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SOLUTIONS OF SOME PROBLEMS IN THE RENEWAL THEORY BY THE MONTE CARLO METHOD

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1. INTRODUCTION

The aim of this paper is to present some new results concerning the random integral equations in the renewal theory. In particular, we introduce the new concepts of effect and potentiality of the renewal process.

Let $\{\tau_n\}_{n=1}^{\infty}$ be a renewal process, i.e. τ_n $(n=1, 2,...)$ are random non-negative, independent variables defined on a probability space (V, Σ_v, P_v) with $P_v\{\tau_n = 0\} < 1$ (cf. [2, 3]).

Suppose that $U(t)$ is the mean renewal quantity on the time interval $[0, t]$. It is known that $U(t)$ is a unique solution in $\mathcal{X}^{\infty}[0, \infty)$ of the following equation:

$$U(t) = K(t) + \int_0^t U(t-x) P_v(dx), \quad \forall\, t \geqslant 0, \tag{1.1}$$

where $K(t)$ is the distribution function of random variable τ_1 and $\mathcal{X}^p[a, \infty)$ $(1 \leqslant p \leqslant \infty)$ is a set of real function defined on $[a, \infty)$, such that for any $c \in (a, \infty)$ and $f \in \mathcal{X}^p[a, \infty)$

$$\int_a^c |f(x)|^p\, dx < \infty.$$

If $k(t)$ is the density function of the random variable τ_1 and

$$K(t) = E_u\, \eta(t), \qquad t \geqslant 0, \tag{1.2}$$

$$K(t) = E_u\, \gamma(t), \qquad t \geqslant 0, \tag{1.3}$$

where $\eta(t) = \eta(t; u)$, $\Upsilon(t) = \Upsilon(t; u)$ are real random processes on a probability space (U, Σ_u, P_u), then the equation (1. 1) can be written in the following form:

$$U(t) = E_u \eta(t) + \int_0^t E_u \Upsilon(x) U(t-x)dx, \; t \in [0, \infty). \tag{1. 4}$$

Our aim is to study some problems connected with the above random integral equation (cf. [7] for the representation of its solutions). Namely, we shall consider the estimation problem for effect and potentiality of the renewal process.

Denote by ν_t^- the age of the renewal process at the moment t (cf. [2, 3]. Suppose that $\rho(x)$ is a real function defined on $[0, \infty)$.

Define

$$V(t) := E_v \rho(\nu_t^-), \tag{1. 5}$$

The function $V(t)$ is called effect of the renewal process. Further, if there exists $t^\star \in (0, \infty)$, such that

$$V(t^\star) = \sup_{o < t < \infty} V(t) \tag{1. 6}$$

then $V(t^\star)$ is called the maximal effect of the renewal process and $t^\star$ is called the optimal moment.

The function $V(t)$ and the value $V(t^\star)$ are used to predict the effect as well as potentiality of an economic - technological system. Here and in the sequel by potentiality we mean the quantity

$$W(t) := E_v \widetilde{W}(t), \qquad t \geqslant 0, \tag{1. 7}$$

where

$$\widetilde{W}(t) := \int_{\nu_t^-}^{\tau_{N(t)+1}} \rho(u)\,du = T(\tau_{N(t)+1}) - T(\nu_t^-), \tag{1. 8}$$

$$T(x) = \begin{cases} \int_0^x \rho(u)du, & \text{if } x > 0, \\ 0, & \text{otherwise}; \end{cases} \tag{1. 9}$$

$$N(t) = \max\{n : \tau_0 + \tau_1 + \ldots + \tau_n \leqslant t\}, \; \tau_0 = 0.$$

The time $t_\star$ for which

$$W(t_\star) = \operatorname{Sup}_{0<t<\infty} w(t) \tag{1. 10}$$

is called the optimal moment for potentiality.

In the sequel we shall establish an algorithm to estimate $V(t)$, $W(t)$, t^*, t_* by the Monte Carlo method.

2. THE EFFECT OF A RENEWAL PROCESS

Suppose that

(2.1) $\rho(x) = 0 \quad \text{if} \quad x < 0$

and $\gamma(x)$, $\eta(x)$ are Hilbert random processes.

Put

(2.2) $$\overline{\varphi}_0(x) = \begin{cases} \frac{1}{b}, & \text{if } x \in [0, b],\ b \in (0, \infty) \\ 0, & \text{if } x \notin [0, b]; \end{cases} \quad \overline{\varphi}(x, y) = \begin{cases} \frac{1}{b}, & \text{if } (x,y) \in [0,b] \times [0,b], \\ 0, & \text{if } (x, y) \notin [0, b] \times [0, b], \end{cases}$$

(2.3) $$\xi(x) = \xi(x;\ u) = \begin{cases} \rho(x)\,[1 - \eta(x;\ u)], & \text{if } x > 0 \\ 0, & \text{otherwise}; \end{cases}$$

Next assume that $(R_+^\infty, \mathcal{B}_+^\infty, P_\infty)$ is the probability space generated by all trajectories of the Markov chain $\{\overline{\varphi}_0, \overline{\varphi}\}$. On the space $(R_+^\infty, \mathcal{B}_+^\infty, P_\infty)$ we define the following function:

(2.4) $$d(\widetilde{x}) = \begin{cases} n, \text{ if } x_0 \geqslant \ldots \geqslant x_n \text{ and } x_n < x_{n+1} \\ \infty, \text{ if } x_n \geqslant x_{n+1},\ n = 0, 1, \ldots \end{cases}$$

$$\widetilde{x} = (x_0, x_1, \ldots, x_n, \ldots) \in R_+^\infty .$$

In [7] we proved that $d(\widetilde{x})$ is a random variable, i. e.

$$P_\infty\{d(\widetilde{x}) = \infty\} = 0,$$

so on the probability space $(\Omega, \mathbf{C}, \lambda)$ we can define the following random variable:

(2.5) $$\xi(t) = \sum_{i=0}^{d(\widetilde{x})} b^{i+1}\, \eta(x_0; u_0)\, \gamma(x_0 - x_1; u_1) \ldots \gamma(x_{i-1} - x_i; u_i)\, \xi(t - x_i; u_{i+1})$$

$$0 \leqslant t \leqslant b,$$

where

(2.6) $$\Omega = R_+^\infty \times U^\infty, \qquad \mathbf{C} = \mathcal{B}_+^\infty \times \Sigma_u^\infty, \qquad \lambda = P_\infty \times P_u^\infty,$$

and $(U^\infty, \Sigma_u^\infty, P_u^\infty)$ is the infinite product of the space (U, Σ_u, P_u).

THEOREM 1. *Let* $\{\tau_n\}_{n=1}^\infty$ *be a renewal process with the probability density*

$k(x) = E_u \gamma(x)$ *and the distribution function* $K(x) = E_u \eta(x)$, *where* $\gamma(x)$, $\eta(x)$ *are random processes continuous in the sense of mean square. Further, let* $\rho(x)$ *be a function in* $\mathcal{X}^1[0, \infty)$.

a. *If for any* $\tilde{x} \in R_+^\infty$ *and* $i = 1, \ldots, d(\tilde{x})$, $\eta(x_0; u_0)$, $\{\gamma(x_k - x_{k+1}; u_{k+1}\}_{k=0}^{i-1}$ *and* $\xi(t - x_i; u_{i+1})$ *are independent samples of random variables* $\eta(x_0)$, $\{\gamma(x_k - x_{k+1})\}_{k=0}^{i-1}$ *and* $\xi(t - x_i)$ *then*

$$E_\lambda \xi(t) = V(t), \ \forall t \in [0 . b]. \tag{2.7}$$

b. *In addition, if*

b1. *the function* $\rho(x)$ *is differentiable,* $\eta(x)$ *is continuously differentiable in the sense of mean square and*

$$\left| \frac{\partial}{\partial x} E_u \xi(x - y) \right| \leqslant Kx + D, \ 0 \leqslant y \leqslant x; \ L, D - \text{constants}, \tag{2.8}$$

b.2 *the condition* a) *is satisfied when* $\xi(t - x_i)$ *is replaced by* $\frac{\partial}{\partial x} \xi(t - x_i)$,

b3. *the function* $V(t)$ *has unique stationary point* $t^* \in (0, b)$ *and attains its maximum on* $(0, b)$,

then

$$\lim_{k \to \infty} E_\lambda (b_k - t^*)^2 = 0 \tag{2.9}$$

$$\lambda \{ \lim_{k \to \infty} b_k = t^* \{ = 1, \tag{2.10}$$

where

$$b_{k+1} = b_k + \gamma_k \tilde{\theta}_o(b_k), \qquad b_o \in (-\infty, \infty) \tag{2.11}$$

$$\tilde{\theta}_o(x) = \begin{cases} \tilde{\theta}_o(x), & \text{if } x \in (o, b), \\ 1, & \text{if } x \leqslant 0, \\ -1, & \text{if } x \geqslant b; \end{cases} \tag{2.12}$$

$$\gamma_k > 0, \quad \sum_{k=0}^{\infty} \gamma_k = \infty, \quad \sum_{k=0}^{\infty} \gamma_k^2 < \infty \tag{2.13}$$

$$\theta_o(X) = \sum_{i=0}^{d(\tilde{x})} b^{i+1} \eta(x_i; u_{i+1}) \gamma(x_o - x_1; u_1) \ldots \tag{2.14}$$

$$\gamma(x_{i-1} - x_i; u_i) \frac{\partial}{\partial x} \xi(x - x_o; u_o).$$

Proof. From the assumptions it follows that the function $k(x) = E_u \gamma(x)$, $K(x) = E_u \eta(x)$ are absolutely continuous and $U(t)$ is continuously differentiable (cf. [5], pages 628—629). Denoting $u(t) = U'(t)$ we get the equation

$$U(t) = (Kt) + \int_0^t E_u \gamma(t-x)\, u(x)\, dx, \qquad t \geqslant 0. \tag{2.15}$$

Denoting

$$\mathcal{G}(x, t) = P_v \left\{ \nu_t^- \leqslant x \right\}$$

We have (cf. [2, 3])

$$\mathcal{G}(x, t) = \begin{cases} \int_{t-x}^{t} [1 - K(t-\tau)]\, u(\tau)\, d\tau, & 0 \leqslant x < t, \\ 1, & \text{if } x \geqslant t. \end{cases} \tag{2.16}$$

On the other hand

$$\begin{aligned} V(t) = E_v\, \rho(\nu_t^-) &= \int_0^t \rho(x)\, \mathcal{G}(dx, t) \\ &= \int_0^t \rho(x)\, [1 - K(x)]\, u(t-x)dx \\ &= \int_0^t \rho(t-x)\, [1 - K(t-x)] u(x) dx \\ &= \int_0^t \rho(t-x)\, [1 - E_u \eta(t-x)]\, u(x) dx \\ &= \int_0^t E_u \xi(t-x)\, u(x)\, dx. \end{aligned} \tag{2.17}$$

By virtue of (2.1), (2.17) and Theorem 2.2 in [7], We obtain (2.7). The relations (2.9), (2.10) are simple consequences of Theorem 3.1 in [7]. The proof is complete.

3. THE POTENTIALITY OF A RENEWAL PROCESS

Put

$$\xi^*(x) = \xi^*(x\,;\, u) = T(x)[1 - \eta(x\,;\, u)] \quad ; \tag{3.1}$$

$$\Omega^* = \Omega \times V\,,\ C^* = C \times \Sigma_v\,,\ \lambda^* = \lambda \times P_v \quad ; \tag{3.2}$$

$$(3.3)\qquad \bar{\xi}(t)=T(\tau_1)+\sum_{i=0}^{d(x)} b^{i+1}\eta(x_o;u_o)\gamma(x_o-x_1;u_1)\ldots$$
$$\gamma(x_{i-1}-x_i;u_i)\,\xi^*(t-x_i;u_{i+1})$$

THEOREM 2. *Let* $\{\tau_n\}_{n=1}^{\infty}$ *be a renewal process defined on the probability space* $(V,\Sigma_v,P_v$ *with density* $k(x)=E_u\gamma(x)$, *distribution function* $K(x=E_u\eta(x)$, $(x\geqslant 0)$, *where* $\gamma(x)$, $\eta(x)$ *are Hilbert random processes, continuous in the sense of mean square.*

1. If $T(x)\in\mathfrak{X}^1[0,\infty)$ *and for any* $\tilde{x}\in R_+^\infty$ *and* $i=1,\ldots,d(\tilde{x})$, $\eta(x_o;u_o)$, $\{\gamma(x_k-x_{k+1};u_{k+1})\}_{k=0}^{i-1}$, $\xi^*(t-x_i;u_{i+1})$ *are independent samples of random variables* $\eta(x_o)$, $\{\gamma(x_k-x_{k+1})\}_{k=0}^{i-1}$, $\xi^*(t-x_i)$ *respectively, then*

$$(3.4)\qquad E_{\lambda^*}\bar{\xi}(t)=W(t),\quad t\geqslant 0.$$

2. In addition, if

a) the function $T(x)$ *is differentiable:* $\eta(x)$ *is continuously differentiable in the sense of mean square and*

$$\left|\frac{\partial}{\partial x}E_u\xi^*(x-y)\right|\leqslant Kx+D,\ 0\leqslant y\leqslant x;\ K,\ D\text{ — constants},$$

b) *the condition l. is satisfied if* $\bar{\xi}^*(t-x_i)$ *is replaced by*

$$\frac{\partial}{\partial x}\xi^*(t-x_i),$$

c) *the function* $W(t)$ *has the unique stationary point* $t_*\in(0,b)$ *and attains its maximum on* $(0,b)$,

then

$$(3.5)\qquad \lim_{k\to\infty}E_{\lambda^*}(C_k-t_*)^2=0$$

$$(3.6)\qquad \lambda^*\{\lim_{k\to\infty}C_k=t_*\}=1,$$

where

$$(3.7)\qquad C_{k+1}=C_k+\gamma_k\,\tilde{\theta}_1(C_k),\ C_o\in(-\infty,\infty)$$

$$(3.8)\qquad \tilde{\theta}_1(x)=\begin{cases}\theta_1(x), & \text{if } x\in(0,b),\\ 1, & \text{if } x\leqslant 0,\\ -1, & \text{if } x\geqslant b;\end{cases}$$

$$\theta_1(x) = \sum_{i=0}^{\tilde{d}(x)} b^{i+1} \eta(x_i; u_{i+1}) \gamma(x_0 - x_1; u_1) \ldots \gamma(x_{i-1} - x_i, u_i) \frac{\partial}{\partial x} \xi^*(x - x_i; u_0). \tag{3.9}$$

Proof. We first note that $N(t)$ is the Markov moment for a stream of σ-fields $\mathcal{G}_n$, $n = 1, 2, \ldots$ generated by the random variables $\tau_1, \tau_2, \ldots, \tau_n$ i. e.

$$\mathcal{G}_n = \sigma\{\tau_1, \ldots, \tau_n\}, \; n = 1, 2, \ldots$$

By Wald's identity, we have

$$E_v \left\{ \sum_{i=1}^{N(t)} T(\tau_i) \right\} = E_v \{ T(\tau_1) \} \, E_v \{ N(t) \} \tag{3. 10}$$

which implies that

$$E_v \{ T(\tau_{N(t)+1}) \} = E_v \{ T(\tau_1) \} \tag{3. 11}$$

By (1. 8), (1. 9), (3. 11) it follows that

$$W(t) = E_v \{ T(\tau_1) \} - E_v \{ T(v_t^-) \} \tag{3. 12}$$

$$= \int_0^\infty T(x) \, E_v \Upsilon(x) dx - \int_0^\infty T(x) \, \mathcal{F}(dx, t).$$

From the above equation the proof can be completed using the same method as in the proof of Theorem 1.

Remark. The relations (2. 7) and (3. 4) imply that if $\tilde{x}^{(k)}$ $k = 1, 2, \ldots, m)$ are independent trajectories of the Markov chain then $\frac{1}{m} \sum_{k=1}^{m} \zeta^{(k)}(t)$, $\frac{1}{m} \sum_{k=1}^{m} \bar{\zeta}^{(k)}(t)$ are unbiased estimators of $V(t)$ and $W(t)$ respectively:

$$V(t) \approx \frac{1}{m} \sum_{k=1}^{m} \zeta^{(k)}(t)$$

$$W(t) \approx \frac{1}{m} \sum_{k=1}^{m} \bar{\zeta}^{(k)}(t)$$

where

$$\xi^{(k)}(t) = \sum_{i=0}^{d(\tilde{x}^{(k)})} b^{i+1} \eta^{i+1} (x_0^{(k)};\ u_0^{(k)})\, \gamma(x_0^{(k)} - x_1^{(k)};\ u_1^{(k)}) \dots$$

$$\dots \gamma(x_{i-1}^{(k)} - x_i^{(k)};\ u_i^{(k)})\, \xi(t - x_i^{(k)};\ u_{i+1}^{(k)})$$

$$\bar{\xi}^{(k)}(t) = T(\tau_1^{(k)}) + \sum_{i=0}^{d(\tilde{x}^{(k)})} b^{(i+1)}\, \eta\, (x_0^{(k)}; u_0^{(k)})\, \gamma(x_0^{(k)} - x_1^{(k)};\ u_1^{(k)}) \dots$$

$$\dots \gamma(x_{i-1}^{(k)} - x_i^{(k)};\ u_i^{(k)})\, \xi^\star(t - x_i^{(k)};\ u_{i+1}^{(k)})$$

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