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A LOGARITHMIC CRITERION FOR THE CONVERGENCE OF MULTIPARAMETER RANDOM SERIES

TRAN QUOC VIẾT

In many limit problems the investigation of the convergence of random power series of the form

$$\sum_{k=0}^{\infty} t^k z_k \quad (1),$$

where z'_k s are i. i. d. r. v. s. and t's are real numbers, is of great importance. The first and well-known result of Zakusilo (Cf, [5]) asserts that if $0 < t < 1$ then (1) is convergent with (P. 1) if and only if

$$E \log (|z_o| + 1) < \infty \quad (2).$$

This logarithmic criterion was generalized by Jurek [2] to the multi-dimensional spaces and by Thu [3] to the multiparameter case. Our aim in the present paper is to give a unified approach of the above-mentioned papers. Namely, we shall prove the following theorem:

THEOREM 1. *Suppose that $T_1, \ldots, T_d$ are some invertible bounded linear operators on a separable Banach space X such that*

$$\lim_{k\to\infty} \| T_j^k \| = 0, j = 1,\ldots, d \quad (3)$$

and z_1, z_n, $n = (n_1, \ldots, n_d) \in N^d$ are i. i. d. X-valued r. v'. s. Then the random power series

$$\sum_{n \in N^d} T_n Z_n \quad (4)$$

where

$$T_n = T_1^{n_1} \ldots T_d^{n_d}.$$

whenever $n = (n_1, \dots, n_d) \in N^d$ is convergent with (P. 1) if and only if

$$F \log^d (1 + \| Z \|) < \infty \tag{5}$$

To prove the above theorem we need several lemmas.

LEMMA 1. *Let V be a bounded linear operator on X. Then the relation*

$$\| V^m \| \to 0 \text{ as } m \to \infty \tag{6}$$

holds if and only if there exist $\alpha > 0$ and $0 < \beta < 1$, such that

$$\| V^m \| \leqslant \alpha\beta^m \qquad m = 1,2,\dots, \tag{7}$$

Proof. Obviously (7) $\Rightarrow$ (6). We shall prove the implication (6) $\Rightarrow$ (7).

From (6) it follows that for some constant $0 < \beta < 1$ there exists a natural number p such that

$$\| V^p \| \leqslant \beta^p.$$

Hence, it follows that

$$\| V^{kp} \| \leqslant \beta^{kp} \quad , \quad k = 1,2,\dots,$$

For arbitrary $m = kp + r$, $k = 0,1,2\dots,$ $r = 0, 1,\dots, p - 1$
We get

$$\| V^m \| \leqslant \| V^r \| \quad \| V^{kp} \| .$$

Putting $\alpha = \max (1, \| V^r \| \beta^{-r}, r = 0, 1,\dots, p - 1)$ and taking into account the above inequalities we get (7). Thus the Lemma is proved.

LEMMA 2. *If $T_1, \dots, T_d$ are operators as in Theorem 1. Then, for every $x \in X$*

$$\| T_n x \| \geqslant \gamma^{|n|} \| x \| \tag{8}$$

where, for $n = (n_1, \dots, n_d) \in N^d$, $\quad | n | = n_1 + \dots + n_d$ and

$$0 < \gamma = \min (\| T_j^{-1} \| ^{-1}, \quad J = 1,\dots, d) < 1 \tag{9}$$

Proof. Since, for $n = (n_1, \dots, n_d)$ and $x \in X$, $\| x \| = \| T_n^{-1} T_n x \| \leqslant \| T_n x \| \| T_n^{-1} \|$ and

$$\| T_n^{-1} \| = \| T_d^{-1} \dots T_1^{-1} \| \leqslant \| T_d^{-1} \| \dots \| T_1^{-1} \|$$

We get

$$\| T_n x \| \geqslant \| T_n^{-1} \| ^{-1} \| x \| \geqslant \| T_1^{-1} \| ^{-n_1} \dots \| T_d^{-1} \| ^{-n_d} \| x \|$$

Consequently, if the condition (3) is satisfied then

$\| T_i^{-1} \| > 1$, $i = 1,\dots, d$ and (8) holds.

Thus the Lemma is proved.

Proof of Theorem 1.

Suppose first that, the series (4) is convergent. Then as $n \to \infty$

$$T_n Z_n \to 0 \quad (P.1) \tag{10}$$

which, by virtue of Lemma 2, implies that

$$\gamma^{|n|} \| Z_n \| \to 0 \qquad (P.1) \tag{11}$$

Consequently, by Lemma 2 in ([1], p. 228), for every $c > 0$

$$\sum_{n \in N^d} P(\{\gamma^{|n|} \| Z_n \| > c) < \infty \tag{12}$$

where γ is the same as in (9).

Then by the same method as in [3] we infer that (12) holds if and only if the condition (5) is satisfied.

Conversely, suppose that (5) is satisfied. By results from [3] it follows that

$$\sum_{n \in N^d} \beta^{|n|} \| Z_n \| < \infty \quad (P.1) \tag{13}$$

for every $0 < \beta < 1$. Further, virtue of Lemma 1 there exist numbers α_j, β_j such that $\alpha_j > 0$, $0 < \beta_j < 1$, and

$\| T_j^m \| \leqslant \alpha_j \beta_j^m$, $m = 1, 2, \ldots$ and $j = 1, \ldots, d$. Put $\beta = \max(\beta_1, \ldots, \beta_j)$. By (13) and by the above inequality we conclude that the series (4) is absolutely convergent with (P. 1). Thus the Theorem is fully proved.

Directly from the proof of Theorem 1 we get

COROLLARY. (i) *If* (4) *is convergent with* (P.1) *then it is absolutely convergent with* (P.1).

(ii) *The series* (4) *is convergent if and only if*

$$T_n Z_n \to 0 \text{ as } n \to \infty \qquad (P.1).$$

By the same method as in the proof of Theorem 1 and Theorem 2.1 in [4] one can prove the following

THEOREM 2. *Let T be an invertible bounded linear operator on X, such that*

$$\| T^m \| \to 0 \text{ as } m \to \infty.$$

Further, let Z_1, Z_2, ... be a sequence of independent X-valued r.v's. *such that for every $k = 1, 2, \ldots$ the distribution of Z_k is $\mu^{*r_{k,\alpha}}$, where μ is an* i.d. *probability measure on X, $\alpha > 0$,*

$$r_{k,\alpha} = \begin{cases} 1 & k = 0 \\ \alpha(\alpha+1)\ldots(\alpha+k-1)/k! & k = 1, 2, \ldots \end{cases}$$

and the power is taken in the convolution sense.

Then, the random series

$$\sum_{k=0}^{\infty} T^k Z_k \tag{14}$$

is convergent with (P. 1) if and only if

$$\int_X \log^{\alpha}(1 + \| x \|)\, \mu\,(dx) < \infty \tag{15}.$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY, DALAT, VIETNAM