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ON CONTROLLABILITIES OF LINEAR DISCRETE-TIME SYSTEMS WITH RESTRAINED CONTROLS AND THE PURSUIT PROCESS IN LINEAR DISCRETE GAMES

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INTRODUCTION

In recent years the study of controllabilities of linear discrete-time systems (DTS) with restrained controls has attracted the attention of many authors. It has been motivated, on one hand, by the fact that this study gives rise to many new problems and, on the other hand, by the great possibility of applications this study offers for studying continuous-time systems described by classical differential equations and some problems of linear games. Controllabilities with geometrical constraints on controls has been studied by Sontag [1], N. K. Son [4]... for finitedimensional DTS, and by N. K. Son, N. V. Chau, N. V. Su [6]. . for infinite-dimensional DTS. Null-controllabilities with energy constraints on controls has been considered to study linear pursuit process of games with many players by N. Yu. Satimov, ... [1]

In this paper we shall consider controllabilities of DTS under constraints of different types imposed on the controls. The criterions for the locally null-controlable and the globally null-controllable are presented. Then we apply the obtained results to solve problems of pursuit in the discrete games with many players.

§ 1. NULL-CONTROLLABILITIES OF LINEAR DISCRETE-TIME SYSTEMS WITH RESTRAINED CONTROLS

We denote the phase vector by z and assume that the motion is described by the difference equation

$$z(k+1) = A\,z(k) + Bu(k);\; z(o) = z_0, \tag{1.1}$$

where $z(k) \in R^n$; $u(k) \in R^m$ is the control, $k = 0, 1, \ldots$; A, B are $n \times n$ — matrix respectively. The controls $u(k)$ must satisfy constraints of the form

$$u(k) \in \Omega, \; k = 0, 1, \ldots, \tag{1.2}$$

or

$$u(k) \in \Omega ; \; \sum_{k=0}^{\infty} \| u(k) \|^p \leqslant \varepsilon^p , \tag{1.3}$$

where Ω is a convex bounded subset in R^m and $0 \in \Omega$, $\varepsilon > 0$ and $p > 1$.

We denote by $K^N(\Omega, \varepsilon)$ (respectively, $K^N(\Omega)$) the set of all initial states $z_0 \neq 0$ such that there exists a sequence of controls $u(o)$, $u(1)$, ..., $u(N-1)$ satisfies

$$u(k) \in \Omega, \; k = 0, 1, \ldots, N-1 ; \; \sum_{k=0}^{N-1} \| u(k) \|^p \leqslant \varepsilon^p ;$$

$$z(N) = A^N z_0 + \sum_{k=0}^{N-1} A^k B u(N-1-k) = 0.$$

(respectively, $u(k) \in \Omega$, $k = 0, 1, \ldots; N-1$; $z(N) = 0$).

Let

$$K(\Omega, \varepsilon) = \bigcup_{N=0}^{\infty} K^N(\Omega, \varepsilon),$$

$$K(\Omega) = \bigcup_{N=0}^{\infty} K^N(\Omega).$$

DEFINITION 1. — *The system (1.1), (1.3) (respectively, (1.1) — (1.2) is said to be locally null-controlable (LC) if $K(\Omega, \varepsilon)$ (respectively, $K(\Omega)$) is a neighbourhood of origin in R^n.*

— *The system (1.1), (1.3) (respectively, (1.1) — (1.2)) is said to be globally null-controllable (GC) if $K(\Omega, \varepsilon) = R^n$ (respectively, $K(\Omega) = R^n$).*

The criterion for (LC) of the system (1.1) — (1.2) had been given in [4]. It is shown that (LC) only depends on geometrical relation between eigenvector of A and $B\Omega$.

It is obvious that if the system (1.1), (1.3) is (LC) then the system (1.1) — (1.2) is (LC) too.

PROPOSITION 1. *If the system* (1.1) — (1.2) *is* (LC), *then the system* (1.1), (1.3) *is (LC) too.*

Proof. Assume that the system (1.1) — (1.2) to be (LC). It is known in [4] that there exists a natural number L such that

$$o \in \text{int } K^L(\Omega)$$

Putting $M = \sup \{ \| u \| : u \in \Omega\}$, we have $M < + \infty$. Denote

$\alpha = \min \left\{ 1, \varepsilon . M^{-1} . L^{-\frac{1}{p}} \right\}$. It is easy to verify that

$$\alpha \, \Omega \subset \Omega \text{ and } K^L (\alpha . \Omega) \subset K^L (\alpha \, \Omega, \varepsilon).$$

On the other hand, by $\alpha \leqslant 1$ we have

$$K^L (\alpha \, \Omega) = \alpha \, K^L (\Omega).$$

Hence, $0 \in \text{int } K^L (\Omega)$ implicate $0 \in \text{int } K^L(\Omega, \varepsilon)$. The proof is complete.

From Proposition 1 and N. K. Son's results in [4], we have

COROLLARY 1. (*The criterion for (L.C) of the system (1.1), (1.3)). The system (1.1), (1.3) is locally null — controllable if and only if the adjoint matrix A^* has neither eigenvectors with real eigenvalues $\lambda > 0$ supporting to $B \, \Omega$ at the origin, nor eigenvectors with complex eigenvalues $\lambda \neq 0$ orthogonal to $B \, \Omega$.*

Note that conditions of Corollary 1 are necessary and sufficient conditions for the system (1.1) — (1.2) to be (LC) too. (see / 4 /).

The criterions for (G.C) of the system (1.1) — (1.2) has been studied by Sontag [1] for the case when Ω is a neighbourhood of origin; N.K. Son [4] for the cace when Ω is convex set and $0 \in B \, \Omega$; N.K. Son, N.V. Chau, N. V. Su /6/ for infinite-dimensional DTS...

In this section the criterions for (G.C) of the system (1.1), (1.3) are presented. The main result is the following.

THEOREM 1. *The system (1.1), (1.3) is globally null-controllablef if and only if.*

i) The system (1.1), (1.3) is locally null — controllable

ii) $\sigma(A) \subset \{\lambda \in C : | \lambda | \leqslant 1\}$, where C is the set of complex numbers and $\sigma(A)$ is the spectrum of A.

Before giving a proof of. Theorem 1, we need consider some lemmas as follows.

LEMMA 1. *Assume that $0 \in \text{int } K(\Omega,\varepsilon)$. Let $x \in R^n$ be a state and M be a constant, $M > 0$, such that $\| A^k x \| \leqslant M$ for all $k = 0,1,...,$. Then $x \in K(\Omega,\varepsilon)$.*

Proof. It is not difficult to verify that $K(\Omega,\varepsilon)$ and $K^N(\Omega,\varepsilon)$ are convex sets and

$$K^N(\Omega,\varepsilon) \subset K^{N+1}(\Omega,\varepsilon).$$

By $0 \in \operatorname{int} K(\Omega,\varepsilon)$, it follows from Lemma 1 in [4] that there exists a natural number L such that $0 \in \operatorname{int} K^L(\Omega,\varepsilon)$, that is, for some $\delta > 0$ we have

$$B(0,\delta) = \{x \in R^n : \| x \| < \delta\} \subset K^L(\Omega,\varepsilon) .$$

On the other hand, it is clear that, for all $0 < \alpha \leqslant 1$

$$\alpha K^L(\Omega,\varepsilon) \subset K^L(\Omega,\varepsilon\alpha) ,$$

i. e.

$$B(0,\delta\alpha) \subset K^L(\Omega,\varepsilon\alpha) . \tag{1.4}$$

Let $\beta \in (0,1)$ a number defined by

$$\beta = \left(\sum_{n=1}^{\infty} \frac{1}{n^p} \right)^{-\frac{1}{p}} \tag{1.5}$$

Take a natural number K and numbers λ_n , $n = 1,2,\ldots, K$ such that

$$0 \leqslant \lambda_n < \frac{1}{n} . \delta . \beta M^{-1} \text{ and } \sum_{n=1}^{K} \lambda_n = 1.$$

Setting $x_n = \lambda_n x$, we have

$$\left\| A^k x_n \right\| = \lambda_n \left\| A^k x \right\| \leqslant \lambda_n M < \frac{1}{n} \delta . \beta ,$$

for all $n = 1,2,\ldots,K$ and $k = 0,1,2,\ldots$.
Hence, by (1.4) ,

$$A^k x_n \in B(0, \frac{1}{n} \delta . \beta) \subset K^L\left((\Omega,\varepsilon . \frac{\beta}{n}\right) .$$

This shows that for every $n = 1,2,\ldots, K$ there exist controls $u_n(0), u_n(1),\ldots u_n(L-1)$ such that

$$u_n(i) \in \Omega ; \quad \sum_{i=0}^{L-1} \| u_n(i) \|^p \leqslant \varepsilon^p \beta^p . \frac{1}{n^p} ; \tag{1.6}$$

and

$$A^L(A^{(n-1)L} x_n) + \sum_{i=0}^{L-1} A^i B u_n(L-1-i) = 0 . \tag{1.7}$$

Let $v(j) = u_n(L - 1 - i)$ if $j = (n - 1)L + L - 1 - i$. Then we obtain the controls $v(0), v(1),\ldots, v(KL - 1)$.

Since (1.6) we have $v(j) \in \Omega$ and

$$\sum_{j=0}^{KL-1} \left\| v(j) \right\|^p = \sum_{n=1}^{K} \sum_{i=0}^{L-1} \left\| u_n(L-1-i) \right\|^p \leqslant \beta^p \varepsilon^p \sum_{n=1}^{K} \frac{1}{n^p} < \varepsilon^p ,$$

Consequently, $\{v(j)\}$, $j = 0,1,\ldots, KL-1$ are admissible controls. On the other hand, we have

$$A^{KL}x + \sum_{j=0}^{KL-1} A^jBv(KL-1-j) = \sum_{n=1}^{K}\left[A^{(K-n)L}.A^L(A^{(n-1)L}x_n)\right] +$$

$$+ \sum_{n=1}^{K}\left[A^{(K-n)L}\sum_{i=0}^{L-1} A^iBu_n(L-1-i)\right] =$$

$$= \sum_{n=1}^{K}\left\{A^{(K-n)L}.\left[\sum_{i=0}^{L-1} A^iBu_n(L-1-i) + A^L(A^{(n-1)L}x_n)\right]\right\}.$$

But in view of (1.7) we have

$$A^{KL}x + \sum_{j=0}^{KL-1} A^jBv(KL-1-j) = 0, \text{ i.e } x \in K(\Omega, \varepsilon).$$

The proof is complete.

Let $E \subset R^n$ be an A-invariant subspace, i.e. $AE \subset E$. Let R^n/E be the quotient space of R^n over E endowed with the quotient norm. Denote by Σ_E the quotient system of (1.1), (1.3) with respect to E be defined by

$$\bar{z}(k+1) = \bar{A}\,\bar{z}(k) + \overline{Bu}(k),$$

$$u(k) \in \Omega; \quad \sum_{k=0}^{\infty} \| u(k) \|^p \leqslant \varepsilon^p,$$

where $\bar{z}(k) \in \overline{R^n}/E$; $\bar{A}$ is the induced mapping of A on R^n/E; $\bar{B} = \pi B$ with π to be the canonical projection from R^n to R^n/E.

We denote by $K_E(N, \Omega, \varepsilon)$ the set of all initial states $\bar{z}(0)$ such that there exists a sequence $u(0),\ldots, u(N-1)$ satisfies $u(k) \in \Omega$, $k = 0, 1,\ldots, N-1$;

$$\sum_{k=0}^{N-1} \| u(k) \|^p \leqslant \varepsilon^p; \ \bar{z}(N) = 0. \text{ Denote}$$

$$K_E(\Omega, \varepsilon) = \bigcup_{N=0}^{\infty} K_E(N, \Omega, \varepsilon).$$

LEMMA 2. *Assume that* $E \subset K(\Omega, \varepsilon)$. *Then*

1) $0 \in \operatorname{int} K_E(\Omega, \varepsilon)$ *if and only if* $0 \in \operatorname{int} K(\Omega, \varepsilon)$

2) If M *is subspace in* R^n *such that* $\pi M \subset K_E(\Omega, \varepsilon)$, *then* $M \subset K(\Omega, \varepsilon)$.

Proof. 1) It is a simple to verify that $\pi K(\Omega, \varepsilon) \subset K_E(\Omega, \varepsilon)$. Since π is open mapping, so if $0 \in \operatorname{int} K(\Omega, \varepsilon)$, then $0 \in \operatorname{int} K_E(\Omega, \varepsilon)$.

In view of the proof of Lemma 1, we have

$$0 \in \operatorname{int} K_E(\Omega, \varepsilon) \text{ if and only if } 0 \in \operatorname{int} K_E\left(\Omega, \frac{\varepsilon}{\sqrt[p]{2}}\right). \tag{1.8}$$

On the other hand, since $K(\Omega, \varepsilon)$ is convex and $E \subset K(\Omega, \varepsilon)$, we have $E \subset K(\Omega, \varepsilon')$ for all $\varepsilon' > 0$. Then it is not difficult to prove that

$$\pi^{-1}\left(K_E\left(\Omega, \frac{\varepsilon}{\sqrt[p]{2}}\right)\right) \subset K(\Omega, \varepsilon). \tag{1.9}$$

From combining (1.8) and (1.9) it implies that if

$$0 \in \operatorname{int} K_E(\Omega, \varepsilon), \text{ then } 0 \in \operatorname{int} K(\Omega, \varepsilon).$$

2) Since M is subspace of R^n and $\pi M \subset K_E(\Omega, \varepsilon)$. Clearly, it follows that $\pi M \subset K_E\left(\Omega, \frac{\varepsilon}{\sqrt[p]{2}}\right)$. Then the proof is immediate from (1.9).

The proof of Theorem 1. From $K(\Omega, \varepsilon) \subset K(\Omega)$ and Theorem 4 in [4] it follows that the «only if» part is proved. We now prove the sufficienty of the conditions. Let X_o denote stable space of the dynamic system $z(k+1) = Az(k)$, i.e.

$$X_o = \left\{ z \in R^n : \lim_{k \to +\infty} A^k x = 0 \right\}.$$

Then X_o is A-invariant subspace. It is known that if E is an A-invariant subspace of R^n, $E \neq R^n$ then there exists an A-invariant subspace E' of R^n such that $E \subset E'$ and E'/E is a minimal invariant space of the linear mapping induced by A from R^n/E to R^n/E. Hence, one allways choise A-invariant subspaces X_i, $i = 0, 1, \ldots, s$, satisfying

1) $X_o \subset X_1 \subset \ldots \subset X_s = R^n$

2) For each $i = 1, \ldots, s$ X_i/X_{i-1} is a minimal invariant subspace of A_i induced by A from R^n/X_{i-1} to R^n/X_{i-1}.

Since $\sigma(A) \subset \{\lambda \in C : |\lambda| \leq 1\}$, so are $\sigma(A_i)$. Hence, orbits $\left\{A_i^k x_i, k = 0, 1, \ldots\right\}$ are bounded for all $i = 1, \ldots, s$ and all $x_i \in X_i/X_{i-1}$.

Now since Lemma 1, the condition if and the definition of X_o it follows that $X_o \subset K(\Omega, \varepsilon)$. Applying Lemma 2 with $E = X_o$, we have $0 \in \operatorname{int} K_{X_o}(\Omega, \varepsilon)$.

Hence, it follows from Lemmas 1, 2 that $X_1 \subset K(\Omega, \varepsilon)$. By continuing this process, on the end, we got $R^n = X_s \subset K(\Omega, \varepsilon)$, i.e. the system (1.1), (1.3) is globally-null controllable. The proof is complete.

COROLLARY 2 *(The criterion for (G.C) of the system (1.1), (1.3). The system (1.1), (1.3) is globally-null controllable if and only if conditions of Corollary 1 and the condition (ii) in Theorem 1 are satisfied.*

The next corollary will be used in next section to build a sufficient condition for the pursuit process in discrete games with many player.

COROLLARY 3. *The system (1.1), (1.3) with $\Omega = B(0, \varepsilon)$ is globally-null controllable if and only if*

i) *range* $[B, AB, \dots, A^{n-1}B] =$ *range* $[B, AB, \dots A^{n-1}B, A^n]$,
ii) $\sigma(A) \subset \{\lambda \in C : |\lambda| \leqslant 1\}$.

Remark 1. For infinite-dimensional DTS whose state space and control space are Banach spaces, Lemmas 1, 2 and Theorem 1 still hold if A is a compact operator.

Remark 2. Assume now that M is A-invariant subspace of R^n. We denote by $K_M(N, \Omega, \varepsilon)$ the set of all initial states z_0 such that there exists a sequence $u(0), \dots, u(N\text{-}1)$ satisfies $u(k) \in \Omega$, $k = 0, 1, \dots, N\text{-}1$;

$\sum_{k=0}^{N-1} \| u(k) \|^p \leqslant \varepsilon^p$; $z(N) \in M$. Denote

$$K_M(\Omega, \varepsilon) = \bigcup_{N=0}^{\infty} K_M(N, \Omega, \varepsilon).$$

The system (1.1), (1.3) is said to be globally M-controllable if $K_M(\Omega, \varepsilon) = R^n$. As an immediate consequence of Theorem 1 we obtain

COROLLARY 4. *The system (1.1), (1.3) is globally M-controllable if and only if*

i) $\{f \in C : f \neq 0, f \perp M\, f \perp B\,\Omega;\ A^*f = \lambda f, \lambda \neq 0\} = \phi$,
ii) $\{f \in R : f \neq 0, f \perp M, \langle f, B\,\Omega\rangle \geqslant 0;\ A^*f = \lambda f, \lambda > 0\} = \phi$,
iii) $\{f \in C : A^*f = \lambda f, f \neq 0, |\lambda| > 1, f \perp M\} = \phi$.

COROLLARY 5. *The system (1.1), (1.3) with $\Omega = B(0, \varepsilon)$ is globally M-controllable if and only if*

i) *range* $[B, AB, \dots, A^{n-1}B] + M =$ *range* $[B, AB, \dots, A^{n-1}B, A^n] + M$,
ii) $\{f \in C : A^*f = \lambda f, f \neq 0, f \perp M, |\lambda| > 1\} = \phi$.

In this section we apply the above results to solve the pursuit problems in discrete games with many players. Without loss of generality we may consider the pursuit games with two pursuers and one evader.

Assume that the motions of the vectors $z_i \in R^n$ are described by the difference equations

$$z_i(k+1) = A_i z_i(k) + B_i u_i(k) - C_i v(k); \; z_i(0) = z_i^0 \tag{2.1}$$

where $i = 1,2$; $k = 0,1,\ldots$; $u_i(k) \in R^{p_i}$ are the pursuit controls and $v(k)$ is the evasion controls; A_i, B_i, C_i are matrices of orders $n \times n$, $n \times p_i$, $n \times q$ respectively. The controls u_i, v must satisfy constraints of the form

$$\sum_{k=0}^{\infty} \| u_i(k) \|^2 \leqslant \rho_i^2 , \quad \sum_{k=0}^{\infty} \| v(k) \|^2 \leq \sigma^2 , \tag{2.2}$$

where $\rho_i > 0$, $\sigma > 0$, $i = 1, 2$.

We shall say that the pursuit process in the discrete games with many players (2.1) — (2.2) is completed after k_1 steps, if for any admissible evasion controls $v(0), \ldots, v(k_1 - 1)$,

$$\sum_{i=0}^{k_1 - 1} \| v(i) \|^2 \leqslant \sigma^2 ,$$

there exist the admissible pursuit controls $u_i(0), \ldots, u_i(k_1 - 1)$; $i = 1,2$

$$\sum_{k=0}^{k_1 - 1} \| u_i(k) \|^2 \leq \rho_i^2$$

such that $z_1(k_1) = 0$ or $z_2(k_1) = 0$.

We shall be interested in computing the value $u_i(k)$ of the pursuit control at each step k when the values $v(o), \ldots, v(k)$ of the evasion control are known. In other words, we shall be interested in finding the functions

$$u_i(k) = u_i(v(0), v(1), \ldots, v(k)).$$

We are now in a position to formulate our basic hypotheses

Hypothesis 1. For $i = 1, 2$ we have

a) range $[B_i, A_i B_i, \ldots, A_i^{n-1} B_i]$ = range $[B_i, A_i B_i, \ldots, A_i^{n-1} B_i, A_i^n]$,

b) $\sigma(A_i) \subset \{\lambda \in C : |\lambda| \leqslant 1\}$.

Hypothesis 2. For $i = 1, 2$ there exist linear operators $F_i : R^q \to R^{p_i}$ such that $B_i F_i = C_i$

Hypothesis 3.
$$\left(\frac{\rho_1}{\gamma_1}\right)^2 + \left(\frac{\rho_2}{\gamma_2}\right)^2 > \sigma^2$$

where $\gamma_i = \| F_i \| = \sup \{ \| F_i v \| : \| v \| = 1\}$.

THEOREM 2. *Under hypotheses 1 — 3, the pursuit process in the discrete games with many players (2.1) — (2.2) is completed after a finite step for any initial position* $z^0 = (z_1^0, z_2^0)$.

Proof. From hypothesis 3, it follows that there exist a number $\varepsilon > 0$ such that

$$0 < \varepsilon < \min(\rho_1, \rho_2); \quad \left[\frac{\rho_2 - \varepsilon}{\gamma_2}\right]^2 > \sigma^2 - \left[\frac{\rho_1 - \varepsilon}{\gamma_1}\right]^2 > 0.$$

Assume now that $\bar{v}(.) = (\bar{v}(0), \bar{v}(1), \ldots, \bar{v}(k), \ldots)$ is an arbitrary evasion control, i.e.

$$\sum_{k=0}^{\infty} \|\bar{v}(k)\|^2 \leqslant \sigma^2 .$$

We consider the following system

$$z_1(k+1) = A_1 z_1(k) + B_1 \omega_1(k) ; \; z_1(0) = z_1^0, \tag{2.3}$$

$$\sum_{k=0}^{\infty} \|\omega_1(k)\|^2 \leqslant \varepsilon^2 . \tag{2.4}$$

By Hypothesis 1 and Corollary 3 there exist the controls $\bar{\omega}_1(0), \ldots, \bar{\omega}_1(N_1 - 1)$ such that

$$\sum_{k=0}^{N_1-1} \|\bar{\omega}_1(k)\|^2 \leqslant \varepsilon^2 ,$$

$$A_1^{N_1} z_1^0 + \sum_{k=0}^{N_1-1} A_1^{N_1-1-k} B_1 \bar{\omega}_1(k) = 0. \tag{2.5}$$

We consider two following cases.

1. If

$$\sum_{k=0}^{N_1-1} \|\bar{v}(k)\|^2 \leqslant \left[\frac{\rho_1 - \varepsilon}{\gamma_1}\right]^2.$$

Then the pursuit controls $\bar{u}_1(0),\dots,\bar{u}_1(N_1-1)$ are defined as follows

$$\bar{u}_1(k) = F_1\bar{v}(k) + \bar{\omega}_1(k).$$

By Minkoswki's inequality we have

$$\left[\sum_{k=0}^{N_1-1}\|\bar{\omega}_1(k)+F_1\bar{v}(k)\|^2\right]^{\frac{1}{2}} \leqslant \left[\sum_{k=0}^{N_1-1}\|\bar{\omega}_1(k)\|^2\right]^{\frac{1}{2}} +$$

$$+\left[\sum_{k=0}^{N_1-1}\|F_1\bar{v}(k)\|^2\right]^{\frac{1}{2}} \leqslant \varepsilon + \left[\gamma_1^2\sum_{k=0}^{N_1-1}\|\bar{v}(k)\|^2\right]^{\frac{1}{2}} \leqslant$$

$$\leqslant \varepsilon + \gamma_1\frac{\rho_1-\varepsilon}{\gamma_1} = \rho_1.$$

Consequently,

$$\sum_{k=0}^{N_1-1}\|\bar{u}_1(k)\|^2 \leqslant \rho_1^2$$

i.e. $\bar{u}_1(0),\dots,\bar{u}_1(N_1-1)$ are the admissible pursuit controls. By formula for the solution of equation (2.1) we have

$$z_1(N_1) = A_1^{N_1}z_1^o + \sum_{k=o}^{N_1-1}A_1^{N_1-1-k}B_1\bar{u}_1(k) - \sum_{k=o}^{N_1-1}A_1^{N_1-1-k}C_1\bar{v}(k) =$$

$$= A_1^{N_1}z_1^o + \sum_{k=o}^{N_1-1}A_1^{N_1-1-k}B_1F_1\bar{v}(k) + \sum_{k=o}^{N_1-1}A_1^{N_1-1-k}B_1\bar{\omega}_1(k) -$$

$$-\sum_{k=o}^{N_1-1}A_1^{N_1-1-k}C_1\bar{v}(k). \tag{2.6}$$

But in view of Hypothesis 2 we have for all $k = 0,\dots, N_1-1$

$$A_1^{N_1-1-k}B_1F_1\bar{v}(k) = A_1^{N_1-1-k}C_1\bar{v}(k).$$

This implies, in view of (2.5), (2.6) $z_1(N_1) = 0$

2. If

$$\sum_{k=0}^{N_1-1}\|v(k)\|^2 > \left(\frac{\rho_1-\varepsilon}{\gamma_1}\right)^2$$

The number k_1 $(k_1 \leqslant N_1 - 1)$ is defined as follows

$$\sum_{k=o}^{k_1-1} \| \overline{v}(k) \|^2 \leqslant \left(\frac{\rho_1 - \varepsilon}{\gamma_1} \right)^2,$$

$$\sum_{k=o}^{k_1} \| \overline{v}(k) \|^2 > \left(\frac{\rho_1 - \varepsilon}{\gamma_1} \right)^2$$

Then the pursuit controls $\overline{u}_i(k)$; $k = 0,1,\ldots, k_1$: $i = 1,2$ are defined as follows

$$\overline{u}_1(k) = F_1 \overline{v}(k) + \overline{\omega}_1(k); \quad k = 0,\ldots, k_1 - 1, \quad \overline{u}_1(k_1) = 0,$$

$$\overline{u}_2(o) = \overline{u}_2(1) = \ldots = \overline{u}_2(k_1) = 0.$$

We consider the following system

$$\overline{z}_2(k+1) = A_2 \overline{z}_2(k) + B_2 \omega_2(k) \; ; \; z_2(0) = \tilde{z}_2^0 \; , \tag{2.7}$$

$$\sum_{k=0}^{\infty} \| \omega_2(k) \|^2 \leqslant \varepsilon^2, \tag{2.8}$$

where

$$\tilde{z}_2^0 = z_2(k_1 + 1) = A_2^{k_1+1} z_2^0 - \sum_{j=0}^{k_1} A_2^{k_1 - j} C_2 \overline{v}(j).$$

By Hypothesis 1 and Corollary then there exist controls $\overline{\omega}_2(0), \overline{\omega}_2(1), \ldots,$ $\overline{\omega}_2(N_2 - 1)$ such that

$$\sum_{k=0}^{N_2-1} \| \overline{\omega}_2(k) \|^2 \leqslant \varepsilon^2,$$

$$A_2^{N_2} \tilde{z}_2^0 + \sum_{k=0}^{N_2-1} A_2^{N_2-1-k} B_2 \overline{\omega}_2(k) = 0. \tag{2.9}$$

Then the pursuit controls $\overline{u}_i(k_1 + 1 + k)$, $k = 0, \ldots, N_2 - 1$; $i = 1, 2$ are defined as follows:

$$\overline{u}_1(k_1 + 1 + k) = 0, \; k = 0,\ldots, N_2 - 1.$$

$$\overline{u}_2(k_1 + 1 + k) = \overline{\omega}_2(k) + F_2 \overline{v}(k_1 + 1 + k).$$

We have

$$\sum_{k=0}^{N_2+k_1} \| \overline{u}_2(k) \|^2 = \sum_{k=0}^{N_2-1} \| \overline{u}_2(k_1+1+k) \|^2 =$$

$$= \sum_{k=0}^{N_2-1} \| \overline{\omega}_2(k) + F_2 \overline{v}(k_1+1+k) \|^2 . \tag{2.10}$$

$$\sum_{k=0}^{N_2-1} \| F_2 \overline{v}(k_1+1+k) \|^2 \leqslant \gamma_2^2 \sum_{k=0}^{N_2-1} \| \overline{v}(k_1+k+1) \|^2 . \tag{2.11}$$

On the other hand, we have

$$\sum_{k=0}^{N_2-1} \| \overline{v}(k_1+k+1) \|^2 \leqslant \sigma^2 - \sum_{k=0}^{k_1} \| \overline{v}(k) \|^2 \leqslant$$

$$\leqslant \sigma^2 - \left(\frac{\rho_1 - \varepsilon}{\gamma_1} \right)^2 < \left(\frac{\rho_2 - \varepsilon}{\gamma_2} \right)^2 . \tag{2.12}$$

From (2.11), (2.12) we deduce

$$\sum_{k=0}^{N_2-1} \| F_2 \overline{v}(k_1+1+k) \|^2 < (\rho_2 - \varepsilon)^2 . \tag{2.13}$$

By Minkowski's inequality and (2.13) we have

$$\sum_{k=0}^{N_2+k_1} \| \overline{u}_2(k) \|^2 \leqslant \rho_2^2 ,$$

i.e. the controls $\overline{u}_2(k)$, $k = 0,\dots, N_2+k_1$ are admissible.

By formula for the solution of equation (2.1) we have

$$z_2(N_2+k_1+1) = A_2^{k_1+N_2+1} z_2^0 +$$

$$+ \sum_{k=0}^{N_2+k_1} A_2^{N_2+k_1-k} B_2 \overline{u}_2(k) - \sum_{k=0}^{N_2+k_1} A_2^{N_2+k_1-k} C_2 \overline{v}(k) =$$

$$= A_2^{N_2} (A_2^{k+1} z_2^0) + \sum_{k=k_1+1}^{N_2+k_1} A_2^{N_2+k_1-k} B_2 \overline{u}_2(k) - \sum_{k=0}^{N_2+k_1} A_2^{N_2+k_1-k} C_2 \overline{v}(k) =$$

$$= A_2^{N_2}(A_2^{k_1+1} z_2^0) - \sum_{k=0}^{k_1} A_2^{N_2+k_1-k} C_2 \overline{v}(k) - \sum_{k=k_1+1}^{N_2+k_1} A_2^{N_2+k_1-k} C_2 \overline{v}(k) +$$

$$+\sum_{k=k_1+1}^{N_2+k_1} A_2^{N_2+k_1-1} B_2\overline{u}_2(k) = A_2^{N_2}(A_2^{k_1+1} z_2^0 - \sum_{k=0}^{k_1} A_2^{k_1-k} C_2\overline{v}(k)) -$$

$$-\sum_{k=k_1+1}^{N_2+k_1} A_2^{N_2+k_1-k} C_2\overline{v}(k) + \sum_{k=k_1+1}^{N_2+k_1} A_2^{N_2+k_1-k} B_2\overline{\omega}_2(k-k_1-1) +$$

$$+\sum_{k=k_1+1}^{N_2+k_1} A_2^{N_2+k_1-k} B_2F_2\overline{v}(k) = A_2^{N_2} z_2^0 + \sum_{k=0}^{N_2-1} A_2^{N_2-1-k} B_2\overline{\omega}_2(k) +$$

$$+\sum_{k-k_1+1}^{N_2+k_1-k} A_2^{N_2+k_1-k} B_2F_2\overline{v}(k) - A_2^{N_2+k_1-k} C_2\overline{v}(k).$$

This implies, in view of (2.9) $z_2(N_2+k_1+1)=0$.

The proof is complete.

In /7/ were considered the pursuit process with two pursuers and one evader:

$$z_i(k+1) = A_i z_i(k) - u_i(k) + v(k);\ z_i(0) = z_i^0,\ i=1,2, \tag{2.14}$$

where $z_i \in R^n$, $A_i = (a_{kj}^{(i)})$ are triangular matrices of order n and assume that

$$|a_{jj}^{(i)}| < 1 \text{ for all } j=1,\dots,n;\ i=1,2.$$

The controls $u_i \in R^n$, $v \in R^n$ must satisfy constraints of the form

$$\sum_{k=0}^{\infty} \|u_i(k)\|^2 \leqslant \rho_i^2,\ \sum_{k=0}^{\infty} \|v(k)\|^2 \leqslant \delta^2, \tag{2.15}$$

where $\rho_i > 0$, $\delta > 0$, $i=1,2$.

We shall say that the pursuit process in the discrete game (2.14) — (2.15) is completed after k_1 steps, if for any the admissible evasion controls $v(0),\dots,v(k_1-1)$, there exist the admissible pursuit controls $u_i(k)=u_i(v(k))$, $k=0,\dots,k_1-1$; $i=1,2$ such that $z_1(k_1)=0$ or $z_2(k_1)=0$.

As an immediate consequence of Theorem 2 we obtain.

COROLLARY 6 (see [7]). *If $\rho_1^2+\rho_2^2>\delta^2$, then the pursuit process in the discrete game (2.14) – (2.15) is completed after a finite step for any initial position $z^0=(z_1^0, z_2^0)$.*

Proof. In this case we have $B_i = C_i = E, i = 1, 2$, where E is the unit matrix of order n. Then Hypothesis 2 is fulfilled with F_i are identity mappings. Since $\gamma_1 = \gamma_2 = 1$ and $\rho_1^2 + \rho_2^2 > \delta^2$ it follows that Hypothesis 3 is fulfilled. By $B_i = E$, we have

$$\text{range}\left[B_i, A_i B_i, \ldots, A_i^{n-1} B_i\right] = \text{range}\left[B_i, A_i B_i, \ldots, A_i^{n-1} B_i, A_i^n\right].$$

On the other hand, since A_i is the triangular matrices and $|a_{jj}^{(i)}| < 1$ implies that Hypothesis 1 is fulfilled too. The proof is complete.

Remark 3. In general there are many linear mappings $F_i : R^q \to R^{p_i}$ such that $B_i F_i = C_i$. It is a simple matter to verify that there exists linear mapping $F_i^* : R^q \to R^{p_i}$ such that

$$B_i F_i^* = C_i \text{ and } \| F_i^* \| \inf_{F_i \in \Delta_i} = \| F_i \|,$$

where Δ_i is the set of linear mappings $F_i : R^q \to R^{p_i}$ such that $B_i F_i = C_i$. We shall then assume that linear mappings F_i in Hypothesis 2 are F_i^*, $i = 1,2$.

Remark 4. Hypothesis 2 is equivalent to the fact that for $i = 1, 2$, we have $C_i R^q \subset B_i R^{p_i}$. This follows from

LEMMA 3. *Let U, V, R be the linear spaces of finite dimensional and $P_i : U \to R$, $Q : V \to R$ are linear mappings. Then $PU \subset QV$ if and only if there exists linear mapping $F : U \to V$ such that $P = QF$.*

Proof. The «only if» part is clear. We now turn to the proof of necessity. Let $PU \subset QV$ and $e_1, \ldots, e_p$ are the basic in U. Then there exist vectors $f_1, \ldots, f_p$, $f_i \in V$. $i = 1,2,\ldots, p$, such that

$$Pe_i = Qf_i \quad , \; i = 1,2,\ldots, p .$$

The linear mappings $F : U \to V$ is defined as follows: Let $x \in U$, $x = \sum_{i=1}^{p} b_i e_i$,

then $Fx = y = \sum_{i=1}^{p} b_i f_i \in V$.

We have

$$Px = \sum_{i=1}^{p} b_i Pe_i = \sum_{i=1}^{p} b_i Qf_i = Q \sum_{i=1}^{p} b_i f_i = QFx,$$

i. e. $P = QF$. The proof is complete.

Now, we consider the pursuit process in the discrete game (2.1)—(2.2) with the terminal set M_i being A_i – invariant subspaces. We shall say that the pursuit process in the discrete game with many players (2.1)—(2.2) is completed after k_1 steps if for any the admissible evasion controls $v(0),\ldots, v(k_1-1)$, there exist the admissible pursuit controls $u_i(0),\ldots, u_i(k_1-1)$, $i = 1, 2$, such that $z_1(k_1) \in M_1$ or $z_2(k_1) \in M_2$.

We are now in a position to formulate our basic hypotheses.

Hypothesis 1'. For $i = 1,2$ we have

a) range $[B_i, A_i B_i, \ldots, A_i^{n-1} B_i] + M_i$ = range $[B_i, \ldots, A_i^{n-1} B_i, A_i^n] + M_i$,

b) $\{f \in C^n : A^* f = \lambda f,\ f \neq 0,\ f \perp M_i,\ |\lambda| > 1\} = \phi$.

Hypothesis 2'. For $i = 1,2$ there exist linear operators $F_i : R^q \to R^{p_i}$ such that $\pi_i B_i F_i = \pi_i C_i$, where $\pi_i : X \to X/M_i$ is canonical projection.

Hypothesis 3'.

$$\left(\frac{\rho_1}{\gamma_1}\right)^2 + \left(\frac{\rho_2}{\gamma_2}\right)^2 > \sigma^2,$$

where $\gamma_i = \| F_i \|$.

THEOREM 3. *Under Hypotheses 1'-3', the pursuit process in the discrete games with many player (2.1) — (2.2) is completed after a finite step for any initial position* $z^0 = (z_1^0, z_2^0)$.

The proof of the Theorem 3 parallels that of Theorem 2 and will be omitted.

EXAMPLE. Assume now that the motions of the vectors $z_i \in R^n$, $i = 1, 2$ are described by the difference equations

$$z_i^1(k+1) = \alpha_i z_i^1(k) + u_i(k);\ z_i^1(0) = z_{i1}^0,$$
$$z_i^2(k+1) = \alpha_i z_i^2(k) + v(k);\ z_2^i(0) = z_{i2}^0, \tag{2.16}$$

where $|\alpha_i| \leqslant 1$, $i = 1,2$. The controls u_i, v must satisfy constraints of the form

$$\sum_{k=0}^{\infty} \| u_i(k) \|^2 \leqslant \rho_i^2,\quad \sum_{k=0}^{\infty} \| v(k) \|^2 \leqslant \sigma^2, \tag{2.17}$$

We shall say that the pursuit process in the discrete game (2.16) — (2.17) is completed after k_1 steps if $z_1^1(k_1) = z_1^2(k_1)$ or $z_2^1(k_1) = z_2^2(k_1)$. We shall be interested in computing the value $u_i(k)$ of the pursuit control at each step k when the value $u(k)$ of the evasion control are known. In other hand we shall be interested in finding the function

$$u_i(k) = u_i(v(k)).$$

Let $z_i = (z_i^1, z_i^2)^T$,

$$A_i = \begin{pmatrix} \alpha_i E & 0 \\ 0 & \alpha_i E \end{pmatrix}; \; B_i = \begin{pmatrix} E \\ 0 \end{pmatrix}; \; C_i = \begin{pmatrix} 0 \\ -E \end{pmatrix}$$

Then we can write (2.16) into a single equation

$$z_i(k+1) = A_i z_i(k) + B_i u_i(k) - C_i v(k) ; z_i(0) = z_i^0 = \left(z_{i1}^0, z_{i2}^0\right)^T,$$

where E is the unit matrix of order n, 0 is the zero matrix of order n and $\left(z_i^1, z_i^2\right)^T$ is transpose of vector $\left(z_i^1, z_i^2\right)$. In this case, we have

$$M_i = \left\{\left(z_i^1, z_i^2\right)^T : z_i^1 = z_i^2\right\}.$$

It is clear that M_i are A_i – invariant subspace of R^{2n}. It can be verified that Hypotheses 1' — 3' of Theorem 3 are satisfied, where F_i, $i = 1, 2$ are identity mappings and then $\gamma_1 = \gamma_2 = 1$. By applying Theorem 3 we get

PROPOSITION 2. *Assume that* $|\alpha_i| \leq 1$, $i = 1,2$ *and* $\rho_1^2 + \rho_2^2 > \sigma^2$. *Then the pursuit process in the discrete game (2.16) — (2.17) is complete after a finite step for any initial position* $z_i^0 = \left(z_{i1}^0, z_{i2}^0\right)^T$.

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