ACTA MATHEMATICA VIETNAMICA

Vol. 10 N° 1 (1985)

REPRESENTATION OF EXTENDED SUPERALGEBRAS

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I. INTRODUCTION :

One of the major problems in physics is to search for an unified theory of all interactions in nature: gravitational, strong, weak and electromagnetic ones. Recently many physicists hope that such a theory might be found in the framework of extended supersymmetry. Many models of supersymmetry have been proposed. But is still unknown that among them which is the law of nature.

In a previous studry [1], we have presented a class of extended superalgebras in the following form :

$$[M_{\mu\nu}, M_{\rho\tau}] = g_{\mu\rho}M_{\nu\tau} + g_{\nu\tau}M_{\mu\rho} - g_{\mu\tau}M_{\nu\rho} - g_{\nu\rho}M_{\mu\tau} \quad (1.1.a)$$

$$[M_{\mu\nu}, P_{\rho}] = i(g_{\mu\rho}P_{\nu} - g_{\nu\rho}P_{\mu}) \quad (1.1.b)$$

$$\left[M_{\mu\nu}, Q^{L}_{\alpha}\right] = \frac{1}{2}(\delta_{\mu\nu})_{\alpha\beta}Q^{L}_{\beta} \quad (1.1.c)$$

$$[M_{\mu\nu}, B_{l}] = 0 \quad (1.1.d)$$

$$[B_{l}, B_{m}] = ic^{k}_{lm}B_{k} \quad (1.1.e)$$

$$[B_{l}, P_{\rho}] = 0 \quad (1.1.f)$$

$$[B_{l}, Q^{L}_{\alpha}] = s^{LM}_{l}Q^{M}_{\alpha} \quad (1.1.g)$$

$$\{Q^{L}_{\alpha}, \overline{Q}^{M}_{\beta}\} = c^{LM}(б_{\mu})_{\alpha\beta}\cdot P^{\mu} \quad (1.1.h)$$

$$\{Q^{L}_{\alpha}, Q^{M}_{\beta}\} = \epsilon_{\alpha\beta}\sum_{l} a^{LM}_{l}B_{l} \quad (1.1.i)$$

$$[P_{\rho}, Q^{L}_{\alpha}] = d^{LM}(б_{\rho})_{\alpha\beta}\cdot\overline{Q}^{M}_{\beta}. \quad (1.1.j)$$

$$[P_{\mu}, P_{\nu}] = 0 \quad (1.1.k)$$

(Here the two component Weyl spinor formalism is used, $\alpha, \beta = 1, 2$ denote the spinor indices; $L, K, M, \ldots = 1, \ldots, N$ denote the extended indices, while $l, m, \ldots = 1, \ldots n$ denote the internal ones)

Because we neglect some constraint, this class of superalgebra includes all the extended superalgebras considered up to now in the literature [1-5]. So we would have a larger choice for a true unified theory.

The Jacobi-identities involving the above extended superalgebra tell us that its structure constants must satisfy a system of self-consistent conditions (see Appendix). This schem is very elegant for classifying the models of extended superalgebras [1]

With certain values of the numeric matrices a_1, s_1, d, c satisfying the self-consistent conditions, we have the well-known models of extended superalgebras.

In this study we consider the problem of representing all these extended superalgebras on the space of superfields (i.e the functions defined on the extended superspace $(x_\mu, \theta^L, \bar{\theta}^L)$, $L = 1, \ldots, N$)

Up to now, the method of representing the extended superalgebras is not uniquely determined. In the case of the simple supersymmetry ($N = 1$), some authors considered the action of one-parameter group elements on the unitary element:

$$L(x_\mu, \theta, \bar{\theta}) = \exp i P^\mu x_\mu \exp i (\theta Q + \bar{Q}\bar{\theta}) \tag{1.2}$$

Applying the group elements on the left of this unitary element, the following relations have been suggested:

$$\exp i c_\mu P^\mu \, L(x, \theta, \bar{\theta}) = L(x + c, \theta \bar{\theta})$$

$$\exp i (\in Q + \bar{Q}\bar{\in}) \, L(x, \theta, \bar{\theta}) = L(x_\mu - \frac{i}{2} \bar{\in} \sigma_\mu \theta + \frac{i}{2} \bar{\theta} \sigma_\mu \in, \theta + \in, \bar{\theta} + \bar{\in}) \tag{1.3}$$

Consider this expression as the action of the group elements on superspace(!), the transformation law of superfields is presented in the following way:

$$\exp i c_\mu P^\mu \, \Phi(x, \bar{\theta}, \theta) \exp -i c_\mu P^\mu = \Phi(x + c, \theta, \bar{\theta})$$

$$\exp i (\in Q + \bar{Q}\bar{\in}) \Phi(x, \theta, \bar{\theta}) \exp - i(\in Q + \bar{Q}\bar{\in}) = \Phi(x_\mu - \frac{i}{2} \bar{\in} \sigma_\mu \theta +$$

$$+ \frac{i}{2} \bar{\theta} \sigma_\mu \in, \theta + \in, \bar{\theta} + \bar{\in}) \quad . \tag{1,4}$$

Expanding both sides in terms of infinitesimal parameters we have a representation of P_μ, Q^L_α in the form of differential operators.

If so, we can easily generalize this method to the case of extended superalgebras. However, because of the formulas $[\partial_\mu, x^\nu]=\delta^\nu_\mu$; $\{\partial_\alpha, \theta_\beta\}=\delta_{\alpha\beta}$ the relations (1.3) are mistaken. So many authors avoid (1.3) and try to come to (1.4) by some special assumptions or by the intuition. However, with such a general algebra as (1.1) to find the actions of supergroup on superspace is not an easy work. Some other authors want to overcome difficulties in the matter of representation by adding to extended superspace some special extra dimensions. But then the number of field components in a representation increases fiercely.

In this paper we give a complete representation of the superalgaera (1.1) without any extra-dimensions. Using this result we define superfields and consider the transformation law of its components.

2. REPRESENTATION OF EXTENDED SUPERALGEBRAS

In certain simple cases, we will show the method as formally as possible so that the method can be generalized easily to more general cases (e. g the semi-simple algebras, the conform space-time symmetry, the impulse space with more dimensions)

2.1 Tsp extended superalgebras:

Tsp extended superalgebras are those superalgebras (1.1) in which $c=0$. Geometrically speaking, those models have some advantages: superspace is linear, superfield is determined uniquely, ... [2]

First, we represent this kind of extended superalgebra in the simplest case when $a_1 = d = 0$.

Generally, an extended superalgebra **G** has its Levi-Malcev expansion $\mathbf{G} = \mathbf{H} \oplus \mathbf{M}$, where **H** is a semi-simple Lie-algebra and **M** is a solvable radical. **H** has its Cartan expansion $\mathbf{H} = \sum \mathbf{H}_i$, where $\mathbf{H}_i$ are simple Lie-algebras.

In our matter $\mathbf{H} = so\,(3,1) \oplus \mathbf{I}$ (**I** is a certain internal algebra), while $\mathbf{M} = \mathbf{P} \oplus \mathbf{R} \oplus \mathbf{R}$ is a graded commutative radical (**P** is the commutative ideal of Boson-type generators, **R** is the anticommutative ideal of Fermi-type generators, while **R** is its conjungate)

Let $\mathbf{M}^0$ denote the tangent space of **M**, then we represent **M** by the derivations on **M**

$$\mathbf{P} \xrightarrow{\text{Hom}} i\partial_{\mathbf{P}^0}$$

$$\mathbf{R} \xrightarrow{\text{Hom}} -i\partial_{\mathbf{R}^0}$$

$$\mathbf{R} \xrightarrow{\text{Hom}} i\partial_{\overline{\mathbf{R}}^0} \tag{2.1.1}$$

Using the Jacobi-identities we can easily verify that the element $g_i \in \mathbf{H}$ then can be represented by

$$g_i = -c^{\beta}_{i\alpha}\, r^{\alpha}\partial_{r_\beta} - c^{\dot\beta}_{i\dot\alpha}\, \bar{r}^{\dot\alpha}\partial_{\bar{r}_{\dot\beta}} - c^{\nu}_{i\mu}x^{\mu}\partial_{x_\nu} + \Lambda_i \tag{2.1.2}$$

where $r \in \mathbf{R}^0$, $\bar{r} \in \overline{\mathbf{R}}^0$, $x \in \mathbf{P}^0$; c^s_{pq} are structure constants of **G**, Λ_i are the hermitian complex matrices satisfying:

$$[\Lambda_i, \Lambda_j] = c^k_{ij}\Lambda_k \tag{2.1.3}$$

In the case of Tsp extended superalgebra (1.1) when $a_1 = d = 0$, we have the following representation:

$$P_\mu = i\,\partial_\mu$$

$$Q^L_\alpha = -i\,\frac{\partial}{\partial\theta^L};\quad \bar{Q}^L_{\dot\alpha} = i\frac{\partial}{\partial\bar\theta^L}$$

$$M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \frac{1}{2}\theta^L\sigma_{\mu\nu}\frac{\partial}{\partial\theta^L} - \frac{1}{2}\bar\theta^L\tilde\sigma_{\mu\nu}\frac{\partial}{\partial\bar\theta^L} + \Lambda_{\mu\nu}$$

$$B_l = -s^{LM}_l\,\theta^L\frac{\partial}{\partial\theta^M} - \bar{s}^{LM}_l\bar\theta^L\frac{\partial}{\partial\bar\theta^M} + \Lambda_l \tag{2.1.4}$$

$\mathbf{M}^0$ then is the extended superspace $(x_\mu, \theta^L, \bar\theta^L)$ $L = 1,\ldots, N$

Due to the presence of Λ_i in (2.1.4) supertransformations can act on the indices of superfields. If superfield has no index (the case of scalar superfield, see Section 3.) $\Lambda_i = 0$ and supertransformations act only on the extended superspace. If $s_l = 0$, internal transformations act only on the indices of superfields as in the usual non-unified theories.

2.2 Extended superalgebras of Wess — Zumino:

This kind of superalgebra was suggested firstly in an original paper of J. Wess and B. Zumino [3] (in the case of simple superalgebra $N = 1$). Extended superalgebras of Wess-Zumino are those special models of (1.1) in which $a_1 = d = 0$ and $c \neq 0$. They have been used widely in the most models of superunified theory because of their simplicity.

In this case **M** in the above Levi-Malcev expansion is a solvable radical and has a commutative ideal **P**.

$$\mathbf{P} = \{\mathbf{R}, \overline{\mathbf{R}}\};\ [\mathbf{P}, \mathbf{P}] = \{\overline{\mathbf{R}}, \overline{\mathbf{R}}\} = [\mathbf{P}, \mathbf{R}] = \{\mathbf{R}, \mathbf{R}\} = [\mathbf{P}, \overline{\mathbf{R}}] = 0 \qquad (2.2.1)$$

$\mathbf{M}^o = \mathbf{P}^o \oplus \mathbf{R}^o \oplus \overline{\mathbf{R}}^o$ is the tangent space of $\mathbf{M} = \mathbf{P} \oplus \mathbf{R} \oplus \overline{\mathbf{R}}$. We have the following representation:

$$\mathbf{P}_\mu \xrightarrow{\text{Hom}} i\,\partial_\mu$$

$$\mathbf{R}_\alpha \xrightarrow{\text{Hom}} -i \frac{\partial}{\partial r_\alpha} - \frac{c^{\mu}_{\alpha\dot\beta}}{2} \overline{r}^{\dot\beta}\,\partial_\mu$$

$$\overline{\mathbf{R}}_{\dot\alpha} \xrightarrow{\text{Hom}} i \frac{\partial}{\partial \overline{r}_{\dot\alpha}} \frac{c^{\mu}_{\dot\alpha\beta}}{2} r^{\beta}\,\partial_\mu \qquad (2.2.2)$$

It is easy to verify that the representation (2.2.2) satisfies the relations (2.2.1) by using:

$$c^{\mu}_{\alpha\dot\beta} = c^{\mu}_{\dot\beta\alpha} \qquad (2.2.3)$$

Then we can represent the generators $g_i \in H$ by

$$g_i = -c^{\alpha}_{i\beta}\, r^{\beta}\partial_{r_\alpha} - c^{\dot\alpha}_{i\dot\beta}\, \overline{r}^{\dot\beta}\, \partial_{\overline{r}_{\dot\alpha}} - c^{\nu}_{i\mu}\, x^{\mu}\, \partial_\nu + \Lambda_i \qquad (2.2.4)$$

Λ_i are the c-number matrices satisfying:

$$[\Lambda_i, \Lambda_j] = c^{k}_{ij}\Lambda_k \qquad (2.2.5)$$

The commutator

$$[g_i, g_j] = c^{k}_{ij}\, g_k \qquad (2.2.6)$$

has been proved in 2.1 because there is no change in the expression of Λ_i.

Using the Jacobi-identities and the conditions (2.2.1) on the structure constants

$$c^{\mu}_{\alpha j} = c^{j}_{i\beta} = c^{\mu}_{ij} = c^{\gamma}_{\alpha\beta} = 0 \qquad (2.2.7)$$

we get: $\quad [g_i, R_\alpha] = c^{\gamma}_{i\alpha} R_\gamma \qquad (2.2.8)$

So the relations (2.2.2) and (2.2.4) are, indeed, the representation of the algebra $\mathbf{G} = \mathbf{H} \oplus \mathbf{M}$

In our matter, when $\mathbf{H} = so(3,1) \oplus \mathbf{I}$ and $\mathbf{M} = \mathbf{P} \oplus \mathbf{R} \oplus \overline{\mathbf{R}}$, we have the following representation:

$$P_\mu = i\partial_\mu$$

$$Q^{L}_{\alpha} = -i \frac{\partial}{\partial\theta^{L}} - \frac{c^{LM}}{2} \left(\widehat{\partial}\,\theta^{M}\right)_{\alpha}$$

$$\bar{Q}^L_{\dot\alpha} = i\frac{\partial}{\partial\bar\theta^L} + \frac{c^{LM}}{2}\left(\theta^M\widehat{\partial}\right)_{\dot\alpha}$$

$$M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \frac{1}{2}\theta^L\sigma_{\mu\nu}\frac{\partial}{\partial\theta^L} - \frac{1}{2}\bar\theta^L\tilde\sigma_{\mu\nu}\frac{\partial}{\partial\bar\theta^L} + \Lambda_{\mu\nu}$$

$$B_l = -s_l^{LM}\theta^L\frac{\partial}{\partial\theta^M} - \bar{s}_l^{LM}\bar\theta^L\frac{\partial}{\partial\bar\theta^M} + \Lambda_l \qquad (2.2.9)$$

M^0 then is the extended superspace $(x_\mu, \theta^L, \bar\theta^L)$ $L=1,..., N$

2. 3. Extended superalgebras with central charges:

Extended superalgebras with central charges are those superalgebras (1,1) in which $a_l = 0$. This kind of superalgebras have been suggested firstly in the paper of R. Haag, J. Lopuszánsky and M, Sohnius [4]. Some authors had presented this algebra by extending the extended superspace with some extra dimensions.

In the superalgebras (1.1) if we introduce $Z^{LM} = a_l^{LM} B_l$ then it is easy to verify that Z^{LM} commute with any generator of the superalgebra (1.1) and form the center of these algebras (Z^{LM} are called the central charges)

We present a representation of this kind of extended superalgebras on the space of superfields defined on the extended superspace $(x_\mu, \theta^L, \bar\theta^L)$, $L=1,...,N$ in the following form

$$P_\mu = i\partial_\mu$$

$$Q^L_\alpha = -i\frac{\partial}{\partial\theta^L} - \frac{c^{LM}}{2}(\widehat{\partial}\bar\theta^M)_\alpha + \frac{i}{2}\epsilon_{\alpha\beta} Z^{LM}\theta^M =$$

$$= -i\frac{\partial}{\partial\theta^L} - \frac{c^{LM}}{2}(\widehat{\partial}\bar\theta^M)_\alpha + \frac{i}{2}\epsilon_{\alpha\beta}\sum_l a_l^{LM}\Lambda_l\theta^M_\beta$$

$$\bar{Q}^L_{\dot\alpha} = i\frac{\partial}{\partial\bar\theta^L} + \frac{c^{LM}}{2}(\theta^M\widehat{\partial})_{\dot\alpha} - \frac{i}{2}\bar\epsilon_{\dot\alpha\dot\beta}\sum_l \overset{*}{a}{}_l^{LM}\Lambda_l\bar\theta^M_{\dot\beta}$$

$$M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \frac{1}{2}\theta^L\sigma_{\mu\nu}\frac{\partial}{\partial\theta^L} - \frac{1}{2}\bar\theta^L\tilde\sigma_{\mu\nu}\frac{\partial}{\partial\bar\theta^L} + \Lambda_{\mu\nu}$$

$$B_l = -s_l^{LM}\theta^L\frac{\partial}{\partial\theta^M} - \bar{s}_l^{LM}\bar\theta^L\frac{\partial}{\partial\bar\theta^M} + \Lambda_l \qquad (2.3.1)$$

Because of the self-consistent condition (see Appendix)

$$\sum_l a_l^{LM} s_l^{LM} = 0 \qquad (2.3.2)$$

we have:

$$Z^{LM} = a_l^{LM}\left(-s_l^{KH}\ \theta^K \frac{\partial}{\partial\theta^H} - \bar{s}_l^{KH}\ \bar{\theta}^K \frac{\partial}{\partial\bar{\theta}^H} + \Lambda_l\right) = \Sigma a_l^{LM}\ \Lambda_l$$

So Z^{LM} can be represented by complex antisymmetric matrices

$$Z^{LM} \xrightarrow{\text{Hom}} \Sigma\ a_l^{LM}\ \Lambda_l = b^{LM} = -\,b^{ML} \tag{2.3.3}$$

Using the following self-consistent conditions (see Appendix)

$$a_l^{LM} = -a_l^{ML} \tag{2.3.4}$$

$$\bar{s}_l\, c = -c\ s_l \tag{2.3.5}$$

we can verify the commutators (1.1.g) and (1.1.i) while the rests have been proved in the previous paragraph.

2.4. Extended semi-simple superalgebras:

We use the word semi-simple to mean those extended superalgebras (1.1) in which $d \neq 0$. Indeed, in this class of superalgebras only the models without central charges are semi-simple in our meaning (i. e they have no commutative ideal). The models with central charges are reductive (they have the form of $G = G_1 + G_2$, where G_1 is semi-simple and G_2 is abelian). A version of this kind of extended superalgebras was suggested by Dao Vong Duc to unify sources and fields into an irreducible representation [5].

We represent the general extended superalgebras (1.1) on the space of superfields defined on the superspace $(x_\mu, \theta^L, \bar{\theta}^L)$ in the following form:

$$P_\mu = i\partial_\mu + i d^{LM}\ \theta^L \sigma_\mu \bar{Q}^M - i\bar{d}^{LM} Q^L \sigma_\mu \bar{\theta}^M = i\partial_\mu - d^{LM}\theta^L \sigma_\mu \frac{\partial}{\partial\bar{\theta}^M}$$

$$+ \frac{i}{2}(dc)^{LM}\theta^L \sigma_\mu (\theta^M \widehat{\partial}) + \frac{1}{2}(da_1)^{LM}\ \Lambda_1 \theta^L\ \sigma_\mu(\in \bar{\theta}^M) -$$

$$- \bar{d}^{LM}\frac{\partial}{\partial\theta^L}\sigma_\mu \bar{\theta}^M + \frac{i}{2}(\overset{*}{d}c)^{ML}(\widehat{\partial}\,\theta^L)\ \sigma_\mu \bar{\theta}^M + \frac{1}{2}\Sigma(\overset{*}{d}a_1)^{LM}\ \Lambda_1 \in \theta^M \sigma_\mu \bar{\theta}^L$$

$$Q^L_\alpha = -i\frac{\partial}{\partial\theta^L_\alpha} - \frac{c^{LM}}{2}(\widehat{\partial\,\bar{\theta}^M})_\alpha + \frac{i}{2}(\in\theta^M)_\alpha\ \Sigma a_l^{LM}\ \Lambda_l$$

$$\bar{Q}^L_{\dot{\alpha}} = i\frac{\partial}{\partial\bar{\theta}^L_{\dot{\alpha}}} + \frac{c^{LM}}{2}(\theta^M\widehat{\partial})_{\dot{\alpha}} - \frac{i}{2}(\bar{\in}\bar{\theta}^M)_{\dot{\alpha}}\ \Sigma a_l^{LM}\Lambda_l$$

$$M_{\mu\nu} = i(x_\mu\,\partial_\nu - x_\nu\partial_\mu) + \frac{1}{2}\theta^L\sigma_{\mu\nu}\frac{\partial}{\partial\theta^L} - \frac{1}{2}\bar{\theta}^L\bar{\sigma}_{\mu\nu}\frac{\partial}{\partial\bar{\theta}^L} + \overset{*}{\Lambda}_{\mu\nu}$$

$$B_l = - s_l^{LM}\, \theta^L \frac{\partial}{\partial\theta^M} - \bar{s}_l^{LM} \bar{\theta}^L \frac{\partial}{\partial\bar{\theta}^M} + \Lambda_l \qquad (2.4.1)$$

where
$$\epsilon_{11} = \bar{\epsilon}_{\dot{1}\dot{1}} = \epsilon_{22} = \bar{\epsilon}_{\dot{2}\dot{2}} = 0$$

$$\epsilon_{12} = \epsilon_{\dot{1}\dot{2}} = -\epsilon_{21} = -\epsilon_{\dot{2}\dot{1}} = 1$$

and
$$\hat{\partial} = \sigma_\mu\, \partial^\mu$$

Because the expression of B_l, $M\mu\nu$, Q^L_α and $\bar{Q}^L_{\dot{\alpha}}$ remains the same as it was in the previous paragraphs, we must verify only those commutators, in which there is the presence of P_μ.

It is easy to verify them, if we use the following self-consistent conditions (see Appendix):

$$\bar{d}.d = 0 \qquad (2.4.2)$$

$$d.\tilde{c} = 0 \qquad (2.4.3)$$

$$d.\tilde{a}_1 = 0 \qquad (2.4.4)$$

$$d.s_1 = -\overset{*}{s}_1.d \qquad (2.4.5)$$

Note that the representation (2.4.1) of the general extended superalgebras (1.1) with certain vanishing values of each numeric matrices will turn into the form (2.1.4), (2.2.9) and (2.3.1) of its special cases. We can see that the extended superspace $(x_\mu, \theta^L \bar{\theta}^L)$ $L = 1, ..., N$ is large enough to represent the extended superalgebras (1.1).

3. THE ACTIONS OF SUPERGROUPS :

Basing on the results of the Section 2., we can derive the actions of supergroups on the extended superspace, then write the transformation laws of superfields components under the actions of supergroups.

3.1 THE ACTIONS OF SUPERGROUPS ON THE EXTENDED SUPERSPACE

Having the expression of the above superalgebras generators in hand we can set up all the possible symmetry transformations corresponding to the following one parameter groups of:

Lorentz – transformations	$\exp i\, \omega^{\mu\nu} M_{\mu\nu}$	(3.1.1)
Internal symmetry transformations	$\exp\, i g_1 B_1$	(3.1.2)
Generalized space-time translation	$\exp i t_\mu P^\mu$	(3.1.3)

Extended spinor coordinate translations

$$\exp i\,(\mathcal{S}\,Q+\overline{Q}\,\overline{\mathcal{S}}) \tag{3.1.4}$$

Now we proceed to consider their actions on the extended superspace.

3.1.1 The actions of Lorentz-transformations

$$x_\mu \longrightarrow x'_\mu = \Lambda_\mu^\nu(\omega)\,x$$

$$\theta^L \longrightarrow \theta'^{\,L} = A_\alpha^\beta(\Lambda)\,\theta_\beta^L$$

$$\overline{\theta}^L \longrightarrow \overline{\theta}'^{\,L} = \widetilde{A}_{\dot\alpha}^{\dot\beta}(\Lambda)\,\overline{\theta}_{\dot\beta}^L \tag{3.1.5}$$

where Λ is space-time rotation matrix in Minkowski-space, $A_\alpha^\beta(\Lambda)$ is its spinor representation, while $\widetilde{A}_{\dot\alpha}^{\dot\beta}(\Lambda)$ is its conjungate.

3.1.2 The actions of extended spinor translations

$$x_\mu \longrightarrow x_\mu + \frac{i}{2}\,c^{LM}\,(\mathcal{S}^L\,\sigma_\mu\,\overline{\theta}^M - \theta^M\,\sigma_\mu\,\overline{\mathcal{S}}^L)$$

$$\theta_\alpha^L \longrightarrow \theta_\alpha^L + \mathcal{S}_\alpha^L$$

$$\overline{\theta}_{\dot\alpha}^L \longrightarrow \overline{\theta}_{\dot\alpha}^L + \overline{\mathcal{S}}_{\dot\alpha}^L \tag{3.1.6}$$

3.1.3. The actions of the internal transformations

$$x_\mu \longrightarrow x_\mu$$

$$\theta^L \longrightarrow \theta^L - i\,s_l^{ML}\,g_l\,\theta_\alpha^M$$

$$\overline{\theta}_{\dot\alpha}^L \longrightarrow \overline{\theta}_{\dot\alpha}^L + i\,s_l^{ML}\,g_l\,\overline{\theta}_{\dot\alpha}^M \tag{3.1.7}$$

when $s_l = 0$, as in the usual non-unified theories these transformations do not action the superspace, but only on the field functions.

3.1.4. The actions of the generalized space-time translations

$$x_\mu \longrightarrow x_\mu + t_\mu - \frac{i}{2}\,(dc)^{LM}\,\theta^L\sigma_\mu\,(\theta^M\,\widehat{t}\,) + \frac{i}{2}\,(cd)^{LM}\,(\widehat{t}\;\overline{\theta}^M)\,\sigma_\mu\,\overline{\theta}^L$$

$$\theta_\alpha^L \longrightarrow \theta_\alpha^L + \overline{d}^{LM}\,(\widehat{t\,\overline{\theta}^M})_\alpha$$

$$\bar{\theta}^{L}_{\dot{\alpha}} \longrightarrow \bar{\theta}^{L}_{\dot{\alpha}} + d^{LM} (\theta^{M} \hat{\partial})_{\dot{\alpha}} \qquad (3.1.8)$$

where $\hat{t} \equiv t_{\mu} \sigma^{\mu}$, If $d = 0$ we get the usual space-time translations.

3.2. SUPERFIELDS AND THE IRREDUCIBILITY CONDITIONS:

As we mentioned in the Section 2., the generators $M_{\mu\nu}$ and B_l are determined exactly to a complex matrix Λ. Λ matrices are finite dimensional linear representations of $M_{\mu\nu}$ and B_l. This complex part of $M_{\mu\nu}$ and B_l does not act on the superspace, but only on the form of field functions. As $[M_{\mu\nu}, B_l] = 0$, we can see that $\Lambda_{\mu\nu}$ and Λ_l matrices act independently. So we can represent

$$\Lambda_{\mu\nu} = \left(\begin{array}{c|c} \Lambda_{\mu\nu} & 0 \\ \hline 0 & 0 \end{array}\right)\begin{array}{l} m \\ n \end{array} \quad \begin{array}{cc} m & n \end{array}$$

$$\text{and } \Lambda_l = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \Lambda_l \end{array}\right)\begin{array}{l} m \\ n \end{array} \quad \begin{array}{cc} m & n \end{array} \qquad (3.2.1)$$

So the field functions can have two kind of indices corresponding to the Lorentz-transformations and the internal transformations. We can define superfields as functions on the extended superspace with two kind of indices. So we have:

Tensor superfields $\Phi^{ij\cdots}_{\alpha\beta\ldots}(x_{\mu}, \theta^{L}, \bar{\theta}^{L})$

Spinor superfields $\Phi^{\gamma\delta\ldots}_{\alpha\beta}(x_{\mu}, \theta^{L}, \bar{\theta}^{L})$

In general, tensor superfields and spinor superfields are reducible so some irreducibility conditions must be required.

Here we consider only the simplest and most interesting superfield: the scalar superfield. This kind of superfield has no index. It is to say, all the matrices are chosen to be 0. We come to the conclusion that the scalar superfield has no central charge.

Expanding $\Phi(x_{\mu}, \theta^{L}, \bar{\theta}^{L})$ into the polynomial series of θ^{L} and $\bar{\theta}^{L}$, we get:

$$\Phi(x_{\mu}, \theta^{L}, \bar{\theta}^{L}) = \sum_{\substack{m_H = 0,1,2 \\ m'_H = 0,1,2}} \prod_{H=1}^{N} \frac{(\theta^{H})^{m_H}}{(m_H!)} \prod_{H'=1}^{N} \frac{(\bar{\theta}^{H'})^{m'_{H'}}}{(m'_{H'}!)}$$

$$(x)$$

$$\Phi\,(m_1, \ldots, m_N;\ m'_1, \ldots, m'_N) \qquad (3.2.2)$$

where the field functions $\Phi\,(m_1 ,..., m_N ; m'_1 ,..., m'_N)$ (x) have their components

$$\Phi_{(m_1 ,..., m_N ; m'_1 ,..., m'_N)}^{\ldots \mathrm{Asym} \prod_{i=1}^{m_H} \alpha_i^H \ldots \mathrm{Asym} \prod_{i=1}^{m'_{H'}} \alpha_i^{H'} \ldots}(x); \quad \alpha_i^H = 1,2$$

The symbol of antisymmetrization is there because of the anticommutation of Grassman variables.

The scalar superfield transforms as follows under the supersymmetry transformations (3.1):

$$\Phi(x'_\mu, \theta'^L, \bar\theta'^L) - \Phi(x_\mu, \theta^L, \bar\theta^L) =$$

$$= \sum_{\substack{m_H = 1, 2, 0 \\ m'_{H'} = 1, 2, 0}} \prod_{H=1}^{N} \frac{(\theta^H)^{m_H}}{(m_H!)} \prod_{H'=1}^{N} \frac{(\bar\theta^{H'})^{m'_{H'}}}{(m'_{H'}!)} \Phi_{(m_1, ..., m_N ; m'_1, ..., m'_N)} \tag{3.2.3}$$

In the next paragraphs we derive the transformation law of the superfields, components $\Phi_{(m_1,..., m_N ; m'_1, ..., m'_N)}$ (x) under the action of (3.1.1) — (3.1.4).

Note that $\Phi\,(x_\mu, \theta^L, \bar\theta^L)$ in general is reducible. To get the irreducible parts we must introduce the covariant derivatives:

$$\mathcal{D}_\alpha^L = \frac{\partial}{\partial\theta_\alpha^L} + \frac{i}{2}\, c^{LM} (\hat\partial\, \bar\theta^M)_\alpha$$

$$\bar{\mathcal{D}}_{\dot\alpha}^L = \frac{\partial}{\partial\bar\theta_{\dot\alpha}^L} - \frac{i}{2}\, c^{LM} (\theta^M \hat\partial)_{\dot\alpha} \tag{3.2.4}$$

The covariant derivatives commute with all the variations of superfields under the actions of (3.1.1) — (3.1.4), so we have the irreducibility for superfields;

$$\mathcal{D}_\alpha^L \Phi_l(x_\mu, \theta^L \bar\theta^L) = 0 \qquad \forall \alpha,\ L \tag{3.2.5}$$

$$\bar{\mathcal{D}}_{\dot\alpha}^L\, \Phi_r(x_\mu, \theta^L \bar\theta^L) = 0 \qquad \forall \alpha,\ L \tag{3.2.6}$$

In the case of simple supersymmetry ($N = 1$) the equations (3.2.5), (3.2.6) have the solutions:

$$\Phi_l(x, \theta, \bar\theta) = \Phi_l(x, \theta) \tag{3.2.7}$$

$$\Phi_r(x, \theta, \bar\theta) = \Phi_r(x, \bar\theta) \tag{3.2.8}$$

3.3 THE ACTIONS OF SUPERGROUPS ON THE COMPONENT FIELDS;

The action of Lorentz-transformations on the fields has been known so well in the literature, we do not discuss it here.

3.3.1. The internal symmetry transformations:

We have the transformation laws as follows:

$$\Phi' = \exp ig_1 B_1 \, \Phi \exp - ig_1 B_1 \tag{3.3.1}$$

Ignoring the high-order terms

$$\Phi' = \Phi + ig_1 [B_l, \Phi] \tag{3.3.2}$$

Hence $\delta\Phi = ig_1 [B_1, \Phi] \equiv -ig_1 \left(s_1^{LM} \theta^L \frac{\partial}{\partial \theta^M} + \bar{s}_1^{LM} \bar{\theta}^L \frac{\partial}{\partial \bar{\theta}^M}\right) \Phi(x_\mu, \theta^L, \bar{\theta}^L)$

$$\tag{3.3.3}$$

Puting the expansion (3.2.2) into (3.3.3) then comparing with (3.2.3) we get

$$\delta \Phi^{\ldots Asym \prod_{i=1}^{m_H} \alpha_i^H \ldots Asym \prod_{i=1}^{m'_H} \dot{\alpha}_i^H \ldots}_{(m_1, \ldots, m_N; m'_1, \ldots, m'_N)}(x) =$$

$$= - g_1 \sum_{M=1}^{N} \Big(S_1^{MM} m^M \, \Phi^{\ldots Asym \Pi \alpha_i^H \ldots Asym \Pi \dot{\alpha}_i^H}_{(m_1, \ldots, m_N;\, m'_1, \ldots, m'_N)}(x) +$$

$$+ \bar{S}_1^{MM} m'^M \, \Phi^{\ldots Asym \Pi \alpha_i^H \ldots Asym \Pi \dot{\alpha}_i^{H'} \ldots}_{(m_1, \ldots, m_N;\, m'_1, \ldots, m'_N)}(x) +$$

$$+ \sum_{L \neq M} (m'_L + 1) \bar{S}_1^{LM} \, \Phi^{\ldots Asym \Pi \alpha_i^H \ldots Asym \Pi \dot{\alpha}_i^H}_{(m_1, \ldots, m_N;\, m'_1, \ldots, m'_M + 1, \ldots, m'_L - 1, \ldots, m'_N)}(x) +$$

$$+ \sum_{L \neq M} (m_L + 1) S_1^{LM} \, \Phi^{\ldots Asym \Pi \alpha_i^H \ldots Asym \Pi \dot{\alpha}_i^{H'} \ldots}_{(m_1, \ldots, m_M + 1, \ldots, m_L - 1, \ldots, m_N;\, m'_1, \ldots, m'_N)}(x) \Big) \tag{3.3.4}$$

(note that $\Phi_{(\ldots m_j \ldots)} = 0$ if $m_j = 0, 1, 2$)

3.3.2. Extended spinor translations

Starting from the following equation

$$\delta\Phi = i\mathcal{S}^L [Q^L, \Phi] + i[\overline{Q}^L, \Phi]\overline{\mathcal{S}}^L = i\mathcal{S}^L\left(-i\frac{\partial}{\partial\theta^L} - \frac{c}{2}^{LM}\widehat{\partial\,\overline{\theta}^M}\right)\Phi +$$

$$+ i\left(\frac{\partial}{\partial\overline{\theta}^L} + \frac{c}{2}^{LM}\theta^M \frac{1}{\partial}\right)\Phi\mathcal{S}^L \tag{3.3.5}$$

then puting the expansion (3.2.2) into (3.3.5) then comparing with (3.2.3) we get:

$$\delta\Phi^{\ldots \text{Asym} \prod_{i=1}^{m_H} \alpha_i^H \ldots \text{Asym} \prod_{i=1}^{m'_{H'}} \dot\alpha_i^{H'} \ldots}_{(m_1,\ldots,\, m_N;\; m'_1,\ldots,\, m'_N)}(x) =$$

$$+ \Sigma\mathcal{S}^L\Phi^{\ldots \text{Asym}\,\Pi\,\alpha_i^H \ldots \text{Asym}\,\Pi\,\dot\alpha_i^{H'} \ldots}_{(m_1,\ldots,\, m_L+1,\ldots,\, m_N;\; m'_1,\ldots,\, m'_N)}(x) -$$

$$-\frac{i}{2}\Sigma c^{LM}\mathcal{S}^L(m'_M+1)\widehat{\partial}\,\Phi^{\ldots \text{Asym}\,\Pi\,\alpha_i^H \ldots \text{Asym}\,\Pi\,\dot\alpha_i^{H'} \ldots}_{(m_1,\ldots,\, m_N;\; m'_1,\ldots,\, m'_M-1,\ldots m'_N)}(x)$$

$$-\Sigma\overline{S}^L\Phi^{\ldots \text{Asym}\,\Pi\,\alpha_i^H \ldots \text{Asym}\,\Pi\,\dot\alpha_i^{H'} \ldots}_{(m_1,\ldots,\, m_N;\; m'_1,\ldots,\, m'_L+1,\ldots\, m'_N}(x) +$$

$$+\frac{i}{2}\sum_{L,M} c^{LM}\mathcal{S}^L(m_M+1)\widehat{\partial}\,\Phi^{\ldots \text{Asym}\,\Pi\,\alpha_i^H \ldots \text{Asym}\,\Pi\,\alpha_i^{H'} \ldots}_{(m_1,\ldots,\, m'_N;\; m'_1,\ldots,\, m'_M-1,\ldots\, n'_N)}(x) \tag{3.3.6}$$

3.3.3 Generalized space-time translations

Starting from the following equation

$$\delta\Phi = i\,t^\mu[P_\mu, \Phi] = i\,t^\mu\,[i\,\partial_\mu + i d^{LM}\theta^L\sigma_\mu Q^M - i\overline{d}^{LM}Q^L\sigma_\mu\overline{\theta}^M) \tag{3.3.7}$$

with the same procedure as above we get

$$\delta\,\Phi^{\ldots \text{Asym}\,\Pi\alpha_i^H \ldots \text{Asym}\,\Pi\dot\alpha_i^H \ldots}_{(m_1,\ldots,\, m_N;\; m'_1,\ldots,\, m'_N}(x) = -\,t_\mu\partial^\mu\,\Phi^{\ldots \text{Asym}\,\Pi\alpha_i^H \ldots \text{Asym}\,\Pi\dot\alpha_i^H \ldots}_{(m_1,\ldots,\, m_N;\; m'_1,\ldots,\, m'_N}(x) +$$

$$+i\hat{t}\,[\Sigma d^{MM} m_M\, \Phi^{\dots\,\mathrm{Asym}\,\Pi\alpha_i^H \dots\,\mathrm{Asym}\,\Pi\dot{\alpha}_i^{H'}\dots}_{(m_1,\dots,m_N;\; m'_1,\dots,m'_N)}(x)+$$

$$+\sum_{L\neq M}\Phi^{\dots\,\mathrm{Asym}\,\Pi\alpha_i^H \dots\,\mathrm{Asym}\,\Pi\dot{\alpha}_i^{H}\dots}_{(m_1,\dots,m_M+1,\dots,m_L-1,\dots,m_N;\; m'_1,\dots,m'_N)}(x)+$$

$$+\sum\frac{(dc)}{2}^{LK}\hat{\partial}(m_L+1)\,(m'_K+1)\;\Phi^{\dots\,\mathrm{Asym}\,\Pi\alpha_i^H \dots\,\mathrm{Asym}\,\Pi\dot{\alpha}_i^{H'}\dots}_{(m_1,\dots,m_L-1,\dots,m_N;\; m'_1,\dots,m'_K-1,\dots,m'_N)}(x)]$$

$$-i\hat{t}\,[\sum_L \bar{d}^{LL} m'_L\, \Phi^{\dots\,\mathrm{Asym}\,\Pi\alpha_i^H \dots\,\mathrm{Asym}\,\Pi\dot{\alpha}_i^{H'}\dots}_{(m_1,\dots,m_N;\; m'_1,\dots,m'_N)}(x)+$$

$$+\sum_{L,M\neq L}(m'_M+1)\;\Phi^{\dots\,\mathrm{Asym}\,\Pi\alpha_i^H \dots\,\mathrm{Asym}\,\Pi\dot{\alpha}_i^{H'}\dots}_{(m_1,\dots,m_N;\; m'_1,\dots,m'_M-1,\dots,m'_L+1,\dots,m'_N)}(x)-$$

$$-\sum\frac{(\bar{dc})}{2}^{LK}\hat{\partial}(m'_L+1)(m_K+1)\;\Phi^{\dots\,\mathrm{Asym}\,\Pi\alpha_i^H \dots\,\mathrm{Asym}\,\Pi\dot{\alpha}_i^{H'}\dots}_{m_1,\dots,m_K-1,\dots,m_N;\; m'_1,\dots,m'_L-1,\dots,m'_N)}(x)$$

Acknowledgement:

Thanks are due to Prof. Dao Vong Duc for numerous helpful discussions. I am also grateful to Prof. Nguyen van Hieu and Dr. Nguyen thi Hong, who helped me so much and provided me a lot of new informations connected to this matter.

I never forget the kindness of my master Prof. Lovas I., who introduced me into elementary physics in first steps.

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Received March 6, 1985

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Appendix (from [1])

The system of self-consistent conditions of the general extended superalgebra (1.1)

$$d.\bar{d} = 0$$

$$d.\tilde{c} = 0$$

$$a_1.\tilde{d} = 0$$

$$d.\overset{*}{s}_1 + s_1.d = 0$$

$$[s_1, s_m] = i\, c^k_{lm}\, s_k$$

$$s_1.\tilde{a}_k - a_k\, .\tilde{s}_1 = i\, c^k_{lm}\, a_m$$

$$s_1.c = c.s^+_1$$

$$\Sigma a^{KL}_1\, s^{NM}_1 = \Sigma a^{LN}_1\, s^{KM}_1$$

$$\Sigma a^{LM}_1\, s^{KH}_1 = 0$$

$$c^{MK}.d^{LH} = c^{LK}.d^{MH}$$

$$a^{LM}_1 = -a^{ML}_1$$

$$c^{LM} = \bar{c}^{LM}$$