ACTA MATHEMATICA VIETNAMICA
Vol. 10 N$_0$1 (1985)

ON THE NULL CONTROLLABILITY OF INFINITE DIMENSIONAL DISCRETE TIME SYSTEMS

NGUYEN KHOA SON* and LE THANH**

I. INTRODUCTION

We shall be concerned with the control system (A, B, Ω) of the form:

$$x_{k+1} = Ax_k + Bu_k, \ k = 1,2,3,.. \tag{1}$$

where the state vector $x_k \in X$, the control vector $u_k \in \Omega \subset U$, X, U being real Banach spaces, and $A : X \to X$, $B : U \to X$ are bounded linear operators.

Throughout this note we assume that Ω is convex and satisfies the condition:

$$\exists u_0 \in \Omega \quad : \quad Bu_0 = 0 \tag{2}$$

For the motivation of qualitative investigations of infinite — dimensional discrete — time systems of type (1) the reader is refered to [1]—[3]. We recall only that (with the aid of a natural method of steps) it is always possible to represent control systems described by equations of retarded type by discrete— time model (1) where the corresponding operators A, B have, in addition, a nice analytic form. This gives, in particular, a good possibility for studying the controllability of hereditary differential systems with restrained control – one problem still rarely considered up to the present in the literature.

In [4] we have studied the local reachability of system (1). In [5] some criteria for global reachalibity and global controlability of the system with bounded Ω have been proved under the assumption that this system is locally reachable or, respectively, locally controllable. It is the main purpose of this paper to obtain the necessary and sufficient conditions for local controllability of the system (A,B,Ω). Also we shall present some criteria for global controleability for the case where Ω is not necessarily bounded.

The results of this note can be considered as an extension of the corresponding controllability properties of (A, B, Ω) investigated in [6] for the finite-dimensional case.

II. LOCAL NULL—CONTROLLABILITY

We recall that the point $x \in X$ is said to be controllable (respectively, reachable) by system (1) if there exists a finite sequence of controls $u_i \in \Omega$, $i = \overline{1, k}$, such that:

$$- A^k x = \sum_{i=1}^{k} A^{k-i} B u_i,$$

(respectively, $x = \sum_{i=1}^{k} A^{k-i} B u_i$).

We shall say that the system is locally (respectively, globaly) controllable if the set S of all controllable points contains some neighbourhood of the origin i.e $O \in \text{int } S$, (respectively, if S coincides with the whole state space, i.e $S = X$). Similarly, the notions of local and global reachability are defined, respectively by the condtions $O \in \text{int } R$ and $R = X$ with R denoting the set of all reachable points.

In [6] the following criterion of local controllability has been proved.

THEOREM 2.1. *For the system (A, B, Ω) with $X = \mathbf{R}^n$, $U = \mathbf{R}^m$ to be locally controllable it is necessary and sufficient that the transpose matrix A^* has neither real eigenvectors with positive eigenvalues supporting to $B\Omega$ nor complex eigenvectors with nonzero complex eigenvalues orthogonal to $B\Omega$ at 0.*

The result just mentioned can be reformulated in the following equivalent form which seems to be more convenient for studying the case of infinite dimension.

THEOREM 2.2. *The system (A, B, Ω) with $X = \mathbf{R}^n$, $U = \mathbf{R}^m$, is locally controllable if and only if:*

(i) *the system*

$$x_{k+1} = Ax_k + Bv_k, \quad v_k \in U = \overline{\text{span}} \, (\Omega - u_0), \tag{3}$$

is globally controllable.

(ii) *There is no real eigenvectors of A^* corresponding to positive eigenvalues and supporting to $B\,\Omega$ at 0.*

Proof. Let $0 \in \text{int } S$. The necessity of (i) being obvious, for the necessity of (ii) we assume the contrary: there exist a nonzero vector $f \in \mathbf{R}^n$ and $\lambda > 0$

such that $A^* f = \lambda f$ and $< f, Bu > \leqslant 0$ ($\forall\, u \in \Omega$). Then, for any $x_0 \in S$ we have $-\ A^k x_0 = \sum_{i=1}^{k} A^{k-i}\ Bu_i$ with $u_i \in \Omega$ ($i = \overline{1, k}$) and hence $< f, A^k x_0 > =$

$= \lambda^k < f, x_0 > = - \sum_{i=1}^{k} \lambda^{k-i} < f, Bu_i > \geqslant 0$ contradicting the assumption that $0 \in \text{int}\, S$.

To show the sufficiency, suppose that (i), (ii) hold. If there exist a nonzero complex vector $f \in \mathbf{C}^n$ and $\lambda \neq 0$ such that $A^*f = \lambda f$ and $< f, Bu > = 0$ ($\forall\, u \in \Omega$) then, since (3) is globally controllable, for any $x_0 \in \mathbf{R}^n$ we derive easily, as above, that $\lambda^k < f, x_0 > = 0$ which is, clearly, absurd. This shows, in combining with (ii) that the sufficient conditions of Theorem 2.1 are satisfied and, consequently, the system is locally controllable.

It is well known that the system

$$x_{k+1} = A\, x_k + Bu_k\ ,\ x_k \in \mathbf{R}^n\ ,\ u_k \in \mathbf{R}^m$$

is globally controllable iff:

$$\text{rank}\,[A^{n-1}B,\ A^{n-2}B, \ldots AB, B] = \text{rank}\ [A^n, A^{n-1}B, \ldots AB, B].$$

with this remark we can state some direct consequences of Theorem 2.2.

COROLLARY 2.1. *Suppose that Ω has non-empty interior in $\mathbf{R}^n$, i . e int $\Omega \neq \phi$. Then the system (A, B, Ω) is locally controllable iff :* (i) *rank* $[B, AB, \ldots, A^{n-1}\ B]$ *rank* $[B, AB, \ldots A^{n-1}\ B, A^n]$, (ii) *There is no real eigenvectors of A^* corresponding to positive eigenvalues and supporting to $B\Omega$.*

COROLLARY 2.2. *The system* in $\mathbf{R}^n$ *with single non-negative input*

$$x_{k+1} = Ax_k + bu_k\ ,\ u_k \geqslant 0,$$

is globally controllable iff :

(i) Every columns of matrix A^n can be written as a linear combination of $b, Ab, \ldots A^{n-1}\, b$.

(ii) The matrix A has no real positive eigenvalues.

THEOREM 2.2, however, does not hold, in general, for infinite — dimensional systems, as shown by the following.

EXAMPLE 2.1. Consider the system (A, B, Ω) in $X = l_2$, with $\Omega = \{u \in X : \| u - u_0 \| \leqslant 1,\ u_0 = (1, 0, 0, \ldots)\}$ and with $A = B$, where A is defined as:

$$A(\zeta_1, \zeta_2 \ldots) = (0, \frac{1}{2}\xi_1, \frac{1}{3}\zeta_2 \ldots\ldots).$$

It can be verified that the operator A is compact, $\ker A = \{0\}$ and $\sigma(A) = \{0\}$. Thus, A has no nonzero eigenvalues and, thereby, conditions (i), (ii) of Theorem 2.2 are fulfilled. However, this system is not locally controllable. To see this we note first that

$$AX \cap \Omega = \{0\} \tag{4}$$

Indeed, if $x = (0, \zeta_1, \zeta_2, ...) \in AX \cap \Omega$, then $\| x - u_0 \| = (1 + \sum_{i=1}^{\infty} |\xi_i|^2)^{1/2} \leqslant 1$

which implies $x = 0$. Now for any $x \in S$ we can write $0 = A^k x + A^k u_1 + ... + Au_k = A(A^{k-1} x + A^{k-1} u_1 + ... + u_k)$, where $u_i \in \Omega \ (i = \overline{1, k})$. Since Ker $A = \{0\}$ we obtain $u_k \in AX \cap \Omega$ which by (4) follows that $u_k = 0$. Repeating this process we find in the end that $x = - u_1$ and hence $S \subset - \Omega$. Actually, we have $S = - \Omega$ and therefore $0 \notin \text{int } S$. It is, therefore of interest to know under that additional conditions on the system data (A, B, Ω) the above criterion of controllability can be extended to the case of infinite dimension.

To this end, we first establish some auxiliary facts. Let denote by Ω^k the set of all vectors $u^k = (u_1, u_2, ..., u_k) \in U^k$ such that $u_i \in \Omega$, $i = \overline{1, k}$ and let consider the operator $F_k : U^k \to X$, defined as $F_k(u^k) = \sum_{i=1}^{k} A^{k-i} Bu_i$. Denote by S_k the controllable set of (A, B, Ω) in k steps, i. e.

$$S_k = \{x \in X : - A^k x \in F_k(\Omega^k)\}$$

Obviously, $S = \bigcup_{k=1}^{\infty} S_k$ and $R = \bigcup_{k=1}^{\infty} F_k(\Omega^k)$.

LEMMA 2.1. *Suppose int $\Omega \neq \phi$ and A is an isomorphism of X onto X. Then the system (A, B, Ω) is locally controllable if and only if it is locally reachable.*

Proof. By definition, $- A^k S_k \subset F_k(\Omega^k)$ and hence $S_k = - A^{-k} F_k(\Omega^k)$. If $0 \in \text{int } R$ then by Theorem 1 of [4], $0 \in \text{int } F_k(\Omega^k)$ for some $k \geqslant 1$. It follows that $0 \in \text{int } S_k$ and therefore $0 \in \text{int } S$. Conversely, suppoting $0 \in \text{int } S$, it is readily verified that the system (A, B, Ω) is globally controllable or, equivalently, by Theorem of Furhman [7], $A^k(X) \subset F_k(U^k)$ for some $k \geqslant 1$. Since A is onto, it follows that $X = F_k(U^k)$ and hence, F_k is onto. By the open mapping Theorem, int $F_k(\Omega^k) \neq \phi$ which shows again int $S_k \neq \phi$. Applying Lemma 1 of [4] we have that there exists m such that $0 \in \text{int } S_m$ which implies readily $0 \in \text{int } F_m(\Omega^m)$.

Now let M be a closed A-invariant subspace of X. We denote by $\widehat{X}$ the factor space X/M equipped with the usual factor norm and by P the canonical projection $P: X \to \widehat{X}$. We define the factor system $(\widehat{A}, \widehat{B}, \Omega)$ of (A, B, Ω) with respect to M as follows

$$\widehat{x}_{k+1} = \widehat{A}\,\widehat{x}_k + \widehat{B}u_k,\ \widehat{x}_k \in \widehat{X},\ u_k \in \Omega \subset U$$

where $\widehat{x} \triangleq Px$, $\widehat{A}\,\widehat{x} \triangleq P(Ax)$, $\widehat{B}u \triangleq P(Bu)$.

LEMMA 2. 2. *Let M be a closed A-invariant subspace contained in the controllability set S of (A, B, Ω). Then system (A, B, Ω) is locally controllable iff the factor system $(\widehat{A}, \widehat{B}, \Omega)$ with respect to M is locally controllable.*

Proof The «is» part is immediate from the fact that P is an open operator and PS belongs to the controllable set $\widehat{S}$ of $(\widehat{A}, \widehat{B}, \Omega)$.

To prove the «only if» part, letting $0 \in \operatorname{int} S$. We take $\varepsilon > 0$ such that $\widehat{x} \in \widehat{S}$ whenever $\|\widehat{x}\| < \varepsilon$, and we set $\varepsilon_1 = \varepsilon / \|P\|$. Then for any $x \in X$ with $\|x\| < \varepsilon_1$ we have $Px \in \widehat{S}$ and hence,

$\widehat{A}^k Px + \widehat{A}^{k-1}\widehat{B}u_1 + \ldots + \widehat{B}u_k = 0$ for some k and some $u_i \in \Omega$, $i = \overline{1, k}$. This implies $A^k x + A^{k-1}Bu_1 + \ldots + Bu_k \in M \subset S$ and, therefore, $x \in S$, completing the proof.

We are now in a position to prove the following assertion which generalizes Theorem 2.2 to infinite dimensional systems.

THEOREM 2.3. *Suppose $\operatorname{int}\Omega \neq \phi$ and there exists $n > 0$ such that*

$$\operatorname{Ker} A^n = \operatorname{Ker} A^{n+1} \text{ and } A^n X = A^{n+1} X \tag{5}$$

Then for the system (A, B, Ω) to be locally controllable it is necessary and sufficient that

(i) $A^m(X) \subset F_m(U^m)$ for some $m > 0$ and

(ii) There is no eigenvector of A^* corresponding to positive eigenvalues and supporting to $B\Omega$.

Proof. The necessity of (i) follows immediately from Theorem Furhman [7] while the proof of the necessity of (ii) is analogous to the one in Theorem 2.2.

To prove the sufficiency, setting $l = \max(m,n)$ and $M = \operatorname{Ker} A^l$. We first observe that the conditions (i), (ii) are also fulfilled for the factor-system $(\widehat{A}, \widehat{B}, \Omega)$ with respect to M. Indeed, since (i) holds, clearly, for $m = l$ we can write

$PA^l X = \widehat{A}^l \widehat{X} \subset P F_l (U^l) = \widehat{F}_l (U^l)$ with $\widehat{F}_l : U^l \to \widehat{X}$ being defined as

$\widehat{F}_l \ (u^l) = \sum_{i=1}^{l} \widehat{A}^{l-i} \widehat{B} u_i$ for $u^l = (u_1, \ldots u_l) \in U^l$. Further, according to Theorem 4.9 of [8], P is an isomorphism of $\widehat{X}^*$ onto M^L and the following diagram

$$\begin{array}{ccc} M^{\perp} & \xrightarrow{A^*} & M^{\perp} \\ {\scriptstyle P^{*-1}} \downarrow & & \uparrow {\scriptstyle P^*} \\ \widehat{X}^* & \xrightarrow{\widehat{A}^*} & \widehat{X}^* \end{array}$$

commutes. Consequently, if there exist $\widehat{f} \in \widehat{X}^*$, $\lambda > 0$ such that $\widehat{A}^* \widehat{f} = \lambda \widehat{f}$ and $< \widehat{f}, \widehat{B} u > \leqslant 0$ for all $u \in \Omega$, then for $f \triangleq P^* \widehat{f}$ we have $(P^* \widehat{A} \ \ P^{*-1}) P^* \widehat{f} = \lambda P^* \widehat{f}$, or $A^* f = \lambda f$ and $< f, Bu > \leqslant 0$, $\forall u \in \Omega$, contradicting (ii). This means that (ii) is satisfied also for the system $(\widehat{A}, \widehat{B}, \Omega)$. On the other hand $\widehat{A}$ is readily verified to be an isomorphism of $\widehat{X}$ onto $\widehat{X}$. Therefore, according Lemmas 2.1, 2.2, to prove the local controllability of (A, B, Ω) it suffices to show that $(\widehat{A}, \widehat{B}, \Omega)$ is locally reachable. Now, since $\widehat{A}$ is onto, by (i) we have $\widehat{A}^l \widehat{X} = \widehat{X} = \widehat{F}^l (U^l)$. Moreover, since $0 \notin \sigma(\widehat{A})$ we can assert, in view of (ii), that $\widehat{A}$ has no eigenvector with *nonnegative* eigenvalue, supporting to $B\ \Omega$ at the origin. Consequently, by Theorem 2 of [4] we conclude that $(\widehat{A}, \widehat{B}, \Omega)$ is locally reachable. This completes the proof.

We note that for finite-dimensional systems, the additonal conditions (5) are trivially satisfied (by taking l equal to the dimension of the state space X). In general, while the second condition in (5) can appear rather restrictive, the first one, called some time «the finiteness condition», is fulfilled for the discrete time models of delays systems (see, e.g. [1]).

The condition (5) is satisfied for example, when $A = I - \lambda C$ with C being of Riesz type and $\lambda \in \sigma(C)$.

Besides, it is worth remarking that for the necessity of Theorem 2.3 we need not assume (5) to be satisfied. Theorem 2.4 below will give other sufficient conditions for local controllability without the mentioned additional assumptions. We first prove the following.

LEMMA 2.3. *Suppose that for some* $k \geqslant 1$

$$A^k X \subset F_k (U^k) \text{ and } F_k (\text{int } \Omega^k) \cap A^k X \neq \phi, \ (6), \text{ then int } S_k \neq \phi.$$

Proof. From the first condition in (6), by the factorization theorem of Douglas [9], there exists an operator $C \in L(X, U^k)$ such that $A^k = F_k \cdot C$. From the second condition in (6), on the other hand, we have $F_k(u_o^k) = A^k y$ for some $u_o^k \in \text{int}\, \Omega^k$ and some $y \in X$. Hence $F_k(u_o^k) = F_k C y$ which implies $(\text{Ker}\, F_k + \text{int}\, \Omega^k) \cap C X \neq \phi$. Consequently, the inverse image $C^{-1}(\text{int}\ \Omega^k + \text{Ker}\, F_k)$ has a nonempty interior in X.

The assertion now follows immediately from the fact that

$$C\,(\text{int}\ \Omega^k + \text{Ker}\ F_k) \subset S_k.$$

Remark that the second condition in (6) is satisfied when, for example, $B(\text{int}\ \Omega) \cap A^k X \neq \phi$. Indeed, in this case we have $Bu_o \in A^k X$ for some $u_o \in \text{int}\ \Omega$ and thus $F_k(u_o^k) \in A^k X$ with $u_o^k = (u_o, u_o, \dots u_o) \in \text{int}\ \Omega^k$. The second condition in (6) means that there exists at least one point $x \in X$ which is controllable to zero by a sequence of *interior controls*.

THEOREM 2.4. *Suppose that int* $\Omega \neq \phi$ *and* (6) *is satisfied. then for the system* (A, B, Ω) *to be locally controllable it is sufficient that* A^* *has no eigenvector corresponding to non negative eigenvalue and supporting to* $B\,\Omega$.

Proof. (6) implies, in view of Lemma 2.3, that $\text{int}\ S_k \neq \phi$. Let define, for every $l \geqslant k$.

$$S'_l = \left\{ x \in X : -A^k x \in F_l\,(\Omega^l) \right\}.$$

It is easy to see that $S'_l \subset S'_{l+1}$, S'_l is convex and $\text{int}\ S'_l \neq \phi$ for all $l \geqslant k$. Besides, $AS'_l \subset S'_{l+1}$, since for any $x \in S'_l$ $-A^k(Ax) = -A\,(A^k x) \in AF_l\,(\Omega^l) \subset F_{l+1}(\Omega^{l+1})$ setting $S' = \bigcup_{l \geqslant k} S'_l$ we prove that $0 \in \text{int}\ S'$. Assuming the contrary we readily verify that the cone $C = \bigcup \lambda > 0\, S'$ is convex, not dense in X and A-invariant, i.e $AC \subset C$. By Krein-Rutman Theorem [10] there exists $\lambda \geqslant 0$ and $f \in X^*$ such that $A^* f = \lambda f$ and $\langle f, c \rangle \leqslant 0$ for all $c \in C$. on the other hand, since, clearly, $A^k B\,\Omega \subset F_{k+1}\,(\Omega^{k+1})$ it follows that $-B\,\Omega \subset S'_{k+1} \subset S' \subset C$ and hence $\langle f, Bu \rangle \geqslant 0$ for all $u \in \Omega$, conflicting with the assumption of Theorem.

Thus $0 \in \text{int}\ S'$. In view of Lemma 1 of [4], there exists $m \geqslant k$ such that $0 \in \text{int}\ S'_m$. If $m = k$ then the assertion is immediate since $S'_k = S_k$. If $m > k$, we consider the operator $A^{m-k};\ X \to X$. Since $o \in \text{int}\ S'_m$ it follow $o \in \text{int}$

$(A^{m-k})^{-1}\left(S'_m\right)$. On the other hand, for any x in the inverse image $(A^{m-k})^{-1}$ S'_m we have $y = A^{m-k}x \in S'_m$ and hence, by the definition of S'_m,

$$-A^k y = -A^k\left(A^{m-k}x\right) = -A^m x \in F_m\,(\Omega^m)$$

This means $x \in S_m$. Therefore, $0 \in \operatorname{int} S_m$ and Theorem is proved.

We illustrate the application of the above criterion by the following two simple examples.

EXAMPLE 2.2. Consider the system (A, B, Ω) with $X = U = l_2$, $A(\xi_1, \xi_2 \ldots) = \left(\xi_1 \cdot -\frac{1}{2}\xi_2, -\frac{1}{3}\xi_3, \ldots\right)$, $B = A$. If we take $\Omega = \{u \in l_2 : \|u - u_0\| \leqslant 1$ with $u_0 = (1, 0, 0\ldots)\}$ then for any $x \in B\ \Omega$, its first coordinate $\xi_1(x) \geqslant 0$. Hence eigenvector $f = (1, 0, 0, \ldots)$ of A^* (corresponding to $\lambda = 1$) is supporting to $B\,\Omega$ and, therefore, the system is not locally controllable. If we take:

$\Omega = \{u \in l_2 : \|u - u_0\| \leqslant 1$ with $u_0 = (0, 1, 0, \ldots)\}$, then since $B\,\Omega$ contains vectors $(1, -1/2, 0\ldots)$ and $(-1, -1/2, 0\ldots)$ we observe that no eigenvector of A^* corresponding to $\lambda = 1$ supports to $B\,\Omega$. Since all conditions of Theorem 2.4 are fulfilled we conclude that the system under consideration is locally controllable.

EXAMPLE 2.3. Consider a control system described by the following integro-difference equation of Voltera type

$$x_{k+1} = \int_0^t M\,(x_k(s) + u_k(s))\,ds,\ x_k(.) \in L_2([0, 1], \mathbf{R}^n)$$

$$u_k \in \Omega \subset L_2([0, 1], \mathbf{R}^n).$$

M being $n \times n$ matrix with $\det M \neq 0$ and

$$\Omega = \left\{u(.) \in L_2([0, 1], \mathbf{R}^n) : \int_0^t \|u(s) - u_0(s)\|^2\,ds \leqslant 1\right\}$$

with

$$\int_0^t \|u_0(s)\|^2\,ds = 1.$$

It can be seen that the adjoint A^* of the system's operator is also of Volterra type and 0 is an only point in $\sigma(A^*)$ moreover, 0 is not an eigenvalue [11]. Thus, by Theorem 2.4. This system is locally controllable (in fact, this system is, actually, globally controllable since for any $x \in X$, $\|A^n x\| \leqslant \frac{M^n}{(n-1)!} \to 0$, as $n \to \infty$).

It is worth noticing that the systems considered in the above two examples are not locally reachable (i.e $0 \notin \operatorname{int} R$) because, in these cases, the operators F_n are compact and hence $\operatorname{int} R = \emptyset$.

We conclude this section with the remark that Theorems 2.3, 2.4 can be strengthened by assuming only that Ω has a non-empty *relative interior*. We formulate for example, the strengthened version of Theorem 2.4.

THEOREM 2.5. *Suppose $ri(\Omega - u_0) \neq \emptyset$ for some $u_0 \in \Omega$.*

If there exists $n > 0$ such that

(i) $A^n X \subset F_n(V^n)$, where $V = \overline{span}\,(\Omega - u_0)$,

(ii) $A^n X \cap F_n(ri(\Omega^n - u_0^n)) \neq \emptyset$,

(iii) there exists no $\lambda \geqslant 0$ and $f \in X^*$ such that $A^*f = \lambda f$ and $< f, Bu > \geqslant 0$, $\forall u \in \Omega$, then the system (A, B, Ω) is locally controllable

III. GLOBAL CONTROLLABILITY.

In this section we shall assume $B = I$ (the identity operator in X). Thus, let us consider the control system (A, Ω):

$$x_{k+1} = Ax_k + u_k, \quad x_k \in X,\ u_k \in \Omega \subset U, \tag{7}$$

where $A \in L(X)$, Ω is a convex set with $\operatorname{int}\Omega \neq \emptyset$ and $0 \in \Omega$.

The criterion of global controllability obtained in this section involves the following notions from convex analysis [12].

DEFINITION 3.1. *Let Ω be a convex subset of X containing 0. The set $C_b(\Omega) = \{f \in X^* : \sup_{u\in\Omega} < f,u > < \infty\}$ is said to be the barrier cone of Ω and the set $C_r(\Omega) = \bigcap_{\lambda>0} \lambda\,\Omega$ is said to be the recession cone of Ω.*

In [6] we have proved the following result.

THEOREM 3.1 *For the system (A, Ω) in $X = \mathbf{R}^n$ to be globally controllable, it is sufficient and, in the case where*

$$C_b(\Omega) = -(C_r(\Omega))^* \tag{8}$$

necessary that

(i) (A, Ω) is locally controllable,

(ii) The transpose matrix A^* has neither real eigenvector f with $\lambda > 1$ supporting to $C_r(\Omega)$ nor complex eigenvector f with $|\lambda| > 1$ orthogonal to $C_r(\Omega)$.

Remark. In t. e above statement we understand by a complex eigenvector of a real matrix A corresponding to $\lambda = \alpha + i\beta$ ($\alpha, \beta \in \mathbf{R}$) a vector $f \in \mathbf{R}^n + i\,\mathbf{R}^n$ such that

$$A(\mathrm{Ref}) = \alpha(\mathrm{Ref}) - \beta(\mathrm{Jmf}),$$
$$A(\mathrm{Jmf}) = \beta(\mathrm{Ref}) + \alpha(\mathrm{Jmf}).$$

Generally, let X be a real Banach space dan let $\widetilde{X} \triangleq X + i\,X$ be the complexification of X with the norm:

$$\| x + iy \|_1 = \sup_\theta \| x \cos\theta + y \sin\theta \| .$$

Then every operator $A \in L(X)$ has a unique *complex extention* $\widetilde{A} \in L(\widetilde{X})$ defined as $\widetilde{A}(x+iy) = Ax + iAy$, and we can define the terms spectrum eigenvalue, eigenvector, ... of A to be the corresponding objects for $\widetilde{A}$ (see, e. g. [13]).

In the sequel, for convenience, we shall identify A, X with their complex extentions wherever the complex spectral objects are involved.

The main goal of this section is to extend the above criterion of global controllability to the class of infinite-dimensional systems (A, Ω) with A satisfing the following spectral condition.

Assumption 1. For some positive number $r < 1$ the set $\delta_1 = \delta(A) \setminus \{z \in \mathbf{C} : \| z \| < r\}$ consists of a finite number of points and the corresponding eigenspace X_1 is finite dimensional (where, by definition,

$$X_1 = PX,\; P = \frac{1}{2\pi i} \int_L (A - \lambda I)^{-1} \, d\lambda \tag{9}$$

and L is a closed simple rectifiable curve on the complex plane $\mathbf{C}$ such that δ_1 contained inside L).

Note that many systems of practical importance can be described by the discrete — time model (A, Ω) or (A, B, Ω) with a compact operator A wich, clearly, satisfies the above assumption. Such are systems of Volterra integral equations and functional-differential equations.

We can now state the main results of this section as follows.

THEOREM 3.2. *Suppose A satisfies Assumption 1. Then for the system (A, Ω) to be globally controllable it is sufficient and, in the case where (8) holds, necessary that*

(i) (A, Ω) is locally controllable,

(ii) The adjoint operator A^* has neither eigenvectors $f \in X^*$ with eigenvalues $\lambda > 1$ supportting $C_r(\Omega)$ nor eigenvectors $f \in X^*$ with eigenvalues $|\lambda| > 1$, such that $\langle f, u \rangle = 0$ for all $u \in C_r(\Omega)$.

Proof. The necessity of (i) being obvious, to prove the necessity of (ii) assume the contrary that (A, Ω) is globally controllable and (8) holds but (ii) is not valid. Then there exists $f_0 \in X^*$ such that $A^* f_0 = \lambda f_0$, $\lambda > 1$ (or $|\lambda| > 1$) and $-f_0 \in (C_r(\Omega))^*$ (respectively, $\langle f_0, u\rangle = 0$, $\forall u \in C_r(\Omega)$). Then, for any $x \in S$, in view of (8) we have, in the first case

$$-< f_0, x > = \sum_{i=1}^{k} \lambda^{-i} < f_0, u_i > \leqslant \left(\sum_{i=1}^{\infty} \lambda^{-i} \right) \sup_{u \in \Omega} < f_0, u > < \infty,$$

and in the second case

$$| < \mathrm{Re}\, f_0, x > | \leqslant | < f_0, x > | \leqslant \Sigma |\lambda|^{-i} | < f, u_i > | \leqslant$$

$$\leqslant \left(\sum_{i=1}^{\infty} |\lambda|^{-i} \right) \left(\sup_{u \in \Omega} | < \mathrm{Re} f_0, x > | + \sup_{u \in \Omega} | < \mathrm{Jm}\, f_0, u > | \right) < \infty$$

(since, obviously, $\pm\,\mathrm{Re} f_0$ and $\pm\,\mathrm{Jm} f_0$ belong to $(C_r(\Omega))^*$), both conflicting with the global controllability of (A, Ω)

To prove the sufficiency we observe that, by Assumption I, there exists a decomposition of X into the direct sum $X = X_1 \oplus X_2$ such that X_i $(i = 1, 2)$ is a closed A-invariant subspace, $\dim X_1 < \infty$, and the spectra of the restrictions A_i of A to X_i $(i = 1, 2)$ coincide, respectively, with $\sigma_1 = \sigma(A) \setminus \{z \in \mathbf{C}\, |z| < r\}$ and $\sigma_2 = \sigma(A) \setminus \sigma_1$, $r < 1$ (See, e. g. [14]). Accordingly, the system (A, Ω) is decomposed into 2 subsystems

$$\begin{cases} x^1_{k+1} = A_1 x^1_k + v^1_k, & x^1_k \in X_1,\ v^1_k \in P\,\Omega \\ x^2_{k+1} = A_2 x^2_k + v^2_k, & x^2_k \in X_2,\ v^2_k \in (I - P)\,\Omega \end{cases}$$

P being the specral projection defined by (9). Observe that $(A_2, (I - P)\,\Omega)$ is locally controllable (by (i)) and A_2 is asymptoticaly stable. This implies that $(A_2, (I - P)\,\Omega)$ is globally controllable and therefore to establish the sufficiency it is enough to prove that $(A_1, P\,\Omega)$ is globally controllable.

On the other hand, since this subsystem is of finite dimensional and is locally controllable (by (i)) our task is reduced, by virture of Theorem 3.1, to showing that the condition (ii) remains valid for $(A_1, P\,\Omega)$. Indeed, suppose there exists $0 \neq f_1 \in X^*$, $\lambda > 1$ (or $|\lambda| > 1$) such that $A_1^* f_1 = \lambda f_1$. Since P is onto we have $P^* f_1 \neq 0$ (by Theorem 4.15 [8]).Besides, notice that $A^* P^* f_1 =$ $= (PA)^* f_1 = P^* A_1^* f_1 = \lambda P^* f_1$.

According to (ii) there exists $\overline{u} \in C_r(\Omega)$ such that $\langle P^*f_1, \overline{u} \rangle = \langle f_1, P\overline{u} \rangle > 0$ (respectively, $\neq 0$). Since $P\overline{u} \in P\,C_r(\Omega) \subset C_r(P\Omega)$ we conclude that (ii) is satisfied for subsystem $(A_1, P\Omega)$. This completes the proof of the Theorem.

Theorem 3.2 yields the following result which has been proved in [5] by a different method.

COROLLARY 3.1 *Suppose A satisfies the Assumption* 1 *and* Ω *is bounded. The system* (A, Ω) *is globally controllable iff*

(i) (A, Ω) *is locally controllable.*

(ii) $|\lambda| \leqslant 1$ *for any* $\lambda \in \sigma(A)$.

Proof. It suffices to note that, Ω being bounded, $C_r(\Omega) = \{0\}$, $C_b(\Omega) = X$ and hence (8) is satisfied.

REFERENCES

[1] K. M Przyluski, *Infinite — Dimensional Discrete — Time Systems: Theory and Applications to Linear Hereditary Control Systems*, IPEPW, Technical Report, Warsaw 1979.

[2] K. M Przyluski *The Lyapunov Equation and The Problems of Stability for Linear Bounded Discrete — Time Systes in Hilbert Spaces*, Appl. Math. Opt.6 (1890), 97 – 112.

[3] A. W Olbrot, A Sosnowski *Duality Theorems on Control and Observation of Discrete — Time Systems*, Math. Systems Theory 94 (1981), 173 – 157.

[4] Nguyen Khoa Son, *Controllability of Linear Discrete Time Systems with Constrained Controls in Banach Spaces*, Control et Cybernetics, 1 (1981), 5 – 17.

[5] Nguyen Khoa Son, Nguyen Van Chau and Nguyen Van Su, *On the Global Null — Controllability of Linear Discrete — Time Systems With Restrained Controls in Banach Spaces*, Control et Cybernetics (to appear).

[6] Nguyen Khoa Son, *A Note on The Null — Controllability of Linear Discrete — Time System With Restrained Controls*, I. Opt. Theory Appl. (to appear).

[7] P. A: Furnman, *On Weak and Strong Reachability and Controllability of Infinite— Dimensional Linear Systems*, J. Opt. Theory Appl. 9 (1972), 77 — 89.

[8] W. Rudin, *Functional Analysis*, Mc Gravo-Hill, New York, 1973.

[9] R. G. Douglas, *On Majoritation, Factorization and Range Inclution of Operators in Hilbert Spaces*, Procc. Amer. Math. Soc., 17 (1966), 413 – 416.

[10] M. G. Krein M.A. Rutman, *Linear Operators Leaving Invariant a Cone in a Banach Space*, Uspehi Math. Nauk, L (23) 1948, 3 – 95.

[11] A. V. Balakrishman, *Applied Functional Analysis*, Springer — Verlarg, New York, 1976.

[12] J.P. Aubin, *Mathematical Method of Game and Economic Theory*, North — Holland, Amsterdam, 1982.

[13] H. Shaffer, *Topological Vector Spaces*, Macmilan, New York, 1964.

[14] T. Kato, *Perbubations Theory of Linear Operators Springer — Verlarg*, Berlin, 1966.

Receiced July 20,1985

* CENTER FOR APPLIED SYSTFMS ANALYSIS.

** INSTITUTE OF MATHEMATICS, P. O. BOX 631, BO HO, 10.000 HANOI, VIETNAM.