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CONVEX PROGRAMS WITH SEVERAL ADDITIONAL REVERSE CONVEX CONSTRAINTS

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1. INTRODUCTION

This paper addresses the general problem of minimizing a convex function subject to convex and reverse convex constraints. Here by a *reverse convex* constraint is meant any constraint of the form

$$g(x) \leqslant 0,$$

where $g: R^n \to R$ is a concave function (i.e $-g$ is a convex function). Clearly, a reverse convex constraint determines a closed subset of R^n, whose complement is an open convex set (therefore, in the literature such constraints are also termed «complementary convex»).

Optimization problems with reverse convex constraints are encountered in a number of economical and engineering applications (see e.g. [1], [5], [12], [13], [14] and also [11]). In recent years they have received an increasing attention from researchers, because of their importance for the applications and perhaps also because of their intrinsic difficulty which, for a long time, has seemed to defy any attempt to solve them numerically. Among the works devoted to special cases of this class of problems let us mention Rosen [5], Avriel and Williams [1], Ueing [12], Hillestad and Jacobsen ([2], [3]), Tuy ([9], [11]), Vidigal [13].

In the present paper we shall focus our attention on the most typical problem of the mentioned class, namely the following problem

$$\text{(P) Minimize } f(x), \text{ s.t. } x \in D,\ g(x) \leqslant 0.$$

where $f: R^n \to R$ is a convex finite function, D is a closed convex subset of R^n given by a convex finite function $h: R^n \to R$:

$$D = \{ x \in R^n : h(x) \leqslant 0 \},$$

while $g: R^n \to R$ is a *concave* finite function.

The case of several reverse convex constraints:

$$g_1(x) \leqslant 0,\ g_2(x) \leqslant 0, \dots,\ g_m(x) \leqslant 0,$$

with g_i $(i = l, \dots, m)$ concave finite functions, can easily be reduced to the previous one see [11]. Indeed, the above system can be written as a single inequality:

$$g(x) \leqslant 0,$$

where $g(x) = \max \{ g_1(x), g_2(x), \dots, g_m(x) \}$. But it is not hard to see that

$$g(x) = p(x) - q(x),$$

with

$$p(x) = g_1(x) + g_2(x) + \dots + g_m(x),$$

$$q(x) = \min \{ \sum_{i \neq j} g_i(x) : j = l, \dots, m \}$$

Since both $p(x)$ and $q(x)$ are concave functions, by introducing an additional variable t we can now rewrite the primary system of reverse convex constraints in the form

$$p(x) \leqslant t,\ q(x) \geqslant t.$$

Here the first inequality is obviously reverse convex, while the second being convex could be incorporated into the constraint $x \in D$.

Thus, at the cost of at most one additional variable, any convex program with several additional reverse convex constraints can be converted into the form (*P*). Also note that, as shown in [11], a very wide class of mathematical programming problems can be cast into this framework. In particular, to this class belongs the problem of minimizing (or maximizing) a d. c. function (i. e. a function which can be rewritten as a difference between two convex functions) under d. c. constraints (i. e. constraints of the form $g(x) \leqslant 0$, with $g(x)$ a d. c. function).

A first systematic study of problem (P) is provided in the recent paper of *H*. Tuy [11]. There, using a general duality relation between constraints and objectives in mathematical programming problems a global optimality criterion is established for problem (P) which reduces the testing for the global optimality of a given feasible solution of (P) to a concave minimization (i. e. convex maximization) subproblem. On the basis of this fundamental result a solution method for problem (P) is developed which consists in solving, e. g. by the method of Thieu-Tam-Ban [6], a connected sequence of linearly constrained concave programs.

Our purpose in the sequel is to present a different, and perhaps more direct, approach to problem (P). By restating the problem appropriately, we first convert it into a problem of finding the lexicographic minimum of a function vector over a compact convex set. Since this function vector enjoys the basic property that its lexicographic minimum over a polytope is always achieved in at least one extreme point, the lexicographic minimization problem can be treated in much the same way as a quasiconcave minimization problem. Thus, from the conceptual point of view the solution method to be proposed is very simple. From the computational point of view, our algorithm, while sharing several common features with the algorithm developed in [11], differs in fact substantially from the latter, offering several improvements upon it. In particular unlike the algorithm in [11], our algorithm will work without requiring any, stability condition for the problem.

The paper consists of 5 sections. After the Introduction we shall establish in Section 2 the reduction of the original problem (P) to a lexicographic minimization problem. In Section 3 we shall present in detail the proposed solution method. Section 4 deals with convergence properties of the algorithm. Section 5 treats the case where the objective function is linear. Section 6 discusses the relation of our method to the method in [11]. Finally, some illustrative examples are given in the Appendix.

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2. REDUCTION TO A LEXICOGRAPHIC MINIMIZATION PROBLEM

For the sake of convenience, let us rewrite the problem we are concerned with :

(P) Minimize $f(x)$, s.t. $x \in D$, $g(x) \leqslant 0$,

where

$$D = \{x \in R^n : h(x) \leqslant 0\},$$

$f, h : R^n \to R$ being convex finite functions, while $g : R^n \to R$ is a concave finite function. As is well known from Convex Analysis (see e. g. [4]), all the functions $f, h, -g$ are then continuous and subdifferentiable everywhere.

Throughout the paper we shall make the following assumptions:

(i) There exists at least one feasible point, i. e. a point x satisfying $x \in D$, $g(x) \leqslant 0$;

(ii) The set $C = \{x \in D : g(x) \geqslant 0\}$ is bounded;

(iii) $\text{Inf}\ \{f(x) : x \in D\} < \text{Min}\ \{f(x) : x \in D,\ g(x) \leqslant 0\}$.

Condition (i) is quite natural. In any case, it can be checked by solving the concave minimization problem:

$$\text{Minimize} \quad g(x) \text{ , s. t. } \quad x \in D$$

(see e. g. [9]). Condition (ii), which generally holds in practical applications, rules out some technical complications connected with the presence of infinite rays over which $g(x)$ may be unbounded below. As for Condition (iii) it simply means that the reverse convex constraint $g(x) \leqslant 0$ is essential: it fails to hold only if the problem reduces to the classical convex program

$$\text{Minimize } f(x) \text{ , s. t. } x \in D.$$

As an immediate consequence of the above assumptions we have

PROPOSITION 1 *Problem* (P) *has an optimal solution and every optimal solution* $\overline{x}$ *of* (P) *must satisfy* $g(\overline{x}) = 0$.

Proof. In view of (iii) there is a point w such that

$$w \in D \text{, } g(w) > 0 \tag{1}$$

$$f(w) < \text{Min}\ \{f(x) : x \in D \text{ , } g(x) \leqslant 0\}. \tag{2}$$

For any $z \in D$ such that $g(z) < 0$ denote by $\pi(z)$ the point where the line segment $[w, z]$ meets the surface $g(x) = 0$ (since g is concave the existence of $\pi(z)$ follows from the inequalities $g(z) < 0$, $g(w) > 0$; moreover, $\pi(z)$ is uniquely defined). From the convexity of f we can write, for $\pi(z) = \theta\, w + (1-\theta)z$, $0 < \theta < 1$:

$$f(\pi(z)) \leqslant \theta f(w) + (1-\theta)f(z) < f(z).$$

This means that z cannot be optimal. Therefore, every optimal solution of (P) must lie on the surfare $g(x) = 0$. Then the compactness of the set $\{x \in D : g(x) = 0\}$ together with the continuity of f ensure the existence of an optimal solution. □

We are now in a position to formulate the key idea of our approach. Let

$$h^+(x) = \max\ \{h(x), 0\} \tag{3}$$

$$C = \{x \in D : g(x) \geqslant 0\} = \{x : g(x) \geqslant 0,\ h^+(x) = 0\} \tag{4}$$

and consider the problem:

(Q) *Find the lexicographic minimum of the function vector* $(g(x) - h^+(x)$ $f(x))$ *over the compact convex set* C.

Recall that a vector $(a_1, a_2) \in R^2$ is said to be lexicographically greater than (b_1, b_2) if either of the following alternatives holds:

1) $a_1 > b_1$; 2) $a_1 = b_1$ and $a_2 > b_2$.

THEOREM 1. *Under Assumptions* (i) (ii) (iii) *problem* (P) *is equivalent to* (Q).

Proof Since $x \in D$ implies $h^+(x) = 0$, we have for all $x \in C$: $g(x) - h^+(x) = g(x)$. This being so, if $\bar{x}$ is an optimal solution to (Q), then

1) $\bar{x} \in D$, $g(\bar{x}) \geqslant 0$;

2) $g(\bar{x}) = \min \{ g(x) : x \in D, g(x) \geqslant 0 \}$;

3) $f(\bar{x}) \leqslant f(x)$ for all $x \in D$ such that $g(x) = g(\bar{x})$.

But by Proposition 1 there exist points $x \in D$ satisfying $g(x) = 0$. Therefore, we must have, from 1) and 2) : $g(\bar{x}) = 0$. Next it follows from 3) that $f(\bar{x}) \leqslant f(x)$ for all $x \in D$ satisfying $g(x) = 0$. By Proposition 1 we then conclude that $\bar{x}$ is an optimal solution to (P).

Conversely, if $\bar{x}$ is an optimal solution to (P) then by Proposition 1 $g(\bar{x}) = 0$ so that $\bar{x}$ satisfies 1) and 2). Since 3) is obvious, $\bar{x}$ is a lexicographic minimum of $(g(x) - h^+(x), f(x))$ over C, hence an optimal solution to (Q). □.

Problem (Q) is merely an alternative restatement of the original problem. However, it is more amenable to solution owing to the following fact.

PROPOSITION 2. *Assume that either $g(x)$ is strictly concave or $f(x)$ is linear. Then the lexicographic minimum of $(g(x) - h^+(x), f(x))$ over any polytope S is achieved in at least one vertex of this polytope.*

Proof. Since $h^+(x)$ is convex, if $g(x)$ is strictly concave then so is $g(x) - h^+(x)$ and the minimum of $g(x) - h^+(x)$ over a polytope S can be achieved only at some vertex of S. Hence the lexicographic minimum of $(g(x) - h^+(x), f(x))$ over S must be attained at some vertex of S. If $g(x)$ is concave but not necessarily strictly concave, the set where $g(x) - h^+(x)$ attains its minimum over S is a union of faces of S (see e.g. [4], Corollary 32.1.1). Therefore, if $f(x)$ is linear, the lexicographic minimum of $(g(x) - h^+(x), f(x))$ must be achieved in at least one vertex of a certain face (among the above mentioned faces) of S. By a well known property of convex polytopes (see e.g. [4], Section 18), this vertex will also be an extreme point of S. □

The above Proposition shows that problem (Q) is essentially similar to the problem of globally minimizing a quasiconcave function over a compact set. This suggests to adapt to problem (Q) the outer approximation method for solving concave minimization problem (see [9]). In the next Section we proceed to describe in detail how this idea can be realized.

3. SOLUTION METHOD

Let us first recall the scheme of outer approximation (see [9]). Given a compact convex set $C \subset R^n$ and a function $F(x)$, with the property that the minimum of F over any polytope is attained in at least one vertex (extreme point), we can find the global minimum of $F(x)$ over C by the following procedure.

Let W be any fixed compact convex subset of C containing at least one point achieving the global minimum of $F(x)$ over C.

We start from a polytope $S_1 \supset W$ and solve the first relaxed problem:

$$\text{Min } F(x), \text{ s.t. } x \in S_1 .$$

(Since the minimum of $F(x)$ over S_1 must be achieved at some vertex of S_1, we have only to search through the vertex set V_1 of S_1). Let z^1 be an optimal solution of the first relaxed problem. If it so happens that $z^1 \in C$ we are done: z^1 is an optimal solution of the original problem. Otherwise we construct an affine function $l_1(x)$ (i.e. a hyperplane) strictly separating z^1 from W, i. e. such that $l_1(z^1) > 0$, while $l_1(x) \leqslant 0$ for all $x \in W$. This is always possible since $z^1 \notin C$ implies $z^1 \notin W$. Adding to S_1 the constraint

$$l_1(x) \leqslant 0,$$

we define a new polytope $S_2 \supset W$ smaller than S_1 since z^1 has been cut away from S_1. Then we consider the second relaxed problem

$$\text{Min } \{F(x) : x \in S_2\}$$

and repeat exactly the same procedure as before.

Continuing this way, we may arrive, after a certain number of iterations, at a point z^k (the optimal solution of the k — th relaxed problem) such that $z^k \in C$: then we stop, having obtained a global minimum of F over C. Otherwise, the process will generate an infinite sequence $\{z^k\}$. We can then prove the following proposition, which is a specialized form of a general theorem established in [9].

PROPOSITION 3. *Let* $l_k(x) = \langle p^k, x\rangle + \gamma_k$ *be the affine function such that* $l_k(z^k) > 0$, *and* $l_k(x) \leqslant 0 \ \forall \ x \in W$. *Assume:*

a) There is a constant r such that for all k:

$$|\gamma_k| \leqslant r \, |p^k| ;$$

b) For any subsequence $\{k_\nu\} \subset \{k\}$, *if* $z^{k_\nu} \to z \notin W$, $p^{k_\nu} \to p$, $\gamma_{k_\nu} \to \gamma$, *then* $l(z) = \langle p, z\rangle + \gamma > 0$.

Then every cluster point z of the sequence z^k will belong to W.

Proof. Since $l_k(z^k) > 0$, $l_k(z^{k+1}) \leqslant 0$, we must have $p^k \neq 0$ and so we can assume that $\| p^k \| = 1$. Let $z = \lim_{\nu \to \infty} z^{k_\nu}$. In view of a) we can assume, by taking subsequences if necessary, that $p^{k_\nu} \to p$, $\gamma_{k_\nu} \to \gamma$, so that $l_{k_\nu}(x) \to l(x) = \langle p, x \rangle + \gamma$ for every x. For all $t > k_\nu$ we have $z^t \in S_t \subset S_k$ i. e. $l_{k_\nu}(z^t) = 0$. Therefore, fixing ν and letting $t = k_\mu$, $\mu \to \infty$, we obtain $l_{k_\nu}(z) \leqslant 0$; then, letting $\nu \to \infty$ we obtain $l(z) = 0$. In view of b) this implies $z \in W$ $\square$.

Thus, if the separating affine functions $l_k(.)$ are constructed in such a way to satisfy a) and b), then every cluster point z of the generated sequence $\{z_k\}$ will belong to $W \subset C$. Since $F(z^k) \leqslant F(x) \; \forall x \in S_k$ and $W \subset S_k$ it follow that $F(z) \leqslant F(x) \; \forall x \in W$, hence z is a global minimum of F over C.

Let us now adapt the above described method to problem (Q), where the function vector $(g(x) - h^+(x), f(x))$ is given the role of $F(x)$ and the set W is taken to be

$$W = \{ x \in D : f(x) \leqslant \alpha \} \tag{5}$$

with

$$\alpha = \min \{ f(x) : x \in D, \; g(x) \leqslant 0 \}. \tag{6}$$

Note that we actually have here $W \subset C$, i. e. $g(z) \geqslant 0$ for every $z \in W$, for if $g(z) < 0$ for some $z \in W \subset D$ then the point $\pi(z)$ defined as in the proof of Proposition 1 would yield $f(\pi(z)) < f(z) \leqslant \alpha$, contradicting (6).

In view of the specific features of problem (Q), two basic points should be clarified for the concrete realization of the above scheme.

The first is concerned with the *relaxed problems* to be solved in the iterations. Following the above scheme, we generate a nested sequence of polytopes S_k enclosing W along with the sequence of relaxed problems

$$\text{Lex-min } (g(x) - h^+(x), \; f(x)), \text{ s . t. } x \in S_k \, . \tag{7}$$

According to Proposition 2, these problems can be replaced by

$$\text{Lex-min } (g(x) - h^+(x), \; f(x)), \text{ s. t. } x \in V_k \, ,$$

where V_k is the vertex set of S_k. However, exploiting the fact that every optimal solution to P must lie in the region $\{x : g(x) \leqslant 0\}$, we shall replace (7) by

$$(Q_k) \qquad \text{Lex-min } (g(x) - h^+(x), \; f(x)), \text{ s.t } x \in V_k \, , \; g(x) \leqslant 0. \tag{8}$$

Actually, it will be shown later (Lemma 2) that any optimal solution z^k to (8) such that $z^k \in C$ will solve the original problem (P).

The second point relates to the construction of the *separating hyperplane* when $z^k \notin C$. This can be done in several different ways. Of course, the simplest way is to construct a hyperplane separating z^k strictly from C. However for our method this kind of cuts would be poorly efficient. In fact, since W is convex and in general much smaller than C, it would be much better to separate z^k from W. But then the difficulty is that α, and hence the set W, is unknown as long as the problem has not been solved yet.

To get round this difficulty we proceed as follows. For the sake of simplicity assume

(iv) $$\text{int } D \neq \phi$$

so that we can pick a point $w \in \text{int } D$ satisfying (1) and (2), i. e. such that

$$h(w) < 0,\ g(w) > 0,\ f(w) < \alpha. \tag{9}$$

At iteration k let $\bar{x}^k$ be the best feasible point known so far, i. e. a feasible point such that $f(\bar{x}^k) \leqslant f(x)$ for all feasible points encountered during the solution procedure up to this iteration. Let

$$\beta_k = f(\bar{x}^k),\ W_k = \{x \in C : f(x) \leqslant \beta_k\} \tag{10}$$

Obviously W_k is a compact convex set such that

$$W \subset W_k \subset C .$$

As the algorithm proceeds, W_k yields a better and better approximation of W. If $z^k \notin C$ then $z^k \notin W_k$ and since $w \in \text{int } W_k$ (see (9)) the line segment $[w, z^k]$ meets the boundary of W_k at a unique point u^k such that

$$\max\{h(u^k),\ -g(u^k),\ f(u^k) - \beta_k\} = 0 .$$

Clearly u^k lies outside W or on the boundary of W, so we can always draw a hyperplane through u^k, strictly separating z^k from W. Specifically let

$$l_k(x) = \langle p^k, x - u^k \rangle ,$$

where $p^k \in \partial h(u^k)$ if $h(u^k) = 0$, or $p^k \in \partial f(u^k)$ otherwise. Later we shall show (Lemma 1) that the function $e_k(x)$ constructed this way actually separates z^k strictly fom the set

$$W_0 = \{x \in C : f(x) \leqslant \beta\} \supset W , \tag{10.a}$$

where $\beta = \lim \beta_k$ (this limit exists because $\beta_k > \alpha$ and $\beta_{k+1} \leqslant \beta_k$); Furthermore, these functions and W_0 will satisfy all the requirements of Proposition 3.

We thus come to the following algorithm.

ALGORITHM 1. (Assuming int $D \neq \phi$)

Initialization. Select a point w satisfying (9). For each point z such that $g(z) < 0$ denote by $\pi(z)$ the point on the line segment $[w; z]$ such that $g(\pi(z))=0$. Select a feasible solution $\bar{x}^1$, and let $\beta_1 = f(\bar{x}^1)$. Select a polytope S_1 containing $\{x \in D : f(x) \leqslant \beta_1\}$, such that the vertex set V_1 of S_1 can readily be computed.

Iteration $k = 1, 2, \ldots$, This iteration is entered with a polytope S_k, knowledge of the vertex set V_k of S_k , a current best feasible point $\bar{x}^k$ and a current best value $\beta_k = f(\bar{x}^k)$

Solve the subproblem:

$$(Q_k) \quad \text{Lex-min } (g(x)-h^+(x) \,,\, f(x)) \,,\ \text{s. t. } x \in V_k \,,\ g(x) \leqslant 0 \qquad (11)$$

obtaining an optimal solution of it: z^k .

If $z^k \in C$ (i. e. $g(z^k) = h^+(z^k) = 0$) , stop .

Otherwise, compute the point u^k on the line segment $[w; z^k]$ that satisfies

$$\text{Max } \{h(u^k) \,,\, -g(u^k) \,,\, f(u^k) - \beta_k\} = 0 \ . \qquad (12)$$

Pick $p^k \in \partial\, h\,(u^k)$ if $h(u^k) = 0$, or $p^k \in \partial\, f\,(u^k)$ otherwise, and generate the new constraint

$$l_k(x) = \langle p^k , x - u^k \rangle \leqslant 0. \qquad (13)$$

Form a new polytope S_{k+1} by adding this new constraint to S_k . Compute the vertex set V_{k+1} of S_{k+1}. Update the current best feasible solution and the current best value by setting $\bar{x}^{k+1}$ equal to the best among $\bar{x}^k$, u^k (if u^k is feasible) and all points of form $\pi(z)$, $z \in V_{k+1} \setminus V_k$, $g(z) \leqslant 0$, that are feasible. Go to iteration k+1.

Remark 1. A point w satisfying 1) and 2) can be obtained by solving the convex program

$$\text{Minimize } f(x), \text{ s. t. } x \in D.$$

If there is an optimal solution w to this program such that $g(w) \leqslant 0$ then w also solves problem (P). Otherwise we shall find an optimal solution w such that $g(w) > 0$ and then w satisfies 1) and 2). Assuming int $D \neq \phi$, we can then slightly move w to satisfy also (9).

Remark 2. Since the polytope S_{k+1} differs from S_k by just one additional constraint, V_{k+1} can be computed from V_k using e.g. the procedure elaborated by Thieu-Tam-Ban in [6] (see also [9]).

Remark 3. A common drawback of outer approximation methods like Algorithm 1 is that the number of constraints $l_k(.) \leqslant 0$ increases as the algorithm proceeds. This drawback can however be partially overcome by using constraint dropping techniques of the type developed in [9] and [10].

4. CONVERGENCE PROPERTIES

This section is devoted to the proof of the convergence of the above Algorithm.

First we observe the following.

LEMMA 1. *The affine functions $l_k(x)$, given by (13) satisfy all the conditions of Proposition 3 for $W_0 = \{x \in C: f(x) \leqslant \beta\}$, where $\beta = \lim \beta_k$.*

Proof. If $p^k \in \partial h(u^k)$ then from the definition of the subdifferential

$$l_k(x) = \langle p^k, x - u^k \rangle \leqslant h(x) - h(u^k) = h(x)$$

hence $l_k(x) \leq 0 \ \forall x \in W_o$. Since $l_k(u^k) = 0$, $l_k(w) \leqslant h(w) < 0$, and since $u^k = t_k w + (1-t_k) z^k$ with $0 < t_k < 1$, it follows from the linearity of $l_k(.)$ that $l_k(z^k) > 0$. If $p^k \in \partial f(u^k)$, then either $f(u^k) - \beta_k = 0$, or $g(u^k) = 0$ and in the latter case $f(u^k) \geqslant \beta_{k+1} = \beta$ because u^k is feasible. Therefore, $l_k(x) \leqslant \leq f(x) - f(u^k) \leq f(x) - \beta \leq 0 \ \forall x \in W_o$, and since $l_k(u^k) = 0$, $l_k(w) \leqslant \leq f(w) - \beta < 0$, it follows that $l_k(z^k) > 0$. Thus the function $l_k(.)$ strictly separates z^k from W_o.

Now, since $l_k(x) = \langle p^k, x\rangle + \gamma_k$ with $\gamma_k = \langle p^k, u^k\rangle$, we have $|\gamma_k| \leq |p^k| \cdot |u^k|$, and the set C being assumed bounded (assumption (ii)), it follows that $|u^k| \leq r \ \forall k$, for some $r > 0$. That is, condition a) in Proposition 3 is fulfilled. It remains to check condition b).

Let $z^{k_\nu} \to z \notin W_o$. Then, noting that the sequence $\underset{k}{\cup}\partial h(u^k)$, $\underset{k}{\cup}\partial f(u^k)$ are bounded (in view of the boundedness of the sequence $\{u^k\}$; see e.g. [4] we can assume, by taking subsequences if necessary, that either of the following cases occurs

1) $p^{k_\nu} \in \partial h(u^{k_\nu})$, $u^{k_\nu} \to u$, (so $h(u^{k_\nu}) \to h(u) = 0$), $p^{k_\nu} \to p \in \partial h(u)$;

2) $p^{k_\nu} \in \partial f(u^{k})$, $u^{k_\nu} \to u$, $p^{k_\nu} \to p \in \partial f(u)$.

Then $l_{k_\nu}(x) \to l(x) = \langle p, x-u\rangle$ with $p \in \partial h(u)$ or $p \in \partial f(u)$, according to the case. In the first alternative $l(w) \leq h(w) - h(u) = h(w) < 0$; In the second alternative, $l(w) \leq f(w) - f(u) \leq f(w) - \beta < 0$. Since u lies in the line segment $[w, z]$, since $l(w) < 0$, $l(u) = 0$, while $z \neq u$ (for $z \notin W$, $u \in W$), it follows that $l(z) > 0$. Thus, condition b) is satisfied as well. □.

CONSEQUENCE. *We have $W_o \subset S_k$ for every k, and any cluster point $\bar{z}$ of the sequence $\{z^k\}$ satisfies*

$$\bar{z} \in D, \quad g(\bar{z}) = 0, \quad f(\bar{z}) \leq \beta. \tag{14}$$

Proof. The inclusion $W_o \subset S_k$ follows from the fact that $W_o \subset \{x \in D : f(x) \leq \beta_1\} \subset S_1$ and $l_j(x) \leq 0$ for all $x \in W_o$, all $j = 1, \ldots, k-1$. Further, from Proposition 3 and the above lemma it follows that $\bar{z} \in W_o$, i.e. $\bar{z} \in D$, $g(\bar{z}) \geqslant 0$, $f(\bar{z}) \leq \beta$. But, since $g(z^k) \leq 0$ for all k, we also have $g(\bar{z}) \leq 0$. Hence $g(\bar{z}) = 0$. □

At this point, it is worthwhile pointing out a peculiar feature of the lexicographic minimization problem (Q). While in the outer approximation scheme for quasiconcave minimization any cluster point $\bar{z}$ of (z^k) necessarily solves the original problem (provided, of course, the objective function is continuous), this may not be true for the lexicographic minimization problem (Q). More precisely, from the fact that z^k solves the relaxed problem (Q_k) (z^k achieves the lexicographic minimum over $S_k \supset C$) it does not necessarily follow that $\bar{z}$ solves (Q) ($\bar{z}$ achieves the lexicographic minimum over C). In other words, we may have $\beta > \alpha$. This umpleasant feature is due to the possible noncontinuity of the lexicographic ordering with respect to the usual topology of R^n.

Fortunately, we can prove the following theorem. Let

$$v^k = \arg\min \{f(x) : x \in V_k, \ g(x) \leqslant 0\}.$$

Clearly v^k exists, for otherwise $V_k \subset \{x : g(x) > 0\}$, hence $S_k \subset \{x : g(x) > 0\}$, contrary to the inclusion $W \subset S_k$, and the fact that W contains at least one optimal solution to (P), i.e. at least one point in the region $g(x) \leqslant 0$.

THEOREM 2. *Assume (i) through (iv) and, moreover, that the function $g(x)$ is strictly concave. If Algorithm 1 terminates at some iteration k, then $z^k = v^k$, and this yields an optimal solution to problem (P). Otherwise, every cluster point $\bar{v}$ of the sequence $\{v^k\}$ yields an optimal solution to problem (P).*

The proof of this fundamental result will follow from several lemmas.

LEMMA 2. *If an optimal solution z^k to (Q_k) satisfies $g(z^k) = h^+(z^k) = 0$, then z^k solves problem (Q) and $z^k = v^k$.*

Proof. We first show that z^k solves the problem

$$\text{Lex-min } (g(z), f(z)), \text{ s.t. } z \in V_k . \tag{15}$$

For any $z \in V_k$ if $g(z) < g(z^k) = 0$, then

$$g(z) - h^+(z) \leqslant g(z) < g(z^k) = g(z^k) - h^+(z^k),$$

contradicting the assumption that z^k solves (Q_k). Furthermore, if $g(z) = g(z^k) = 0$, then $g(z) - h^+(z) \leqslant g(z) = g(z^k) = g(z^k) - h^+(z^k)$. Since z^k solves (Q_k), this implies $f(z) \geqslant f(z^k)$. Therefore, z^k solves (15) and hence, by Proposition 2, z^k solves the problem

$$\text{Lex-min } (g(z), f(z)), \text{ s.t. } z \in S_k.$$

From the inclusion $W \subset S_k$ and the fact that $g(\bar{x}) = g(z^k) = 0$ for every optimal solution $\bar{x}$ of (P) (Proposition 1) it then follows that $f(z^k) \leqslant \alpha$, and hence that z^k is an optimal solution to (P), because z^k is feasible. Further, since $g(z^k) = 0 = \min\{g(z) : z \in V_k, g(z) \leqslant 0\}$, we have $g(z) = 0$ for all $z \in V_k$ satisfying $g(z) \leqslant 0$. This implies $z^k = \arg\min\{f(z) : z \in V_k, g(z) \leqslant 0\} = v^k$, completing the proof. □

LEMMA 3. *Any cluster point $\bar{v}$ of $\{v^k\}$ satisfies*

$$\bar{v} \in D, \; g(\bar{v}) = 0, \; f(\bar{v}) \leqslant \beta \tag{16}$$

Proof. From the definition of v^k it follows that $f(v^k) \leq f(z^k)$, hence $f(\bar{v}) \leqslant f(\bar{z}) \leqslant \beta$ (where $\bar{z}$ is any cluster point of the sequence $\{z^k\}$). Further, since z^k solves (Q_k) we have

$$g(v^k) - h^+(v^k) \geqslant g(z^k) - h^+(z^k),$$

hence, in view of (14):

$$g(\bar{v}) - h^+(\bar{v}) \geqslant g(\bar{z}) - h^+(\bar{z}) = 0.$$

But from $g(v^k) \leqslant 0$, $h^+(v^k) \geqslant 0$ we get $g(\bar{v}) \leqslant 0$, $h^+(\bar{v}) \geqslant 0$. Therefore, $g(\bar{v}) = 0$, $h^+(\bar{v}) = 0$, i. e. $\bar{v} \in D$. □

LEMMA 4. *For every $u \in W$ such that $g(u) = 0$ there exists a sequence $\{u^k\}$ such that $u^k \in V_k$, $g(u^k) \leqslant 0$, $u^k \to u$ $(k \to \infty)$.*

Proof. Consider any hyperplane $H = \{x : \langle p, x - u \rangle = 0\}$, with $p \in \partial(-g(u))$. Let $G = \{x : g(x) \geqslant 0\}$. For any $x \in H \cap G$ we have $g(x) \geq 0$ and $0 = \langle p, x - u \rangle = -g(x) + g(u) = -g(x)$, hence $g(x) = 0$. If $x \neq u$ then obviously $x' = 1/2\,(x + u) \in H \cap G$, and therefore $g(x') = 0$, but because of the strict concavity of g, $g(x') > (g(x) + g(u))/2 = 0$. This contradiction shows that $H \cap G = \{u\}$. For any $x \in G$, we have $\langle p, x - u \rangle \leqslant -g(x) \leqslant 0$. Since $u \in W \subset S_k$ the linear function $\langle p, x - u \rangle$ attains its maximum over S_k at some vertex $u^k \in V_k$ such that $\langle p, u^k - u \rangle \geq 0$, hence $g(u^k) \leqslant 0$. Let $\overline{u}$ be a cluster point of the sequence $\{u^k\}$. Then

$$\langle p, \overline{u} - u \rangle \geqslant 0, \ g(\overline{u}) \leqslant 0. \tag{18}$$

On the other hand, from the definition of z^k we have

$$g(u^k) - h^+(u^k) \geq g(z^k) - h^+(z^k),$$

hence $g(\overline{u}) - h^+(\overline{u}) \geq g(\overline{z}) - h^+(\overline{z}) = 0$ with $\overline{z}$ being some cluster point of $\{z_k\}$ (see (14)). Therefore $g(\overline{u}) \geq h^+(\overline{u}) \geqslant 0$ and hence, $\overline{u} \in G$, $\langle p, \overline{u} - u \rangle \leq 0$. The latter inequality implies, in view of (18), $\langle p, \overline{u} - u \rangle = 0$. Thus $\overline{u} \in H \cap G$ and consequently $\overline{u} = u$. This shows that $u^k \to u$. □

LEMMA 5. *Every cluster point $\overline{v}$ of the sequence $\{v^k\}$ solves problem (P).*

Proof. By Lemma 3, $\overline{v} \in D$, $g(\overline{v}) = 0$. By Lemma 4, for any $u \in W$ such that $g(u) = 0$ there is a sequence u^k satisfying the conclusion of Lemma 4. But from the definition of v^k, $f(v^k) \leqslant f(u^k)$, hence $f(\overline{v}) \leqslant f(u)$. Therefore, by Proposition 1, $\overline{v}$ is an optimal solution to (P). □

Theorem 2 now immediately follows from Lemmas 2 and 5.

In actual practice, we must, of course, stop the solution procedure at some iteration k. Theorem 2 ensures that for k sufficiently large v^k will be sufficiently near to an optimal solution. However, it should be borne in mind that v^k may not be feasible, even though it is nearly feasible as the following lemma shows

LEMMA 6. *If*

$$g(z^k) - h^+(z^k) > -\varepsilon \tag{19}$$

then $-\varepsilon < g(v^k) \leqslant 0$, $h(v^k) < \varepsilon$ (in fact these inequalities hold for every $v \in V_k$ such that $g(v) \leqslant 0$).

Proof. If $v \in V_k$ is such that $g(v) \leqslant -\varepsilon$ then $g(v) - h^+(v) \leqslant -\varepsilon - h^+(v) \leqslant -\varepsilon$, contradicting (19), and the definition of z^k. Therefore.

$$g(v) > -\varepsilon \ \forall\ v \in V_k \tag{20}$$

Since $v \in V_k$, and $g(v) \leqslant 0$ imply $g(v) - h^+(v) \geqslant g(z^k) - h^+(z^k) > -\varepsilon$ it follows that $h^+(v) < g(v) + \varepsilon < \varepsilon$. □

Thus, for k large enough, v^k is nearly feasible and approximately optimal. However sometimes we may want to have an approximate optimal solution which is feasible. This motivates considering in each iteration the current best feasible solution $\overline{x}^k$. We can then prove the following proposition.

THEOREM 3. *Under the same conditions as in Theorem 2, for any given* $\varepsilon > 0$ *the situation* (19) *must occur after finitely many iterations. If* (19) *holds then the current best solution* $\overline{x}^k$ *is approximately optimal in the sense that.*

$$f(\overline{x}^k) < \min \{f(x) : x \in D,\ g(x) + \varepsilon \leqslant 0\}. \tag{21}$$

Proof. That (19) necessarily occurs after finitely many iterations follows from the fact (14) (consequence of Lemma 1). Suppose now that (19) holds. First from (20) and the concavity of g it follows that.

$$g(x) > -\varepsilon \quad \text{for all } x \in S_k. \tag{22}$$

On the other hand it can easily be seen that

$$D_k = \{x \in D : f(x) \leqslant \beta_k\} \subset S_k. \tag{23}$$

Indeed $D_1 \subset S_1$ and for any $j < k$, either $h(u^j) = 0$, and then $p^j \in \partial h(u^j)$, $l_j(x) = \langle p^j, x - u^j\rangle \leqslant h(x) - h(u^j) \leqslant h(x) \leqslant 0$, for all $x \in D$; Or $h(u^j) < 0$ and then $p^j \in \partial f(u^j)$,

$$l_j(x) = \langle p^j. x - u^j\rangle \leqslant f(x) - f(u^j) \leqslant f(x) - \beta_k \leqslant 0$$

for all x such that $f(x) \leqslant \beta_k$. Here we used the fact (established in the proof of Lemma 1) that either $f(u^j) = \beta_j \geqslant \beta_k$ or $f(u^j) \geqslant \beta_{j+1} \geqslant \beta_k$.

From (22) and (23) it is easily seen that for any $x \in D$ such that $g(x) \leqslant -\varepsilon$ we must have $f(x) > \beta_k$. This proves (21). □

As a consequence of the above Theorem we obtain for any $\varepsilon > 0$:

$$\beta < \min \{f(x) : x \in D,\ g(x) + \varepsilon \leqslant 0\},$$

(recall that $\beta = \lim f(\overline{x}^k)$).

The question arises as to under which conditions we have

$$\min \{f(x) : x \in D,\ g(x) + \varepsilon \leqslant 0\} \xrightarrow[\varepsilon \to 0+]{} \alpha$$

or equivalently, $\beta = \alpha$? If this holds, then following [11] we say that the problem (P) is *stable*. It has been shown in [11] that stability is ensured, provided the following condition is satisfied:

(v) No point $x \in D$ with $g(x) = 0$ is a local minimum of g over D. Thus we can state:

THEOREM 4. *Assume* (i) *through* (v). *Then any cluster point* $\bar{x}$ *of the sequence* $\bar{x}^k$ *generated by Algorithm 1 is an optimal solution to problem* (P).

5. CASE WHERE THE OBJECTIVE FUNCTION IS LINEAR

If the function $g(x)$ is concave but not strictly concave, while $f(x)$ is linear, then to guarantee convergence, the algorithm should be modified as follows.

Consider at iteration k the set

$$V_k^* = V_k \cup U_k$$

where V_k denotes, as previously, the vertex set of S_k, and U_k is the set of all points x such that x is the intersection of the surface $g(x) = 0$ with an edge of S_k joining two vertices v, v' satisfying $g(v) < 0$, $g(v') > 0$.

Let us call Algorithm 2 the algorithm which proceeds exactly as Algorithm 1, except that we use V_k^* instead of V_k in the definition of the subproblem (Q_k). In other words, the subproblem to be solved in iteration k is

$$(Q_k)\ \text{Lex-min}\ (g(x) - h^+(x),\ f(x)),\ \text{s.t}\ x \in V_k^*,\ g(x) \leqslant 0. \tag{24}$$

THEOREM 5. *Assume in addition to* (i) *through* (iv) *that the function* $f(x)$ *is linear. Then the conclusions of Theorem 1 hold for Algorithm 2, with*

$$v^k = \arg\min \{ f(x) : x \in V_k^*,\ g(x) \leqslant 0 \}. \tag{25}$$

Proof. The first part of the Theorem (asserting that when the Algorithm stops at iteration k, $z^k = v^k$ is an optimal solution) is proved exactly as before, replacing everywhere V_k by V_k^*. To establish the second part of the Theorem, we proved the following property as a substitute to Lemma 4:

For any $u \in W$ such that $g(u) = 0$, there exists a sequence $\{\bar{u}^k\}$ such that

$$\bar{u}^k \in V_k^*,\ \ g(\bar{u}^k) \leqslant 0,\ \ f(\bar{u}^k) \leqslant f(u). \tag{26}$$

Indeed, consider any $p \in \partial(-g(u))$, so that $g(x) \leqslant 0$ for all x of the halfspace $\langle p, x-u\rangle \geqslant 0$. Then the linear function f achieves its minimum over the convex polytope

$$T_k = \{x \in S_k : \langle p, x-u\rangle \geqslant 0\}.$$

at some vertex u^k of T_k, which is either an element of V_k, or the intersection of the hyperplane $\langle p, x-u\rangle = 0$ with two vertices v, v' of S_k. In the former case we let $\bar{u}^k = u^k$. In the latter case, if $g(v) \leqslant 0$, $g(v') \leqslant 0$, we take as $\bar{u}^k$ the vertex v or v' that corresponds to the smallest value of $f(x)$ over $[v, v']$; if $g(v) > 0$ then necessarily $g(v') \leqslant 0$ (otherwise $[v, v']$ would lie outside T_k), so that $u^k \in V_k^*$ and we take $\bar{u}^k = u^k$. In any case we thus have for each k, a point $\bar{u}^k$ satisfying (26).

Now suppose that the algorithm is infinite and let $\bar{v}$ be any cluster point of v^k. Then, by the same argument as that used for proving Lemma 3, $\bar{v} \in D$, $g(\bar{v}) = 0$ and $f(\bar{v}) \leqslant \beta$. For any $u \in W$ such that $g(u) = 0$, we have, as shown just above, a sequence $\bar{u}^k$ satisfying (26). Since

$$f(v^k) \leqslant f(\bar{u})^k \leqslant f(u),$$

by passing to the limit we get $f(\bar{v}) \leqslant f(u)$. Therefore, by Proposition 1 $\bar{v}$ is an optimal solution of (P). □

6. DISCUSSION

1) In contrast to the method of [11] the method presented above does not require stability condition (v). For the convergence of Algorithm 1, we only need, aside from conditions (i) through (iv), that the function $g(x)$ be strictly concave (Theorem 2). Note that the latter condition does not imply stability, as can be shown by simple counter-examples.

For the convergence of Algorithm 2 we only need that the objective function $f(x)$ be linear (Theorem 4). Also note that at the expense of an additional variable, every problem (P) can be converted into one with a linear objective function, since it can always be written in the form.

Minimize t, s. t. $f(x) \leqslant t$, $x \in D$, $g(x) \leqslant 0$.

Therefore, Algorithm 2 can be used to solve the problem in the general case, even if the stability condition (v) is not satisfied.

We need condition (v) only when we want to produce a sequence of feasible solutions $\bar{x}^k$ converging to an optimal solution (Theorem 3).

2) The basic idea underlying the approach in [11] is the following. Using a general duality principle between objective and constraint, a global optimlity criterion is established which says that, in the presence of stability, a feasible point $\bar{x}$ is globally optimal if and only if

$$0 = \min \{g(x) : x \in D, f(x) \leqslant f(\bar{x})\}.$$

Now, giving a point $x^1 \in D$ such that $g(x^1) = 0$ we can check whether x^1 is optimal or not by solving the concave program

$$(Q(x^1)) \text{ Minimize } \{g(x), \text{ s. t. } x \in D, f(x) \leqslant f(x^1)\}$$

If the optimal value in this concave program is zero, x^1 is optimal to (P) (assuming the problem stable). Otherwise, a point $z^1 \in D$ is obtained with $g(z^1) < 0$. Then $x^2 = \pi(z^1)$ is a feasible point better than x^1, and the procedure can be repeated, starting from x^2 ...

Since each subproblem $(Q(x^k))$ is solved by the outer approximation method, i. e. by replacing its constraint set with a sequence of polytopes approximating it more and more closely from the outside, the procedure in this primary form would imply solving a double sequence of linearly constrained concave programs. Therefore, to make the procedure more implementable, the double sequence is replaced by a « diagonal » one. This leads to a procedure consisting in solving at iteration k a single concave program of the form

$$\min \{g(x) : x \in S_k\},$$

or, equivalently

$$\min \{g(x) : x \in V_k\}, \tag{27}$$

where V_k is the vertex set of S_k, and S_k is an outer approximation of the constraint set

$$D_k = \{x \in D : f(x) \leqslant \beta_k\} \tag{28}$$

(β_k being the best feasible value of f known up to iteration k).

Thus, as regards the formal structure, our Algorithm 1 shares many common features with the algorithm developed in [11], inasmuch as both proceed through a sequence of concave minimization subproblems over polytopes. The conceptual bases, however, are quite different: while each subproblem in the algorithm of [11] is conceived primarily as a test for global optimality, it is in our Algorithm 1 a relaxation of the equivalent lexicographic minimization problem (Q). Since the test is sure only for stable problems, the algorithm in [11] does not work for unstable problems. On the other hand, the possible discontinuity of the lexicographic minimization in our algorithm is overcome by the devise of the v^k sequence.

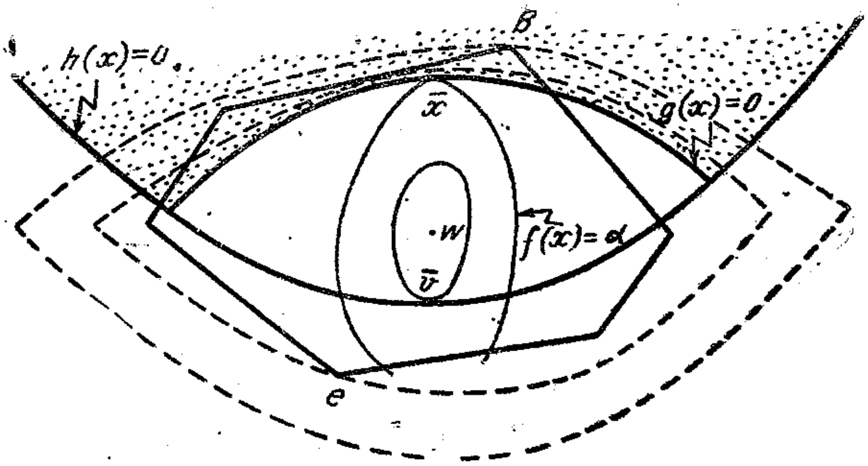

FIG. 1

3) From a computational point of view, the two algorithms (Algorithm 1 and the one in [11]) are not equivalent to each other. First, even if at some iteration k the polytope S_k happens to be the same in both methods, our subproblem (Q_k) (11) differs from (27) (the corresponding subproblem in [11]) by the objective function as well as by the constraints. For example, in the case of fig. 1 where a part of the set $g(x) = 0$ lies on the boundary of D and the broken lines represent the level sets of the function $g(x)$, the optimal solution of the subproblem (Q_k) in our algorithm is e, whereas it would be b in the algorithm of [11]. Under these conditions, the algorithm of [11] would generate a sequence tending to a point $\bar{x}$ satisfying the criterion

$$0 = \min \{ g(x) : x \in D,\ f(x) \leqslant f(\bar{x}) \},$$

without being a global optimum, whereas our Algorithm 1 will necesssarily produce a sequence converging to a global optimum $\bar{v}$ (Theorem 2). Furthermore if at some iteration k the optimal solution z^k to the subproblem at iteration k satisfies $z^k \in D, g(z^k) = 0$, then for the above Algorithm 1 we are sure that z^k is already a global optimum, but this may fail to be true for the algorithm of [11] unless the problem is stable.

The second difference lies in the construction of the cutting planes. In our method, the cutting plane at iteration k separates the set

$$D_k = \{x \in D: f(x) \leqslant \beta_k \}$$

from the portion of $[w, z^k]$ outside the set

$$W_k = \{x \in D : g(x) \geqslant 0,\ f(x) \leqslant \beta_k \}.$$

(see (23)), whereas in the method of [11] it separates D_k from the portion $[w, z^k]$ outside the set

$$F_k = \{x : g(x) \geqslant 0,\ f(x) \leqslant \beta_k\}.$$

Since $W_k \subset F_k$, our cuts are generally deeper, which may result in a more rapid convergence.

Thus, in a certain sense, the present method could be considered as an improved version of the method developed in [11].

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APPENDIX

ILLUSTRATIVE EXAMPLES

EXAMPLE 1.

$$\text{Minimize } f(x_1, x_2) = (x_1 - 3.68)^2 + (x_2 - 12)^2, \text{ s. t.} \tag{29}$$

$$h(x_1, x_2) = \max\{x_1 + x_2 - 30, (0.1x_1 - 3)^2 + (0.1x_2 - 2.5)^2 - 11.25, -x_1 + 18x_2^2/484 - 10, -x_1, -x_2\} \leqslant 0 \tag{30}$$

$$g(x_1, x_2) = (-x_1^2 - x_2^2 + 484)/10 \leqslant 0 \tag{31}$$

It is easily seen that $f(x_1, x_2)$ is the distance from (x_1, x_2) to $w = (3.68, 12)$ (Fig . 2), so w is an optimal solution of the convex program (29) — (30):

$$\text{Minimize } f(x_1, x_2), \text{ s. t. } h(x_1, x_2) \leqslant 0.$$

Now, we apply Algorithm 1 to solve the program (29) — (31).

Initialization

$w = (3.68, 12)$,

$S_1 = \{(x_1, x_2) : x_1 + x_2 \leqslant 30,\ x_1 \geqslant 0,\ x_2 \geqslant 0\}$

$V_1 = \{(0,0), (30,0), (0,30)\}$

$\overline{x}_1 = (21.6697, 3.79801)$

$\beta_1 = 390.905$, (Fig . 2)

Iteration 1.

$z^1 = (0,30)$ solves (Q_1).

$g(z^1) - h^+(z^1) = -65.0711$

$u^1 = (2.3969,\ 18.2576)$

$l_1(x_1, x_2) = -x_1 + 1.358x_2 - 22.4$

$S_2 = S_1 \cap \{(x_1, x_2) : l_1(x_1, x_2) \leqslant 0\}$

$V_2 = \{(0,0), (30,0), (7.78,22.22), (0,16.49)\}$

$v^2 = (7.78,22.22)$

$\bar{x}^2 = (7.2,20.79),\ \beta_2 = 89.63;$ (Fig. 3).

Iteration 2.

$z^2 = (30,0)$ solves (Q_2).

$g(z^2) - h^+(z^2) = -41.6$

$u^2 = (12.2943,\ 8.0725)$

$l_2(x_1, x_2) = 17.23x_1 - 7.855x_2 - 148.505$

$S_3 = S_2 \cap \{(x_1, x_2) : l_2(x_1, x_2) \leqslant 0\}$

$V_3 = \{(0,0), (15.31,14.69), (7.78,22.22),$
$(0,16.49), (8.61,0)\}$

$v^3 = (7.78,22.22)$

$\bar{x}^3 = (7.2,20.79),\ \beta_3 = 89.63$ (Fig. 4).

Continuing in this way we obtain the results, given in Tableau 1. ($\bar{x}^k$ denotes the current best feasible solution, β_k the current best value, v^k the current best approximate solution, z^k the solution of (Q_k), N_k the number of constraints of (Q_k) ; the stopping criterion is $g(z^k) - h^+(z^k) \geqslant -0.001$).

EXAMPLE 2.

$$\text{Minimize } f(x_1, x_2) = (x_1 - 3.68)^2 + (x_2 - 12)^2, \quad \text{s.t.} \tag{32}$$

$$h(x_1, x_2) = \max\{x_1 + x_2 - 30,\ (0.1x_1 - 3)^2 + (0.1x_2 - 2.5)^2 - 11.25,\ -x_1 + 18x_2^2/484 - 10,\ -x_1, -x_2\} \leqslant 0 \tag{33}$$

$$g(x_1, x_2) = \min\{(-x_1^2 - x_2^2 + 484)/10,\ -(.1x_1 - 6)^2 - (.1x_2 - 4)^2 + 45\} \leqslant 0. \tag{34}$$

This problem is not stable. Note that Program (32) — (34) differs from Program (29) — (31) only by the reverse convex constraint.
The computational results are shown in Tableau 2.

Tableau 1

k	$\lvert V_k \rvert$	N_k	$\bar{x}^k$	β_k	v^k	$f(v^k)$	$g(z^k) - h^+(z^k)$
1	4	4	(21.6697,3.7980)	390.90	(0,30)	337.54	− 65.07
2	5	5	(7.2044,20.7870)	89.632	(7.7794,22,2206)	121.27	− 41.6
3	6	6	»	»	»	»	− 7.6108
4	7	7	»	»	(6.3161,21.1432)	90.548	− .6021
5	7	7	(6.4907,21.0207)	89.273	(6.5052,21.0674)	90.199	− .5458
6	7	7	»	»	»	»	− .2162
7	8	8	»	»	(6.4516,21.0329)	89.275	− .0308
8	9	9	»	»	»	»	− .0059
9	10	10	»	»	»	»	− .0013
10	11	11	(6.4520,21.0326)	89.272	(6.4520,21.0328)	89.275	− .0006

Tableau 2.

k	$\lvert V_k \rvert$	N^k	$\bar{x}^k$	β_k	v^k	$f(v^k)$	$g(z^k) - h^+(z^k)$
1	4	4	(21.6697,3.7980)	390.90	(0,0)	157.54	− 65.07
2	5	5	(7.2044,20.7870)	89.632	(7.7794,22.2206)	121.27	− 41.6
3	6	6	»	»	»	»	− 11
4	6	6	»	»	(0,9.5655)	19.469	− 7.6108
5	7	7	»	»	»	»	− .6021
6	8	8	(6.4 07,21.0207)	89.273	»	»	− .5458
7	7	7	»	»	»	»	− .3949
8	8	8	»	»	(0,9.99566)	17.559	− .2162
9	9	9	»	»	»	»	− .0308
10	10	10	»	»	»	»	− .0059
11	11	11	»	»	»	»	− .0039
12	11	11	»	»	(0,9.99999)	17.542	− .0013
13	12	12	(6.4520,21.0207)	89.272	»	»	− .0006

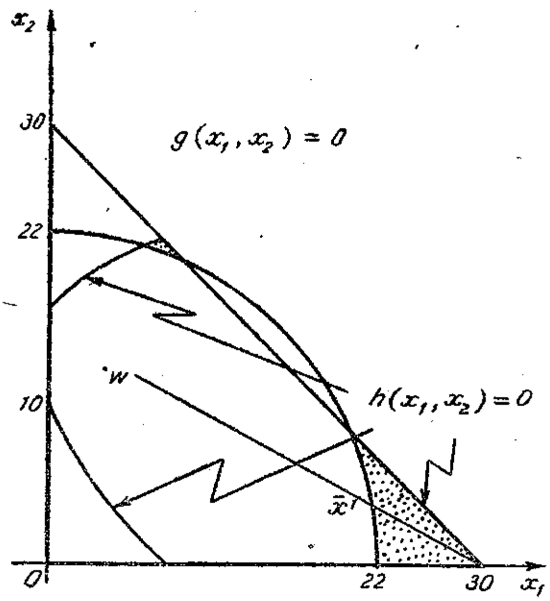

FIG. 2

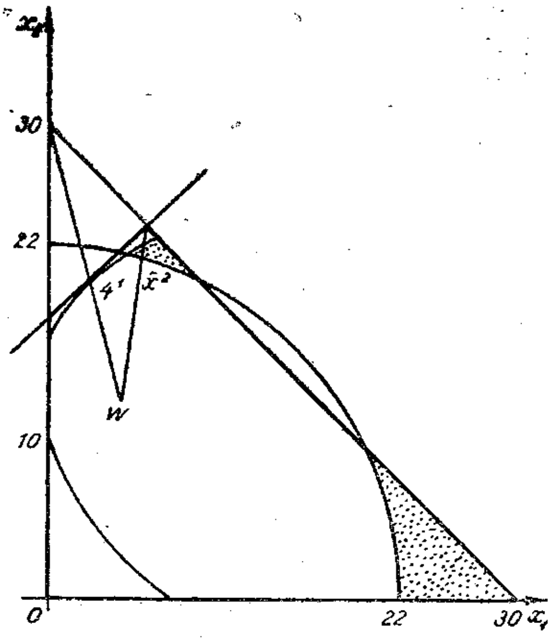

FIG. 3

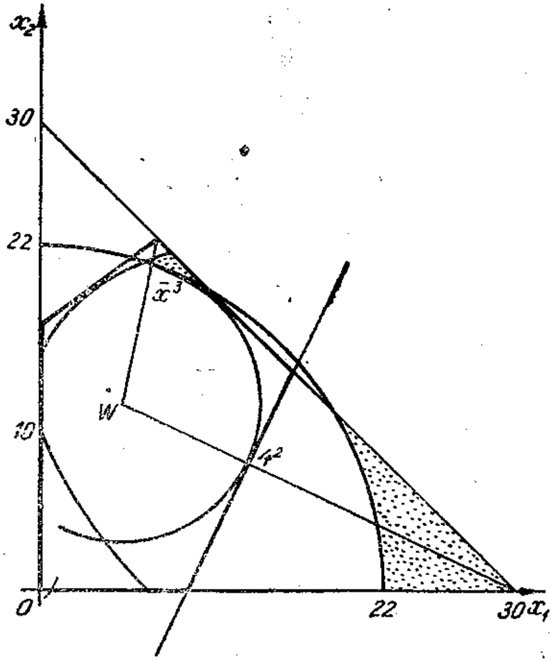

FIG. 4

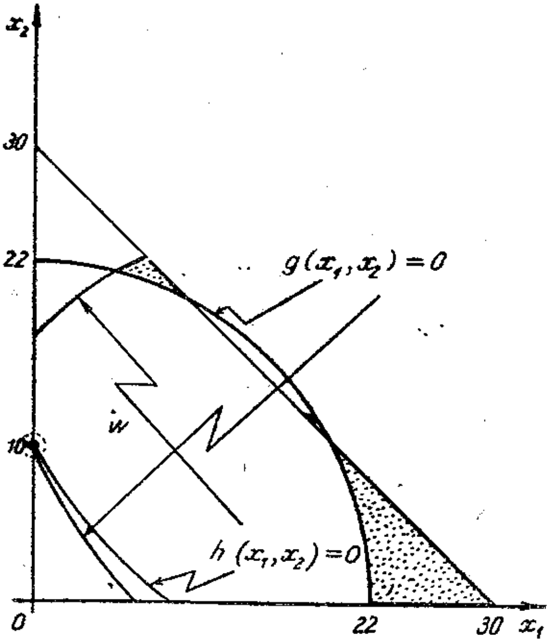

FIG. 5

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