies Condition 2 for every pair $\langle a_1, b_1; c, d_1 \rangle$ and $\langle a_2, b_2; c, d_2 \rangle$. Hence, we can assume that $c > a, b$ for every $\langle a, b; c, d \rangle$ in L. Then $K_1 = \{t \mid t > c, t \in L\}$ and $K_2 = \{t \mid t \leqslant c, t \in L\}$ are sublattices of L and they form a proper linear decomposition of L, which contradicts (i). Thus, Condition 1 is satisfied for some pair $< a_1, b_1; c, d_1 >$ and $\langle a_2, b_2; c, d_2 \rangle$.

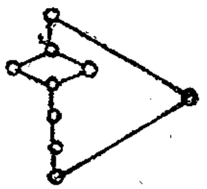
(*ii*) $\Rightarrow$ (*i*) By hypothesis L is of locally finite length, hence, every its interval $[p, q]$ is complete. Suppose that L has a proper linear decomposition. Then, we have $L = K_1 \cup K_2$, where K_1 and K_2 are sublattices of L satisfying $a > b$ for every $a \in K_1$, $b \in K_2$. Thus, $\inf K_1 > \operatorname{Sup} K_2$. Since $|L| > 2$, we can assume that $c = \inf K_1 \neq 1$ (or $c = \sup K_2 \neq 0$). Therefore, it is clear that for every $\langle a, b; c, d \rangle$ in L we have $c < a, b$, hence c satisfies Condition 2. This contradicts (ii).

(*ii*) $\Leftrightarrow$ (*iii*) Because $a \in L$ is linear iff $\{a\} \in \operatorname{Sub}(L)$ is isolated; $\langle a, b; c, d \rangle$ is square in L iff $\operatorname{Sub}(\{a, b, c, d\}) \cong N_{13}$; and a_1, b_1 are comparable with a_2, b_2 iff $h(\{a_1\} \vee \{a_2\}) = h(\{a_1\} \vee \{b_2\}) = h(\{b_1\} \vee \{a_2\}) = h(\{b_1\} \vee \{b_2\}) = 2$ From this it follows at once that (ii) and (iii) are equivalent.

Using Theorem 1 we see that two non-modular lattices L_1 and L_2 shown in the following figures are neither isomorphic nor dually isomorphic, but $\operatorname{Sub}(L_1) \cong \operatorname{Sub}(L_2)$



L_1



L_2

From Theorem 1, Lemma 3 and Lemma 4 we observe that only modular lattices of locally finite length which have no linear decompositions are interesting for us, first of all.

Before expressing and proving the main theorem we have to introduce some necessary concepts and to prove some auxilinary statements.

Now we assume that L and L' are lattices and $\operatorname{Sub}(L) \cong \operatorname{Sub}(L')$. Then we have

Statement 1. Let $\langle a_1, a_2; a_3, a_4 \rangle$ be a square in L. If $a_{i_0} < a_{j_0}$ implies that $\varphi(a_{i_0}) < \varphi(a_{j_0})$ for some $i_0, j_0 \in \{1, 2, 3, 4,\}$, then the lattice $\{a_1, a_2, a_3, a_4\}$ is isomorphic to the lattice $\{\varphi(a_1), \varphi(a_2), \varphi(a_3), \varphi(a_4)\}$

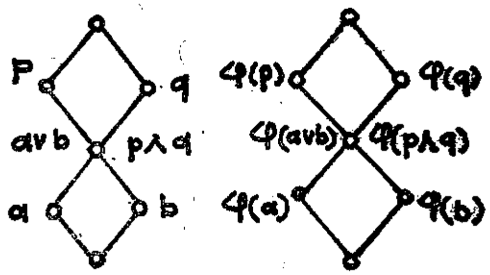
Proof. It is obvious by Theorem 1.

We say that finitely many squares $\langle a_i, b_i; c_i, d_i \rangle$ in L ($i \in I$, I being finite) are associate, if for any pair $\langle a_j, b_j; c_j, d_j \rangle$ and $\langle a_k, b_k; c_k, d_k \rangle$ $j, k \in I$ there is a serie $\langle a_j, b_j; c_j, d_j \rangle = \langle a_{i_1}, b_{i_1}; c_{i_1}, d_{i_1} \rangle, \ldots, \langle a_{i_n}, b_{i_n}; c_{i_n}, d_{i_n} \rangle = \langle a_k, b_k; c_k, d_k \rangle$, $i_1, \ldots, i_n \in I$, such that, either $a_{i_t} \wedge b_{i_t} = a_{i_{t+1}} \vee b_{i_{t+1}}$ (or $a_{i_t} \vee b_{i_t} = a_{i_{t+1}} \wedge b_{i_{t+1}}$) or $|\{a_{i_t}, b_{i_t}, c_{i_t}, d_{i_t}\} \cap \{a_{i_{t+1}}, b_{i_{t+1}}, c_{i_{t+1}}, d_{i_{t+1}}\}| \geqslant 2$, $t = 1, \ldots, n-1$.

Associate unit squares are associate squares, each of which is unit.

Statement 2. Assume that $\langle a, b; a \vee b, a \wedge b \rangle$ and $\langle p, q; p \vee q, p \wedge q \rangle$ are two squares, such that, $a \vee b = p \wedge q$. If $\{a, b, a \wedge b, a \vee b\}$ is isomorphic to $\{\varphi(a), \varphi(b), \varphi(a \vee b), \varphi(a \wedge b)\}$ then $\{p, q, p \vee q, p \wedge q\}$ is also isomorphic to $\{\varphi(p), \varphi(q), \varphi(p \vee q), \varphi(p \wedge q)\}$.

Proof We have show that $\varphi(p) \wedge \varphi(q) = \varphi(p \wedge q)$. Clearly, $p, q > a, b$ imply that $\varphi(p), \varphi(q)$ are comparable with $\varphi(a), \varphi(b)$. Because $\varphi(p) // \varphi(q)$, $\varphi(a) // \varphi(b)$, we have either $\varphi(p), \varphi(q) > \varphi(a), \varphi(b)$ or $\varphi(p), \varphi(q) < \varphi(a), \varphi(b)$. But $\varphi(p), \varphi(q) < \varphi(a), \varphi(b)$ imply that $\varphi(p) \wedge \varphi(q) < \varphi(p) \vee \varphi(q) \leqslant \varphi(a) \wedge \varphi(b) < \varphi(a) \vee \varphi(b) = \varphi(a \vee b)$, which is impossible for $\varphi(a \vee b) = \varphi(p \wedge q)$. Thus, we obtain $\varphi(p), \varphi(q) > \varphi(a), \varphi(b)$, hence $\varphi(p) \vee \varphi(q) > \varphi(p) \wedge \varphi(q) \geqslant \varphi(a \vee b) = \varphi(p \wedge q)$, which means that $\varphi(p) \wedge \varphi(q) = \varphi(p \wedge q)$.

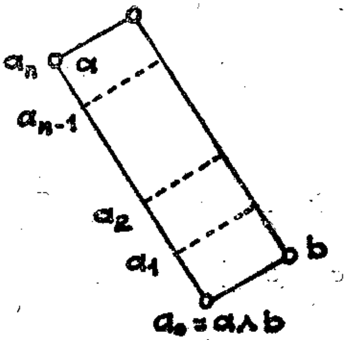


Statement 3. Let $\langle a_i, b_i; c_i, d_i \rangle$ $i \in I$ (I being finite) be associate squares in L If there is $j_0 \in I$, such that, the lattice $\{a_{j_0}, b_{j_0}, c_{j_0}, d_{j_0}\}$ is isomorphic to the lattice $\{\varphi(a_{j_0}), \varphi(b_{j_0}), \varphi(c_{j_0}), \varphi(d_{j_0})\}$ then the lattice $\{a_i, b_i, c_i, d_i\}$ is isomorphic to the lattice $\{\varphi(a_i), \varphi(b_i), \varphi(c_i), \varphi(d_i)\}$ for every $i \in I$.

Proof. It is derived immediately from the definition of associate squares. Statement 1 and Statement 2.

Statement 4. Let L be a modular lattice and $\{a, b; a \vee b, a \wedge b\}$ be a square in L, such that, $a \wedge b \prec b$ (or dually, $a \prec a \vee b$). Then from a serie $a \wedge b = a_0 \prec a_1 \ldots \prec a_n = a$, we have:

$\langle a_1, b; a_0, a_1 \vee b\rangle$, $\langle a_2, a_1 \vee b; a_1, a_2 \vee b\rangle, \ldots$, $\langle a_i, a_{i-1} \vee b; a_{i-1}, a_i \vee b\rangle, \ldots \langle a_n, a_{n-1} \vee b; a_{n-1}, a_n \vee b\rangle$ are associate unit squares.



Proof. It is evident.

LEMMA 5. *Assume that L is of locally finite length, modular lattice and has no linear decompositions. If $\langle a, b; a \vee b, a \wedge b\rangle$ is a unit square in L, then for each $s \in L$ there are associate unit squares in L, one of which is $\langle a, b; a \vee b, a \wedge b\rangle$ and amongst which we can find a square containing s.*

Proof. Here we have four cases:

— Case 1: $s \parallel a$, $s > a \wedge b$ (or dually, $s \parallel a$, $s < a \vee b$).

— Case 2: $s \parallel a \vee b$, $s > a$ (or dually, $s \parallel a \wedge b$, $s < a$).

— Case 3: $s < a \vee b$ (or dually, $s < a \wedge b$).

— Case 4: $s \parallel a, b, a \vee b, a \wedge b$.

Case 1. Because $s \parallel a$, $s > a \wedge b$, we have a square $\langle s, a; a \wedge b, s \vee a\rangle$ with $a \wedge b \prec a$. Thus, because of finte length of $[a \wedge b, s]$ and Statement 4 we have found the desired associate unit squares (see Figure 1).

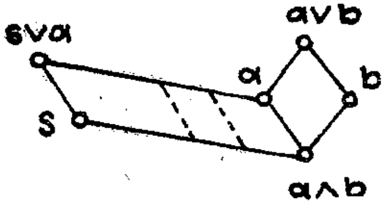


Figure 1

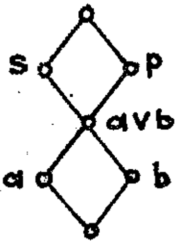


Figure 3

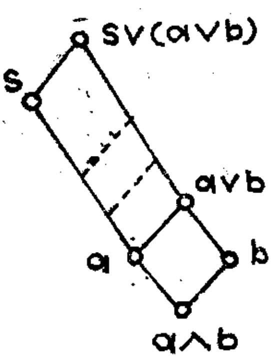


Figure 2

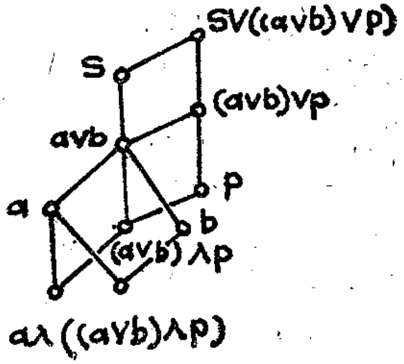


Figure 4

Case 2. Because $s \parallel a \vee b, s > a$, we have a square $\langle s, a \vee b; a, s \vee (a \vee b)\rangle$ with $a \prec a \vee b$. Thus, because of finite length of $[a, s]$ and Statement 4 we have found the desired associate unit squares (see Figure 2).

Case 3 It is proved by finite induction on the length of $[a \vee b, s]$. For $h([a \vee b, s]) = 1$, the statement is clear, because if $a \vee b$ is a linear element, then there is an element p, such that, $p \succ a \vee b$ and $p \neq s$, and we obtain the desired unit square $\langle s, p; a \vee b, s \vee p\rangle$ (see Figure 3); or if $a \vee b$ is not linear, then by modularity and locally finite length of L there is an element p, such that $(a \vee b) \wedge p \prec a \vee b$, $p \prec (a \vee b) \vee p$, and so, we obtain at most three desired unit squares: $\langle a, (a \vee b) \wedge p; -, -\rangle$, $\langle a \vee b, p; -, -\rangle$ and $\langle s, (a \vee b) \vee p, -, -\rangle$ (see Figure 4). Thus, we have proved the statement for $h([a \vee b, s]) = 1$.

Now we assume that $s \in L$, $s > a \vee b$ and $h([a \vee b, s]) = n$. Then, by locally finite length of L there is an element r, such that, $r \in [a \vee b, s]$ and $r \prec s$. Since $h([a \vee b, r]) = n-1$, by the induction there are associate unit squares $\langle a_i, b_i; c_i, d_i\rangle$, $i \in I$ (I being finite), such that, r is an element of some square $\langle a_{i_0}, b_{i_0}; c_{i_0}, d_{i_0}\rangle$ with $i_0 \in I$ (We can assume that $s \notin \{a_i, b_i, c_i, d_i\}/i \in I$). Here we have three possibilities:

(i) $r = a_{i_0} \wedge b_{i_0}$. This is Case 1

(ii) $r = a_{i_0}$ (b_{i_0}). This is Case 2.

(iii) $r = a_{i_0} \vee b_{i_0}$. This is proved by an argument analogous to that used for the proof of Case 3 for $h([a \wedge b, s]) = 1$.

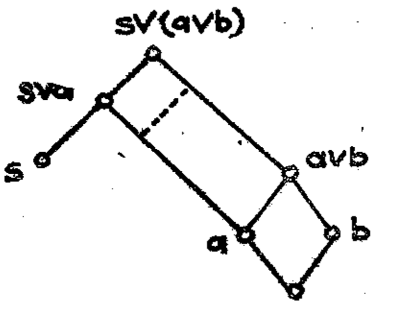


Figure 5

Case 4. Because of modularity of L we have either $s \vee a \neq s \vee (a \vee b)$ and $\langle s \vee a, a \vee b; s \vee (a \vee b), a\rangle$ is a square (see Figure 5); or $s \vee a = s \vee (a \vee b)$, but $s \wedge (a \vee b) \neq s \wedge a$ and $\langle s \wedge (a \vee b), a; a \vee b, s \wedge a\rangle$ is a square (see Figure 6). Hence, from Statement 4 and Case 2, the desired conclusion follows at once.

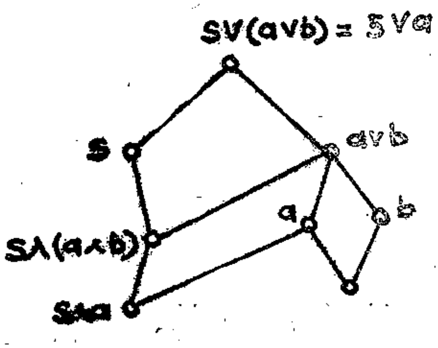


Figure 6

We are now in a position to formulate the main result of the paper.

THEOREM 2. (main result).

Let L be a modular lattice of locally finite length which has no linear decompositions. Then Sub (L) determines L up to isomorphism and dual isomorphism.

By Lemmas 2 and 4, this result can be restated as follows:

THEOREM 2'. *Let L be a lattice. If* $|Sub(L)| = 2$ *or Sub(L) satisfies condition (iii) of*

Lemma 4 and does not contain a principal ideal which is isomorphic to Sub (N_5), then, Sub(L) determines L up to isomorphism and dual isomorphism.

Proof of main theorem.

If $|\mathrm{Sub}(L)| = 2$, then $|L| = 1$ and the proof of the theorem is trivial. Thus, let us assume that $|L| > 2$. As for the proof of Theorem 1, we define the map $\varphi: L \to L'$ as follows: $\varphi(a) = a'$ iff $f(\{a\}) = \{a'\}$. By the supposition of the theorem there is a unit square in L, for example, $\langle a, b; a \vee b, a \wedge b\rangle$. Consider two cases:

Case A: The lattice $\{a, b, a \vee b, a \wedge b\}$ is isomorphic to $\{\varphi(a), \varphi(b), \varphi(a \vee b), \varphi(a \wedge b)\}$ that is,

$$\varphi(a \vee b) = \varphi(a) \vee \varphi(b),$$
$$\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b).$$

Case B: $\varphi(a \vee b) = \varphi(a) \wedge \varphi(b)$,
$\varphi(a \wedge b) = \varphi(a) \vee \varphi(b)$.

For Case A we shall prove that L is isomorphic to L'. By locally finite length of L it is enough to show that for $r, s \in L$, $r \prec s$ implies that $\varphi(r) < \varphi(s)$. Thus, let us assume that $r, s \in L$ and $r \prec s$. By Lemma 5 there are associate unit squares in L, one of which is $\langle a, b; a \vee b, a \wedge b\rangle$ and amongst which we can find a square $\langle a_0, b_0; a_0 \vee b_0, a_0 \wedge b_0\rangle$ containing r. Taking account of Statement 3 we see that $\{a_0, b_0, a_0 \vee b_0, a_0 \wedge b_0\}$ is isomorphic to $\{\varphi(a_0), \varphi(b_0), \varphi(a_0 \vee b_0), \varphi(a_0 \wedge b_0)\}$. Because $r \prec s$, we can apply the argument used in the proof of Lemma 5 to Case 1, Case 2 and Case 3, shown in the Figure 7.

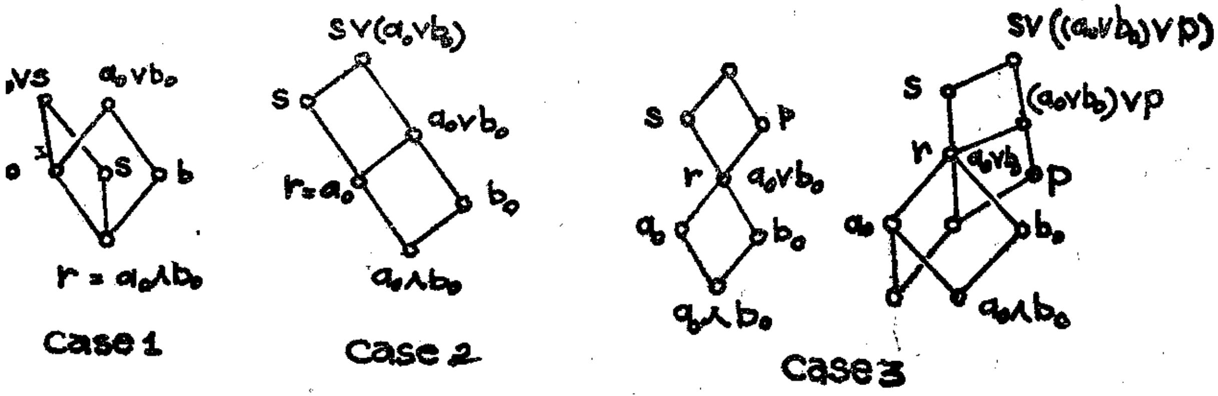


Figure 7

So we obtain an unit square

($\langle a_0, s; r, a_0 \vee s\rangle$ for Case 1,

$\langle s, a_0 \vee b_0; r, s \vee (a_0 \vee b_0)\rangle$ for Case 2,

$\langle s, p; r, s \vee p\rangle$ or $\langle s, (a_0 \vee b_0) \vee p; r, s \vee ((a_0 \vee b_0) \vee p)\rangle$ for Case 3) which is associate to $\langle a_0, b_0; a_0 \vee b_0, a_0 \wedge b_0\rangle$. Hence, by Statement 3 we get $\varphi(r) < \varphi(s)$. We have proved that L is isomorphic to L' for Case A.

For Case B the argument used above shows that $\hat{L}$ is isomorphic to the dual lattice of L', hence L is dually isomorphic to L'. The proof of the main theorem is thus complete.

Question. Is there an infinite lattice L (L is not of locally finite length) for which Sub (L) determines L', up to isomorphism and dual isomorphism?

REFERENCES

[1] L.N. Sevrin, *Projectivities of semilattices (in Russian)*, Dokl. Akad. Nauk SSSR, 154: 538 — 541 (1964).

[2] N.D. Filippov, *Projectivity of lattices (in Russian)*, Mat. Sb. 70 (112): 36—45 (1966).

[3] G. Grätzer, *General Lattice Theory*, Akademie — Verlag — Berlin, (1978).

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