DEGREE BOUND FOR THE DEFINING EQUATIONS OF
PROJECTIVE MONOMIAL CURVES

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1. INTRODUCTION

Projective monomial curves often occur in Algebraic Geometry. They are
given parametrically by sets of monomials of some degree in two indeterminates.
One also calls them projections of one-dimensional Veronese varieties. For
example, the first imperfect prime ideal given by Macaulay is the defining prime
ideal of a monomial curve in $P^3$. To study such a curve one should know its
defining equations, i.e. a basis of its defining prime ideal. For example,
Störckrad and Vogel [5] could show that arithmetically Cohen-Macaulay monomial
curves in $P^3$ are set-theoretically complete intersections only after they knew
certain minimal bases of the defining prime ideals of these curves given in [2].
Usually, it is very difficult to compute a basis for the defining prime ideal of
a projective monomial curve, even in $P^3$, see [1].

The aim of this paper is to show that there is an effective way to determine
the defining equations of a large class of projective monomial curves. That is
based on a degree bound for the elements of all minimal bases of the defining
prime ideal of such curves.

2. MAIN RESULT

Throughout this paper, let $V$ be a projective monomial curve given para-
metrically by a set $M_v$ of monomials of some degree, say $d$, in two indetermin-
ates $t$ and $s$ such that

$$t^d, t^{d-1}s, t^{d-1}s^d \in M_v.$$

Let $S$ denote the multiplicative monoid generated by the monomials of $M_v$.
Then it is easily seen that there exist positive integers \( n \) such that \( t^{nd-a} s^a \in \mathcal{S} \) for all \( a = 0, 1, \ldots, nd \). Let \( n_\mathcal{V} \) denote the minimum of all such integers \( n \). Then we have the following

**Theorem 1.** There is a basis for the defining prime ideal of \( \mathcal{V} \) consisting of binomials of degree \( \leq n_\mathcal{V} + 1 \).

By a binomial we understand a difference of two monomials of the same degree. Clearly, Theorem 1 yields an effective way to determine the defining equations of \( \mathcal{V} \) because one can easily establish an algorithm to compute \( n_\mathcal{V} \) and all binomials of degree \( \leq n_\mathcal{V} + 1 \) vanishing at \( M_\mathcal{V} \). These binomials will form a basis for the defining prime ideal of \( \mathcal{V} \). It should be noticed that a basis found by this way need not be a minimal basis and \( n_\mathcal{V} \) is not always the maximal degree appearing in every minimal basis of the defining prime ideal of \( \mathcal{V} \), see example 3 below.

**Remark.** The Hilbert polynomial of \( \mathcal{V} \) always has the following form:

\[
P(n) = nd + 1,
\]

and \( n_\mathcal{V} \) is the last integer \( n \) such that \( P(n) \) coincides with the Hilbert function of \( \mathcal{V} \) at \( n \).

**Proof of Theorem 1.** To each element \( t^{d-a} s^a \) of \( M_\mathcal{V} \) we assign an indeterminate \( X_a \). Let \( k[X] \) denote the polynomial ring over a field \( k \) in the indeterminates \( X_a \).

Let \( k[S] \) denote the subalgebra of the polynomial ring \( k[l, s] \) generated by the elements of \( M_\mathcal{V} \). Then there is a natural homomorphism \( \varphi \) from \( k[X] \) onto \( k[S] \) which sends \( X_a \) to \( t^{d-a} s^a \). Let \( P_\mathcal{V} \) denote the kernel of \( \varphi \). Then \( P_\mathcal{V} \) is the defining prime ideal of \( \mathcal{V} \) in \( k[X] \). Since \( P_\mathcal{V} \) has a basis consisting of binomials [3], to prove Theorem 1 we need only show that every binomial of \( P_\mathcal{V} \) of degree \( n \geq n_\mathcal{V} + 2 \) can be expressed as a linear combination of binomials of \( P_\mathcal{V} \) of degree \( n - 1 \).

Let \( F, G \) be two monomials of degree \( n \geq n_\mathcal{V} + 2 \) such that \( F - G \) belongs to \( P_\mathcal{V} \). Then we have

\[
\varphi(F) = \varphi(G) = t^{nd-m} s^m
\]

for some non-negative integer \( m \leq nd \). For convenience, we write

\[
F = X_a X_b X_c F_1,
\]

\[
G = X_e G_1,
\]

where \( F, G \) are monomials of degree \( n-3, n-1 \), respectively. Since \( m = a + b + c + \cdots = e + \cdots \), we may assume, without restriction, that \( a \geq b \geq e \).

Note that

\[
t^{(n-2)d-m+b+c} s^m-b-c = \varphi(X_a F_1),
\]

\[
t^{(n-2)d-m+a+c}s^m-a-c = \varphi(X_b F_1)
\]
are elements of $S$. Then, since $n-2 \geq n_v$ and

$$0 \leq m-a-c \leq m-e-c \leq m-b-c \leq (n-2)d,$$

we must have $t_{(n-2)d-m-e-c} s_{m-e-c} \in S$. Thus, we can find a monomial $H$ of degree $n-2$ in $k[X]$ such that

$$t_{(n-2)d-m-e-c} s_{m-e-c} = \varphi(H).$$

Now, looking at the powers of $t$ and $s$, we can easily verify that

$$\varphi(X_e H) = \varphi(X_a X_b F_1),$$

$$\varphi(X_c H) = \varphi(G_1).$$

From this it follows that $X_e H = X_a X_b F_1$ and $X_c H = G_1$ are binomials of $P_v$ of degree $n-1$, and we have the expression

$$F - G = X_c (X_a X_b F_1 - X_e H) - X_e (X_c H - G_1),$$

as required. The proof of Theorem 1 is now complete.

3. APPLICATIONS

It is known [6] that $V$ is an arithmetically Buchsbaum curve if and only if $n_v \leq 2$. Hence we have the following interesting consequence of Theorem 1:

**COROLLARY 1.** Let $V$ be an arithmetically Buchsbaum curve. Then there is a basis for the defining prime ideal of $V$ consisting of binomials of degree $\leq 3$.

A famous example for this result is the prime ideal of Macaulay which defines the projective monomial curve given parametrically by the set $\{t^4, t^3s, t^2s^2, ts^3, t^4\}$. This is an arithmetically Buchsbaum curve and the prime ideal of Macaulay has the following basis

$$x_0 x_4 - x_1 x_3, x_0 x_3^2 - x_1^2 x_4, x_0^2 x_3 - x_1^2, x_1 x_4^2 - x_3^3.$$

There also exist projective monomial curves whose defining prime ideals are generated by binomials of degree $\leq 3$ but they are not arithmetically Buchsbaum curves, see Example 3 below.

Now, we will apply Theorem 1 to determine minimal bases for the defining prime ideals of some projective monomial curves. We are interested in curves given parametrically by sets of monomials of the form

$$\{t^d, t^{d-1}s, \ldots, t^d-\alpha s^\alpha, t^{d-b} s^b, \ldots, ts^{d-1} s^d\}.$$

$a < b \leq d-1$ Without restriction, we may assume that $a + b \leq d$. Then we can compute $n_v$ in terms of $a$ and $b$.

**COROLLARY 2.** Let $V$ be as above. Then the defining prime ideal of $V$ has a basis consisting of binomials of degree $\leq \lfloor (b-2)/a \rfloor + 2$. 

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Proof. Let \( t^{nd - c}s^c \) be an arbitrary element of \( S \). Then \( t^{nd - c}s^c \) is a product of \( n-m \) monomials of the set

\[
M_1 = \{ t^d, t^{d-1}s, \ldots, t^{d-a}s^a \}
\]

with \( m \) monomials of the set

\[
M_2 = \{ t^{d-b}s^b, \ldots, ts^{d-1}, s^d \}
\]

for some \( m \leq n \). Clearly, we have \( mb \leq c \leq (n-m) a + md \). Conversely, for every integer \( c \) satisfying the above inequalities for some \( m \leq n \), the element \( t^{nd - c}s^c \) belongs to \( S \) and can be written as a product of \( n-m \) monomials of \( M_1 \) with \( m \) monomials of \( M_2 \). Thus, to compute \( n_V \), we need only check when the intervals \([mb, (n-m)a + md], m = 0, \ldots, n\) contain the integers 0, \ldots, \( nd \). That happens if and only if \( mb - 1 \) is contained in the interval \([(m-1)b, (n-m+1)a + (m-1)d]\), or, equivalently,

\[
mb - 1 \leq (n-m+1)a + (m-1)d
\]

for all \( m = 1, \ldots, n \). From this it follows that

\[
b + a - 2 < (n+1)a + (m-1)(d-a-b)
\]

for all \( m = 1, \ldots, n \). Since \( d-a-b > 0 \), the above condition is satisfied if and only if

\[
b + a - 2 < (n+1)a
\]

or, equivalently,

\[
[(b + a - 2)/a] < n + 1.
\]

Therefore, \( n_V = [(b - 2)/a] + 1 \), and the statement of Corollary 2 follows from Theorem 1.

Now, since the maximal degree of the elements of a minimal basis of a homogeneous ideal is an invariant of this ideal \(^4\text{, Satz } 2\text{, p. 37}\) and since every basis can be reduced to a minimal one, we need only determine the type of the binomials of a basis of the defining prime ideal of \( V \) because by restricting the degree of this type \( \leq [(b - 2)/a] + 2 \), we will find a minimal basis.

In the following examples, we shall use the notations of the proof of Theorem 1.

Example 1. Let \( V \) be a projective monomial curve given parametrically by the set

\[
\{ t^d, t^{d-1}s, ts^{d-1}, s^d \},
\]

\( d \geq 3 \). Then \( n_V = d - 2 \). It is easy to see that \( P_V \) has a basis consisting of the binomial

\[
X_0X_d - X_1X_{d-1}
\]

and binomials of the type

\[
X_0^{n-r}X_1^rX_d - X_0^{n-s}X_1^sX_d, n \geq r, s \geq 1.
\]
Now we will check which positive integers \( n, r, s \) satisfy the equation

\[
  r(d - 1) = n - s + sd.
\]

Since we may assume \( n \leq d - 1, \) \( 0 \leq n - s \leq d - 1. \) Hence

\[
  s = [r(d - 1)/d] = r - 1.
\]

From this it follows that

\[
  r(d - 1) = n - r + 1 + (r - 1)d.
\]

Hence \( n = d - 1. \) Thus, \( P_V \) has the following basis:

\[
  X_0 X_d - X_1 X_{d-1},
\]

\[
  X^{d-r-1} X_r - X^{d-r} X_{d-r}, \quad r = 1, \ldots, d - 1.
\]

It is easily seen that this basis is a minimal one, of. [4, Beispiel 1, p. 182].

**Example 2.** Let \( V \) be a projective monomial curve given parametrically by the set

\[
  \{ t^d, t^{d-1} s, t^{d-2} s^2, t s^{d-1}, s^d \},
\]

\( d \geq 4. \) Then \( n_V = d - 3. \) It is easy to see that \( P_V \) has a basis consisting of the binomials

\[
  X_0 X_d - X_1^2,
\]

\[
  X_0 X_d - X_1 X_{d-1},
\]

\[
  X_1 X_d - X_2 X_{d-1},
\]

\[
  X_2 X^{d-3}_d - X^{d-2}_{d-1},
\]

and binomials of the following types

\[
  X^{n-r-1}_0 X_r X^r_{d-1} - X^n_2, \quad n > r > 1,
\]

\[
  X^{n-r}_0 X^{r}_{d-1} - X^{n-r-s}_2 X^s_d, \quad n > r, s > 1.
\]

For the first type, we have to solve the equation

\[
  1 + r(d - 1) = 2n.
\]

It has solutions if and only if \( d \) is even. In that case, we only get a binomial associated with the solution \( r = 1, n = d/2 \) because \( n \) should be chosen as small as possible.

For the second type, we have the equation

\[
  r(d - 1) = 2(n - s) + sd,
\]

or, equivalently,

\[
  (r - s)(d - 1) + s = 2n.
\]

Thus, since \( n > s > 1, \) \( n \) takes the minimum value if \( r = s + 1, \) i.e.

\[
  n = [(d + s - 1)/2].
\]

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According to our analysis, $P_V$ then has the following basis:

**Case $d = 2t$:**

\[
\begin{align*}
X_0 X_2 - X_1^2 \\
X_0 X_d - X_1 X_{d-1}, \\
X_1 X_d - X_2 X_{d-1}, \\
X_2 X_{d-3} - X_{d-1}, \\
X_0^{t-2} X_1 X_{d-1} - X_2^t, \\
X_0^{t-1} X_{2i} - X_2^{t-1} X_{d-1}, i = 1, \ldots, t - 1,
\end{align*}
\]

**Case $d = 2t + 1$:**

\[
\begin{align*}
X_0 X_2 - X_1^2 \\
X_0 X_d - X_1 X_{d-1}, \\
X_1 X_d - X_2 X_{d-1}, \\
X_2 X_{d-3} - X_{d-1}, \\
X_0^{t-1} X_{2i} - X_2^{t-1} X_{d-1}, i = 0, \ldots, t - 1.
\end{align*}
\]

It is easily seen that this basis is a minimal one.

**Example 3.** Let $V$ be a projective monomial curve given parametrically by the set

\[
\{t^d, t^{d-1} s, t^{d-2} s^2, t^2 s^{d-2}, t s^{d-1}, s^d\},
\]

$d \geq 5$. Then $r_V = \lceil (d - 2) / 2 \rceil$. It is easy to see that $P_V$ has a basis consisting of the quadrics

\[
\begin{align*}
X_0 X_2 - X_1^2, \\
X_0 X_{d-1} - X_1 X_{d-2}, \\
X_0 X_d - X_2 X_{d-2}, \\
X_0 X_d - X_1 X_{d-1}, \\
X_1 X_d - X_2 X_{d-1}, \\
X_{d-2} X_d - X_{d-1},
\end{align*}
\]

and binomials of the following types

\[
\begin{align*}
X_0^{n-r-1} X_1 X_{d-2} - X_2^{n-s} X_d, \\
X_0^{n-r} X_{d-2} - X_2^{n-s} X_d,
\end{align*}
\]

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\[ X^n_{0} - X^n_{r_{d-2}} X^r_{2} X^{n-s-1}_{d-1} X^s_{d}, \]
\[ X^n_{0} - X^n_{r_{d-1}} X^r_{1} X^{n-1}_{2}, \]
\[ X^n_{1} - X^n_{r_{d-2}} X^{n-1}_{d-1}. \]

Processing as in Example 2, we can easily verify that only the following binomials come into consideration:

Case \( d = 2t \):
\[ X^{t-r-1}_{0} X^r_{d-2} - X^{t-r}_{2} X^{r-1}_{d}, \]
\( r = 1, \ldots, t-1 \).

Case \( d = 2t + 1 \):
\[ X^{t-1}_{0} X^r_{d-1} - X^t_{1} X^{t-1}_{2}, \]
\[ X^t_{1} X^{t-1}_{d-2} X^{d-1}_{d}, \]
\[ X^{t-r}_{0} X^r_{d-2} - X^{t-r+1}_{2} X^{r-1}_{d}, \]
\( r = 1, \ldots, t-1 \).
\[ X^{r-1}_{0} X^{t-r+1}_{d-2} X^{t-r}_{d-1} X^r_{d}, \]
\( r = 1, \ldots, t-1 \).

These binomials together with the above quadrics will form a minimal basis for \( P_V \).

In particular, if \( d = 2t \), the maximal degree of the elements of this basis is equal to \( n_V = t - 1 \). This shows that the bound \( n_V + 1 \) for this maximal degree is not always attained.

Moreover, if \( d = 8 \), we get an example of a projective monomial curve defined by binomials of degree \( \leq 3 \) but it is not an arithmetically Buchsbaum curve.

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REFERENCES

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