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ON TOPOLOGICAL PROPERTIES OF NOETHERIAN TOPOLOGICAL MODULES

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All algebras in this paper are assumed to be commutative with unit element. A topological algebra B is called m-convex if its topology can be defined by a family of submultiplicative seminorms. By a topological B-module we mean a B-module M equipped with a topology such that the maps:

$$M \times M \longrightarrow M \quad : \quad (m, n) \longrightarrow m + n$$
$$B \times M \longrightarrow M \quad : \quad (b, m) \longrightarrow bm$$

are continuous.

The aim of this paper is to study topological properties of noetherian topological B-modules, and to apply the results to the study of algebraic properties of algebas of germs of holomorphic functions on a neighbourhood of a compact set in a complex space [10]. Some results on this problem for topological algebras have been established by Ferreira and Tomassini [5], Ballico and Ferreira [2].

In §1 we investigate the closedness of submodules of noetherian topological modules. We shall describe the classes of noetherian topological modules for which every submodule is closed. A condition for a noetherian topological module to be finite dimensional is proved in §2. The noetherian topological modules over algebras of holomorphic functions on complex spaces are investigated in §3 and §4.

There we prove that every Hausdorff Noether topological module over a Stein algebra is finite dimensional. Moreover we prove that over every infinite dimensional Stein algebra there exists an infinite dimensional noetherian topological module.

§ I. THE CLOSEDNESS OF SUBMODULES OF NOETHERIAN TOPOLOGICAL MODULES:

Let B be a m-convex algebra and M a topological B-module Following [5] we say that M satisfies the open mapping theorem if for every submodule N of M of finite codimension and for all continuous surjective B — linear map θ: $B^n \longrightarrow N$ the $\mathbf{C}$ — linear map $\theta \oplus \beta$:

$B^n + \mathbf{C}^n \longrightarrow M$, where $\beta: \mathbf{C}^m \to N$ is given by a vector basis of M/N is open.

I.I. SOME EXAMPLES:

1) From the open mapping theorem [12] it follows that every barrelled module over a B — complete algebra the open maping.

2) Let V be an algebraic set in $\mathbf{C}^n$ and let $\mathbf{C}[V] = \mathbf{C}[Z_1,\dots,Z_n] / I[V]$ where $I(V) = \{\sigma \in \{C[z_1,\dots, z_n]: \sigma \mid V = 0\}$ denote the m — convex algebra of regular functions on V. This algebra is equipped with the compact open topology. Then $\mathbf{C}[V]$ satisfies the open mapping theorem.

Consider the commutative diagram

$$\begin{array}{ccccccccc} O & \longrightarrow & I(V) & \longrightarrow & C[z_1, \dots, z_n] & \xrightarrow{\eta} & C[V] & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & J(V) & \longrightarrow & O(C^n) & \xrightarrow{R} & O(V) & \to & 0 \end{array}$$

in which R and η are restriction maps.
It is known [9] that

a) R is open

b) $J(V) = I(V)\, O(\mathbf{C}^n)$

From b) we have

c) $\overline{I(V)} = J(V)$

Therefore, from a) and b) we infer that η is open

Let I be a given ideal in $C[V]$, $\varphi: C[V]^q \to I$ a continuous surjective homomorphism. We have to prove that φ is open.

We first consider the case where $V = \mathbf{C}^n$. Let $\tilde{\varphi}$ denote the canonical extension of φ to $O(\mathbf{C}^n)$. Then $\operatorname{Im} \tilde{\varphi} = IO(\mathbf{C}^n)$ is close in $O(\mathbf{C}^n)$ and $\operatorname{Ker} \tilde{\varphi} = \operatorname{Ker} \varphi O(\mathbf{C}^n)^q$. Hence, as above we infer that φ is open

In the general case, take $m_1, \dots, m_p \in I(V)$ and $f_1, \dots, f_p$, such that $\eta(f_i) = \varphi(e_i)$ for every $i = 1, \dots, p$ where $e_i = (\underbrace{0, \dots, 1}_{i}, \dots, 0) \in C[V]^q$ and $(f_1, \dots, f_p, m_1, \dots, m_p)$ generates $\eta^{-1}(I)$. Consider the commutative diagram

$$\begin{array}{ccc} \mathbf{C}[z_1, \ldots, z_n]^{q+p} & \xrightarrow{\ \psi\ } & \overset{-1}{\eta}(I) \\ \downarrow \tilde{\eta} & & \downarrow \\ \mathbf{C}[V]^q & \xrightarrow{\qquad\qquad} & I \end{array}$$

where ψ is defined by $(f_1, \ldots, f_p; m_1, \ldots, m_p)$ and

$$\tilde{\eta}(a_1, \ldots, a_q, b_1, \ldots, b_p) = (a_1 \mid V, \ldots, a_q \mid V)$$

Since ψ and η are open and $\tilde{\eta}$ is continuous we infer that φ is open.

3) Let $B \cong \mathcal{O}_n / J$ be an analytic algebra, where $\mathcal{O}_n$ is the algebra of germs of holomorphic functions at $O \in \mathbf{C}^n$, equipped with the topology induced by the topology of $C[[z_1, \ldots, z_n]]$. Then B satisfies the open mapping theorem [9].

4) Consider the noetherian normed algebra $[C[z], \|\cdot\|]$, with $\|\sigma\| = \sup\{ |\sigma(z)| ; |z| \leqslant 1/2 \}$ and the maximal ideal $I = (z+1)\mathbf{C}[z]$ in $C[z]$. Since

$$z^{2n+1} + 1 \in I \text{ and } z^{2n+1} + 1 \to 1 \text{ in } [\mathbf{C}[z], \|\cdot\|]$$

we infer that I is not closed. Hence $[C[z], \|\cdot\|]$ does not satisfy the open mapping theorem.

1. 2. THEOREM. *Let B be a m-convex algebra and M a Hausdorff Noether topological B-module satisfying the open mapping theorem. Then the following conditions are equivalent:*

(i) *every maximal ideal in B containing AnnM is closed*

(ii) *every submodule of M is closed.*

Proof. (i) $\to$ (ii). Since every complex m-convex algebra which is a field is isomorphic to C [13], by the Noetherianess of $B / AnnM$ [4] it is easy to see that $\dim B / I^n < \infty$ for every maximal ideal $I \supset AnnM$ and for every $n \geqslant 0$. Then, since M is finitely generated it follows $I^n M < \infty$ for all $n \geqslant 0$ and for every maximal ideal $I \supset AnnM$.

Assume now that N is a submodule of M of finite codimension. Since N is finitely generated there exists a continuous surjective B-linear map $\theta : B^n \to N$. From the hypothesis that M satisfies the open mapping theorem the map $\theta \oplus \beta : B^n \oplus \mathbf{C}^m \to M$, where $\beta : \mathbf{C}^m \to M/N$ is defined by a vector basis of M/N, is open. This implies that the map $\mathrm{id} \oplus \beta : N \oplus C^n \to M$ is isomorphic Hence, N is closed in M.

In general we write $N = \bigcap_{i=1}^{p} Q_i$, where Q_i are primary ideals of M. Then it suffices to show that Q_i is closed for every $i = 1, \ldots, p$.

Let Q be a given primary ideal in M. Take a maximal ideal I in B containing $r_M(Q)$, where $r_M(Q)$ denotes the radical of Q in M. Note that $I \supset AnnM$. By Krull intersection theorem [4] we have $Q = \bigcap_{n=0}^{\infty} (Q + I^n M)$. Hence Q is closed, since codim $(Q + I^n M) < \infty$ for every $n \geqslant 0$.

(ii) $\rightarrow$ (i). Let I be a given maximal ideal in B containing $AnnM$. Then $I \in$ Supp$M$ [4] and hence by Nakajama lemma [4], $IM \neq M$. Take $m_0 \in M \setminus IM$. Consider the continuous B-linear map $\widehat{m_0}: B \rightarrow M$ induced by m_0. By the maximality of I and since $m_0 \overline{\in} IM$ we infer thac $m_0^{-1}(Im_0) = I$. Hence I is closed. The theorem is proved.

1. 3. THEOREM. *Let B be a m-convex algebra and M a Hausdorff topological B-module. Then*

(ii) *If every submodule of M is a Baire closed space, then M is noetherian.*

(ii) *If B is barrelled and M is a noetherian complete B-module satisfying the open mapping theorem, then every submodule of N is closed.*

Proof. (i) Let $\{M_n\}$ be a given increasing sequence of submodules of M. From the hypothesis $\bigcup_{n=1}^{\infty} M_n$ is Baire, so we have $IntMn_0 \neq \phi$ for some n_0. This implies that $M_n = M_{n_0}$ for all $n \geqslant n_0$.

To prove (ii) we need the following

1. 4. LEMMA. *Let B be a barrelled m-convex algebra and M a Noether Hausdorff complete B-module. Then*

(i) *The spectrum $S(B/AnnM)$ is equicontinuous ;*

(ii) *Every maximal ideal in $\widehat{B/AnnM}$ is closed.*

Proof. (i) Since $B/AnnM$ is barrelled it suffices to show that $\sup | \hat{b}(\omega) | : w \in S(B/AnnM)\} < \infty$ for every $b \in B$. Conversely, there exists $b \in B$ and a sequence $\{\omega_n\} \subset S(B/AnnM)$ such that $| \hat{b}(\omega_n) | \rightarrow \infty$ as $n \rightarrow \infty$ For each n we find $\varphi_n \in O(C)$ such that $\varphi_n \;\; (\widehat{b(\omega_m)}) = \delta_{n,m}$. Since $\{\omega_n\} \subset$ supp$M$ it follows that for each n there exists $m_n \in M \setminus w_n M$. Let $\varphi_n(Z) = \sum_{i=b}^{\infty} a_i^n Z^i$. and $b_n = \sum_{i=0}^{\infty} a_i^n b^i \in \widehat{B}$... Since M is also a B-module we can consider the submodule N of M generated by $\{b_n m_n\}$. Then for some n_0 we have $b_{n_0+1} = \sum_{i=1}^{n_0} b_i \, m_i \, a_i$

Hence $0 \neq m_{n_0} \bmod \omega_{n+1} M = b_{n_0+1}(\omega_{n_0+1})\, m_{n_0+1} \bmod \omega_{n_0+1} M = b_{n_0+1}$ $m_{n_0+1} \bmod \omega_{n_0+1} M = \sum_{i=1}^{n_0} b_i\, m_i\, a_i \bmod \omega_{n_0+1} M = \sum_{i=0}^{n_0} b_i(\omega_{n_0+1})\, m_i\, a_i \bmod$ $\omega_{n_0+1} M = 0.$

This contradictihn shows that $S(B/AnnM)$ is equicontinuous.

(ii) Since $S(\widehat{B/AnnM}) = S(B/AngM)$ and the seminorm on $\widehat{B/AnnM}$ defined by $S(B/AnnM)$ is continuous the set of invertible elements of $\widehat{B/AnnM}$ is open. Hence every maximal ideal in $\widehat{B/AnnM}$ is closed.

As in the proof of Theorem 1. 2 it suffices to show that every primary submodule Q of M is closed. Let $r_M(Q)$ and $\widetilde{r_M(Q)}$ denote the radicals of Q in $B/AnnM$ and $\widehat{B/AnnM}$ respectively. Take a maximal ideal $\widetilde{I}$ in $\widehat{B/AnnM}$ containing $\widetilde{r_M(Q)}$. By lemma I. 4 $\widetilde{I}$ is closed and hence $I = \widetilde{I} \cap B/AnnM$ is a maximal ideal in $B/AnnM$ containing $r_M(Q)$. Thusby Krull intersection theorem we infer that $Q = \bigcap_{n=0}^{\infty} (Q + I^n M)$ is closed. The statement (ii) is proved.

I. 5. COROLLARY. *Let B be a Frechet m-convex algebra and M a Frechet B-module. Then M is noetherian of and only if every submodule of M is closed.*

Let K be a compact set in a complex space. By $O(K)$ we denote the algebra of germs of holomorphic functions on a neighbourhood of K. The algebra $\mathcal{O}(K)$ is equipped with induction topology:

$$\mathcal{O}(K) = \lim_{\rightarrow} \{\mathcal{O}(U) : U \supset K\}$$

Then $\mathcal{O}(K)$ is a DFN - space [3]. Since the sets of the form
$\mathrm{Conv}\,[U\,\{6 \in \mathcal{O}(U) : \|6\|_U < \varepsilon_U < 1,\ U \supset K\}]$
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form an idempotent neighbourhood basis of zero in $\mathcal{O}(K)$ it follows that $\mathcal{O}(K)$ is m-convex [13].

1. 6. COROLLARY. Let K be a compact set in a Riemann domain over a Stein manifold. Then every ideal in $\mathcal{O}(K)$ is closed.

Proof. For each neighbourhood U of K we denote by $\widetilde{U}$ the envelope of holomorphy of U[11] Put $\widehat{K}_{\widetilde{U}} = \{z \in \widetilde{U} : |f(z)| \leqslant \|f\|_K, \forall f \in \mathcal{O}(\widetilde{U})\}$

Then it is easy to check that

$$\mathcal{O}(K) = \lim_{\rightarrow} \mathcal{O}(U) = \lim_{\rightarrow} \mathcal{O}(\widetilde{U}) = \lim_{\rightarrow} \mathcal{O}(\widehat{K}_{\widetilde{U}})$$

Thus we can assume that K is holomorphically convex.

Then K can be written in the form $K = \bigcap_{1}^{\infty} P_i$, where P_i are analytic polyhedrons and hence P_i are semianalytic. Then $\mathcal{O}(P_i)$ are noetherian. Hence by theorem 1. 3 every ideal in $\mathcal{O}(K)$ is closed

2. NOETHERIAN TOPOLOGICAL OF FINITE DIMENSION

It is known [1] that every Banach-Noether algebra is finite dimensional. By considering either the Frechet m-convex algebra $C[[z]]$ of formal series of variable z or the m-convex DFN-algebra O_n of germs of holomorphic functions at $0 \in \mathbf{C}n$ we see that in general, this statement is not true. In this section we give a condition for a noetherian topological module to be finite dimensional.

Let B be a m-convex algebra and M a topological B-module. For each $b \in B$ we denote by b_M the continuous B-linear map of M into M induced by b. We say that b is a topological divizor of zero for M if there exists $\{m_\alpha\} \subset M$ such that $m_\alpha \not\rightarrow 0$ but $b_M \; (m_\alpha) \rightarrow 0$

By Spb_M we denote the spectrum of bM$\in$ HOM (M,M)

$Spb_M := \{ \lambda \in \mathbf{C} : b_M - \lambda$ is not invertible$\}$. We can prove the following.

2. 1. THEOREM. *Let B be a barrelled m-convex algebra and M a Hausdorff — Noether B-complete B-module such that every submodule of M is barrelled. Suppose that every closed submodule N of M and for every* $b \in B$ *there exists* $\lambda_b \in \partial Spb_{M/N}$. *such that* $b_{M/N} - \lambda_b$ *is a topological divizor of zero for M/N. Then M is finite dimensional.*

Proof. a) We first consider the case where $B/AnnM$ does not contain a divizor of zero for M. For each $b \in B$ take $\lambda_b \in \partial \, Spb_M$ such that $b_M - \lambda_b$ is a topological divizor of zero for M. By Theorem 1.2, Im $(b_M - \lambda_b)$ is closed. Since $B/AnnM$ does not contain a divizor of zero for M, it follows from the open mapping theorem that $b_M = \lambda_b$. It is easy to see that the form: $b$mod-$AnnM \rightarrow \lambda_b$ defines an isomorphism of $B/AnnM$ onto $\mathbf{C}$. Hencé dim$M < \infty$.

b) Assume now that every divizor of zero for M in $B/AnnM$ is nilpotent. Let $J = J(B/AnnM)$ denote the nilradical ideal of $B/AnnM$. Since $B/AnnM$ is noetherian we have $J^q = 0$ for sufficiently large q. Consider the Hausdorff — Noether topological B/J—module MJ/M. It is easy to check that $Ann(M/JM) = 0$ and $J(B/J) = 0$.

Thus we can assume $0 \neq \bigcap_{i=1}^{p} Q_i$ where Q_i are submodules of M/JM such that B/J does not contain a divizor of zero for $M/JM/Q_i = M/JM + Q_i$. Since $M/JM + Q_i$ is B—complete and every its closed submodule is also barrelled, by a, we have $\dim M/JM + Q_i < \infty$ for every $i = 1, \ldots, p$. Hence $\dim M/JM < \infty$. Then, in view of the relation $Ann M/JM = 0$, it follows that $\dim B/J < \infty$. Hence $\dim J^m M^{m+1} M < \infty$ for every $m \geqslant 0$. Since $J^q = 0$ we infer that $\dim M < \infty$

c) In the general case, we write $0 = \bigcap_{i=1}^{p} M_i$, where M_i are closed submodules of M such that every divizor of zero in B for M/M_i is nilpotent, Then by b) we have $\dim M/M_i < \infty$ for every $i = 1, \ldots, p$ and hence $\dim M < \infty$.

The theorem is proved.

Since every Frechet space (resp. every closed subspace of a DFN—space) is B — complete and barrelled [12] by Theorem 2.1 we can deduce the following

2.2. COROLLARY. *Let B be a barrelled m-convex algebra and M a Frechet (resp. a DFN —) B-module and let M be noetherian. Then* $\dim M < \infty$ *if and only if for every* $b \in B$ *there exists* $\lambda_b \in \partial Spb\, M/N$ *such that* $b_{M/N} - \lambda_b$ *is a topological divizor of zero for* M/N.

2.4. **Remark.** It is known [1] that when M is a Banach B-module, $b_M - \lambda$ is not a topological divizor of zero for all $b \in B$ and $\lambda \in \partial Spb M$

3. NOETHERIAN TOPOLOGICAL MODULES OVER $\mathcal{O}(X)$

Let X be a complex space having a countable topology

We denote by $\mathcal{O}(X)$ the Frechet algebra of holomorphic functions on X equipped with $\mathcal{O}$ the compact—open topology. In this section we investigate noetherian topological modules over $\mathcal{O}(X)$ We first prove the following

3.1. THEOREM. *Let X be a complex reduced space and B a closed subalgebra of* $\mathcal{O}(X)$. *Then every noetherian ideal in B is finite dimensional.*

Proof. We first prove that

$$\text{Supp } \sigma \subset V\,(AnnI) \text{ for all } \sigma \in I$$

Where I is a closed ideal in B and $V(AnnI) \equiv \{z \in X : \sigma(z) = 0 \ \forall \sigma \in I\}$.

Let $\sigma \in I$ and $z \in X$, $\sigma(z) \neq 0$, Let $\beta \in AnnI$. Take an irreductible branch V of X such that $z \in V$. Then by the relation $\beta\sigma = 0$ we infer that $\beta \mid V = 0$. Hence $z \in V\,(AnnI)$. Thus by Lemma 1.4 we have

$\| \sigma \| = \sup \{ | \sigma (z) | : z \in X \} = \sup \{ | \sigma(z) | : z \in V (Ann I) \} < \infty$ for all $\sigma \in I$ and

$\| b \| = \sup \{ | b(z) | : z \in V (Ann I) \} < \infty$ for all $b \in B$ Observe that I is complete with respect to the norm $\| . \|$ and it is a Banach – Noether A-module, where A denotes the completion of $B / \| \ \| (o)^{-1}$ with respect to the norm $\| . \|$. By Theorem 2.1 we have $\dim I < \infty$.

The theorem is proved.

Let X be a complex space. We say that X is K – separable if $\dim \{y \in X: f(x) = f(y), \forall f \in \mathcal{O}(X)\} = 0$ for every $x \in X$. The space X is called perfect if $\mathcal{O}_x$ is a Cohen – Macaulay ring for every $x \in X$.

3.2. THEOREM. *Let X be a perfect complex space which is K-separable. Let $H^p(X, \mathcal{O})$ be noetherian for all $p \geq 1$ and suppose $\bigcap_{p=0}^{\infty} NilH^p(X, \mathcal{O})$ is closed. Then $H^p(X, \mathcal{O}) =$ for every $p > 1$.*

Proof. a) We first prove that $\dim \bigcup_{p=0}^{\infty} V(NilH^p(X, \mathcal{O})) = 0$. Since $H^p(X,\mathcal{O}) = 0$ for every $p > 2\dim X$, it suffices to show that $\dim V(NilH^p(X, \mathcal{O})) = 0$ for every $p \geq 0$. By the relation $V(NilH^p(X,\mathcal{O})) = V(AnnH^p(X, \mathcal{O}))$ and by the Noetherianess of $\mathcal{O}(X)/AnnH^p(X, \mathcal{O})$ we infer that $\delta(V(NilH^p(X,\mathcal{O}))$, where $\delta: X \to SO$ denotes the canolical map, is relatively compact in $SO(X)$. Hence every holoporphic function on X is constant on every irreducible branch of $V(NilH^p(X, \mathcal{O}))$. Since X is K – separable it follows that $\dim V(NilH^p(X,\mathcal{O})) = 0$.

b) Let Z be a given irreducible branch of X. Put

$$A_Z = \{f \in \bigcap_{p=0}^{\infty} NilH^p(X, \mathcal{O}): f | Z \neq \text{const}\}$$

We claim that A_Z is dence in $\bigcap_{p=0}^{\infty} NilH^p(X, \mathcal{O})$. For suppose there exists $g \in \bigcap_{p=0}^{\infty} NilH^p(X, \mathcal{O}) \setminus \overline{A}_Z$. Then there exist a neighbourhood U of g in $\bigcap_{p=0}^{\infty} NilH^p(X, \mathcal{O})$ such that $U \cap \overline{A}_Z = \emptyset$.. Thus $f | Z = \text{const}$ for every $f \in U$ and hence $f | Z = \text{const}$ for every $f \in \bigcap_{p=0}^{\infty} NilH^p(X, \mathcal{O})$. This is imposible, by a).

c) By Baire theorem $\cap_Z A_Z$ is dense in $\bigcap_{p=0}^{\infty} NilH^p(X, \mathcal{O})$.

d) Let $f \in \bigcap_Z A_Z$. Take n such that $f^n \; H^p(X, \mathcal{O})$ 0 for every $p \geqslant 1$

Since X is perfect, the sequence

$$0 \to \mathcal{O} \xrightarrow{f^n} \mathcal{O} \to \mathcal{O}/f^n \to \mathcal{O}$$

is exact. From the exactness of cohomology sequence

$$\ldots \to H^p(X, \mathcal{O}) \xrightarrow{f^n} H^p(X, \mathcal{O}) \to H^p(X, \mathcal{O}/f^n\mathcal{O}) \to H^{p+1}(X, \mathcal{O})) \to \ldots$$

we infer that for every $p \geqslant 0$ the sequence

$$0 \to H^p(X, \mathcal{O}) \to H^p(X, \mathcal{O}/f^n\mathcal{O}) \to H^{p+1}(X, \mathcal{O}) \to 0$$

is exact.

Then

$$SuppH^p(X, \mathcal{O}/f^n\mathcal{O}) = SuppH^p(X, \mathcal{O}) \, U SuppH^{p+1}(X, \mathcal{O})$$

[4] Hence,

$$NilH^p(X, \mathcal{O}/f^n\mathcal{O}) = NilH^p(X, \mathcal{O}) \cap NilH^{p+1}(X, \mathcal{O})$$

[4] for every $p \geqslant 0$.

Thus we have:

$$\bigcap_{p=0}^{\infty} NilH^p(X, \mathcal{O}/f^n\mathcal{O}) = \bigcap_{p=0}^{\infty} NilH^p(X, \mathcal{O}).$$

and hence $\bigcap_{p=0}^{\infty} H^p(X, \mathcal{O}/f^n\mathcal{O})$ is closed in $H^0(X, \mathcal{O}/f^n\mathcal{O})$.

Since $\dim V(f) < \dim X$ and the space $(V(f), \mathcal{O}/f), \mathcal{O}/f^n\mathcal{O})$ satisfies the conditions in Theorem 3. 2, by induction hypothesis we have $H^p(X, \mathcal{O}/f^n\mathcal{O}) = .0$ for every $p \geqslant 1$

The theorem is proved.

4. NOETHERIAN TOPOLOGICAL MODULES OVER STEIN ALGEBRAS:

We recall that a Frechet algebra B is called Stein if it is isomorphic to the Frechet algebra of holomorphic Functions on a Stein space. Thus B is a Stein algebra if and only if $S(B)$ has a Stein space structure such that $B \cong \mathcal{O}(S(B))$.

4. 1. THEOREM. *Let B be a Stein algebra. Then*

i) *If M is a noetherian B-module such that AnnM is closed in B. then* $\dim M < \infty$

ii) *If* $\dim B = \infty$, *then there exists an infinite dimensional noetherian B-module.*

Proof. (i) let $J = AnnM$. Then M is a noetherian module over the Frechet Noether—algebra B/J. To prove $\dim M < \infty$ it suffices to show that $\dim B/J < \infty$.

Let X be a Stein space such that $B \cong O(X)$. Let $\tilde{J}$ denote the coherent ideal sheaf on X generated by J. By Cartan theorem B, we have.

$$\dim B/J = \dim H^0(X, O/\tilde{J})$$

Since $\operatorname{Supp} O/J = V(J) = S(O(X)/J)$ we infer that $V(J)$ is compact. Hence $\dim H^0(X, O/J) < \infty$

(ii) Take a discrete sequence $\{z_n\} \subset X$. Then $O(V) \cong C^\infty$, where $V = \{z_n\}$. Let J denote the ideal in $O(V)$ generated by $\{\chi A: \# (V \setminus A) = \infty\}$. Put $M = O(X)/R^{-1} J$. where $R: O(X) \to O(V)$ is the restriction map. Then $M \cong C^\infty/J$. We prove that M is a noetherian $O(X)$-module and $\dim M = \infty$. Obviously $\dim M = \infty$.

Let I be a given non-zero ideal in M.

$$\text{Take } h = \sum_{i=1}^{r} \chi_{A_i} f_i \in R^{-1}(I) \setminus J$$

We write

$$h = \chi_{(A_1 \setminus A_2) \cap S_1} f_1 + \chi_{A_1 \cap A_2 \cap S_2}(f_1 + f_2) + \chi_{(A_2 \setminus A_1) \cap S_3} f_2 + \\ + \sum_{i=3}^{r} \chi_{A_i} f_i$$

where $S_1 = \operatorname{Supp} f_1$, $S_2 = \operatorname{Supp}(f_1 + f_2)$, $S_3 = \operatorname{Supp} f_3$,.

We may assume that $\chi_{(A_1 \setminus A_2) \cap S_1} f_1 \cdot \chi_{A_1 \cap A_2 \cap S_2}(f_1 + f_2)$.

$\chi_{(A_2 \setminus A_1) \cap S_3} f_2 \overline{\in} J$. Then

$$\chi_{(A_1 \setminus A_2) \cap S_1} + \chi_{(A_1 \cap A_2 \cap S_2} + \chi_{(A_2 \setminus A_1) \cap S_2} + \sum_{i=3}^{r} \chi_{A_i} g f_i \in R^{-1}(I) \text{for}$$

some $g \in O(V)$ and

$$\chi_{(A_1 \setminus A_2) \cap S_1 \cup A_1 \cap A_2 S_2 \cup (A_2 \setminus A_1) \cap S_3} \overline{\in} J.$$

Continuing this process we find a subset A of V such that

$\chi_A \in R^{-1}(I) \setminus J$ and hence $\# (V \setminus A) < \infty$.

Since $V = A \cup (V \setminus A)$ and $\# (V \setminus (V \setminus A)) = \infty$ it follows that $\chi_V \bmod J = \chi_A \bmod J$. Thus $\chi_A \bmod J$ generates I, hence M is noetherian.

4. 2. THEOREM. *Let B be a Stein algebra. Then every closed ideal in B is finitely generated if and only if $\dim S(B) \leqslant 1$.*

Proof. Let $\dim S(B) \leqslant 1$. Let I be a given closed ideal in B. We denote by $\widetilde{I}$ the coherent ideal sheaf on $S(B)$ generated by I. Then by Cartan theorem B we have $H^0(S(B), \widetilde{I}) = I$. To prove that I is finitely generated it suffices by a result of Forster [6] to show that

$$d = \sup \{ \widetilde{I}_x / m_x \widetilde{I}_x \; : \; x \in S(B) \} < \infty$$

where m_x denotes the maximal ideal in O_x

Let $x \in S(B)$, Take a neighbourhood U of x and $b \in B$ such that $b: U \to V$, with V being an open disc in $\mathbf{C}$, is an analytic covering and $b^{-1} bx = \{x\}$. Since the map $(b_* \widetilde{I})_y / m_y (b_* \widetilde{I})_y \to I_x / m_x I_x$, where $y = b(x)$, is surjective and $\dim (b_* \widetilde{I})_y / m_y (b_* \widetilde{I})_y = 1$ we infer that $\dim I_x / m_x I_x \leqslant 1$. Hence $d \leqslant 1$.

Assume now that $\dim S(B) \geqslant 2$. Take a discrete sequence $\{x_n\} \subset R(S(B))$, where $R(S(B))$ is the regular locus of $S(B)$. Since $\dim S(B) \geqslant 2$, we have

$$\dim m^n_{x_n} / m^{n+1}_{x_n} \geqslant n \text{ for all } n \geqslant 1.$$

Consider the coherent analytic sheaf $\mathcal{S}$ on $S(B)$ given by

$$\mathcal{S}x = \begin{cases} 0 & \text{if } x \neq x_n \\ m^n_{x_n} & \text{if } x = x_n \end{cases}$$

Then $H^0(S(B), \mathcal{S})$ is a closed ideal in B which is not finitely generated.

The theorem is proved.

4. 3. **Remark.** According to result of Gleason [7] it follows that every closed ideal of a uniform algebra B is finitely generated if and only if $\dim B < \infty$.

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