ACTA MATHEMATICA VIETNAMICA

Vol. 9 No 2 (1984)

DEGREE BOUND FOR THE DEFINING EQUATIONS OF PROJECTIVE MONOMIAL CURVES

NGO VIET TRUNG

1. INTRODUCTION

Projective monomial curves often occur in Algebraic Geometry. They are given parametrically by sets of monomials of some degree in two indeterminates. One also calls them projections of one-dimensional Veronese varieties. For example, the first imperfect prime ideal given by Macaulay is the defining prime ideal of a monomial curve in P^3. To study such a curve one should know its defining equations, i.e. a basis of its defining prime ideal. For example, Stückrad and Vogel [5] could show that arithmetically Cohen—Macaulay monomial curves in P^3 are set-theoretically complete intersections only after they knew certain minimal bases of the defining prime ideals of these curves given in [2]. Usually, it is very difficult to compute a basis for the defining prime ideal of a projective monomial curve, even in P^3, see [1].

The aim of this paper is to show that there is an effective way to determine the defining equations of a large class of projective monomial curves. That is based on a degree bound for the elements of all minimal bases of the defining prime ideal of such curves.

2. MAIN RESULT

Throughout this paper, let V be a projective monomial curve given parametrically by a set M_V of monomials of some degree, say d, in two indeterminates t and s such that

$$t^d,\ t^{d-1}s,\ ts^{d-1},\ s^d \in M_V.$$

Let S denote the multiplicative monoid generated by the monomials of M_V.

Then it is easily seen that there exist positive integers n such that $t^{nd-a}s^a \in \dot{S}$ for all $a = 0,1,..., nd$, Let n_V denote the minimum of all such integers n. Then we have the following

THEOREM 1. *There is a basis for the defining prime ideal of V consisting of binomials of degree* $\leqslant n_V + 1$.

By a binomial we understand a difference of two monomials of the same degree. Clearly, Theorem 1 yields an effective way to determine the defining equations of V because one can easily establish an algorithm to compute n_V and all binomials of degree $\leqslant n_V + 1$ vanishing at M_V. These binomials will form a basis for the defining prime ideal of V. It should be noticed that a basis found by this way need not be a minimal basis and n_V is not always the maximal degree appearing in every minimal basis of the defining prime ideal of V, see example 3 below.

Remark. The Hilbert polynomial of V always has the following form:

$$P(n) = nd + 1,$$

and n_V is the last integer n such that $P(n)$ coincides with the Hilbert function of V at n.

Proof of Theorem 1. To each element $t^{d-a}s^a$ of M_V we assign an indeterminate X_a. Let $k[X]$ denote the polynomial ring over a field k in the indeterminates X_a. Let $k[S]$ denote the subalgebra of the polynomial ring $k[t, s]$ generated by the elements of M_V Then there is a natural homomorphism φ from $k[X]$ onto $k[S]$ which sends X_a to $t^{d-a}s^a$. Let P_V denote the kernel of φ. Then P_V is the defining prime ideal of V in $k[X]$. Since P_V has a basis consisting of binomials [3], to prove Theorem 1 we need only show that every binomial of P_V of degree $n \geqslant n_V + 2$ can be expressed as a linear combination of binomials of P_V of degree $n - 1$.

Let F, G be two monomials of degree $n \geqslant n_V + 2$ such that $F - G$ belongs to P_V. Then we have

$$\varphi(F) = \varphi(G) = t^{nd-m}s^m$$

for some non-negative integer $m \leqslant nd$. For convenience, we write

$$F = X_a X_b X_c F_1,$$

$$G = X_e G_1,$$

where F, G are monomials of degree $n-3$, $n-1$, respectively. Since $m = a + b + c + ... = e + ...$, we may assume, without restriction, that $a \geqslant e \geqslant b$. Note that

$$t^{(n-2)d-m+b+c}s^{m-b-c} = \varphi(X_a F_1),$$

$$t^{(n-2)d-m+a+c}s^{m-a-c} = \varphi(X_b F_1)$$

are elements of S. Then, since $n-2 \geqslant n_V$ and

$$0 \leqslant m-a-c \leqslant m-e-c \leqslant m-b-c \leqslant (n-2)d,$$

we must have $t^{(n-2)d-m-e-c}\, s^{m-e-c} \in S$. Thus, we can find a monomial H of degree $n-2$ in $k[X]$ such that

$$t^{(n-2)d-m-e-c}\, s^{m-e-c} = \varphi(H).$$

Now, looking at the powers of t and s, we can easily verify that

$$\varphi(X_e H) = \varphi(X_a X_b F_1),$$
$$\varphi(X_c H) = \varphi(G_1).$$

From this it follows that $X_e H - X_a X_b F_1$ and $X_c H - G_1$ are binomials of P_V of degree $n-1$, and we have the expression

$$F - G = X_c(X_a X_b F_1 - X_e H) - X_e(X_c H - G_1),$$

as required. The proof of Theorem 1 is now complete.

3. APPLICATIONS

It is known [6] that V is an arithmetically Buchsbaum curve if and only if $n_V \leqslant 2$. Hence we have the following interesting consequence of Theorem 1:

COROLLARY 1. *Let V be an arithmetically Buchsbaum curve. Then there is a basis for the defining prime ideal of V consisting of binomials of degree* $\leqslant 3$.

A famous example for this result is the prime ideal of Macaulay which defines the projective monomial curve given parametrically by the set $\{t^4, t^3s, ts^3, t^4\}$. This is an arithmetically Buchsbaum curve and the prime ideal of Macaulay has the following basis

$$X_0 X_4 - X_1 X_3\ ,\ X_0 X_3^2 - X_1^2 X_4\ ,\ X_0^2 X_3 - X_1^3,\ X_1 X_4^2 - X_3^3.$$

There also exist projective monomial curves whose defining prime ideals are generated by binomials of degree $\leqslant 3$ but they are not arithmetically Buchsbaum curves, see Example 3 below.

Now, we will apply Theorem 1 to determine minimal bases for the defining prime ideals of some projective monomial curves. We are interested in curves given parametrically by sets of monomials of the form

$$\{t^d, t^{d-1}s, \ldots t^{d-a}s^a, t^{d-b}s^b, \ldots, ts^{d-1}, s^d\}.$$

$\leqslant a < b \leqslant d-1$ Without restriction, we may assume that $a + b \leqslant d$. Then can compute n_V in terms of a and b.

COROLLARY 2. *Let V be as above. Then the defining prime ideal of V has a basis consisting of binomials of degree* $\leqslant [(b-2)/a] + 2$.

Proof. Let $t^{nd-c}s^c$ be an arbitrary element of S. Then $t^{nd-c}s^c$ is a product of n-m monomials of the set

$$M_1 = \{t^d, t^{d-1}s, \dots, t^{d-a}s^a\}$$

with m monomials of the set

$$M_2 = \{t^{d-b}s^b, \dots, ts^{d-1}, s^d\}$$

for some $m \leqslant n$. Clearly, we have $mb \leqslant c \leqslant (n-m)a + md$. Conversely, for every integer c satisfying the above inequalities for some $m \leqslant n$, the element $t^{nd-c}s^c$ belongs to S and can be written as a product of $n-m$ monomials of M_1 with m monomials of M_2. Thus, to compute n_V, we need only check when the intervals $[mb, (n-m)a + md]$, $m = 0, \dots, n$, contain the integers $0, \dots, nd$. That happens if and only if $mb - 1$ is contained in the interval $[(m-1)b, (n-m+1)a + (m-1)d]$, or, equivalently,

$$mb - 1 \leqslant (n-m+1)a + (m-1)d$$

for all $m = 1, \dots, n$. From this it follows that

$$b + a - 2 < (n+1)a + (m-1)(d-a-b)$$

for all $m = 1, \dots, n$. Since $d - a - b \geqslant 0$, the above condition is satisfied if and only if

$$b + a - 2 < (n+1)a$$

or, equivalently,

$$[(b+a-2)/a] < n+1.$$

Therefore, $n_V = [(b-2)/a] + 1$, and the statement of Corollary 2 follows from Theorem 1.

Now, since the maximal degree of the elements of a minimal basis of a homogeneous ideal is an invariant of this ideal [4, Satz 2, p. 37] and since every basis can be reduced to a minimal one, we need only determine the type of the binomials of a basis of the defining prime ideal of V beeause by restricting the degree of this type $\leqslant [(b-2)/a] + 2$, we will find a minimal basis.

In the following examples, we shall use the notations of the proof of Theorem 1.

Example 1. Let V be a projective monomial curve given parametrically by the set

$$\{t^d, t^{d-1}s, ts^{d-1}, s^d\},$$

$d \geqslant 3$. Then $n_V = d-2$. It is easy to see that P_V has a basis consisting of the binomial

$$X_0X_d - X_1X_{d-1}$$

and binomials of the type

$$X_0^{n-r}X_{d-1}^r - X_1^{n-s}X_d^s,\ n \geqslant r, s \geqslant 1.$$

Now we will check which positive integers n, r, s satisfy the equation

$$r\,(d-1)=n-s+sd.$$

Since we may assume $n\leqslant d-1$, $0\leqslant n-s\leqslant d-1$. Henee

$$s=[r(d-1)/d]=r-1.$$

From this it follows that

$$r\,(d-1)=n-r+1+(r-1)\,d.$$

Hence $n=d-1$. Thus, P_V has the following basis:

$$X_o X_d - X_1 X_{d-1},$$

$$X_0^{d-r-1}\,X_{d-1}^r - X_1^{d-r}\,X_d^{r-1},\ r=1,\dots,d-1.$$

It is easily seen that this basis is a minimal one, of. [4, Beispiel 1, p. 182].

Example 2. Let V be a projective monomial curve given parametrically by the set

$$\{t^d,\ t^{d-1}s,\ t^{d-2}s^2,\ ts^{d-1},\ s^d\},$$

$d\geqslant 4$. Then $n_V=d-3$. It is easy to see that P_V has a basis consisting of the binomials

$$X_o\,X_2 - X_1^2$$

$$X_o\,X_d - X_1\,X_{d-1}$$

$$X_1\,X_d - X_2\,X_{d-1},$$

$$X_2\,X_d^{d-3} - X_{d-1}^{d-2},$$

and binomials of the following types

$$X_0^{n-r-1}X_1X_{d-1}^r - X_2^n,\qquad n>r\geqslant 1,$$

$$X_0^{n-r}X_{d-1}^r - X_2^{n-s}X_d^s,\qquad n\geqslant r,\ s\geqslant 1.$$

For the first type, we have to solve the equation

$$1+r\,(d-1)=2n.$$

It has solutions if and only if d is even. In that case, we only get a binomial associated with the solution $r=1$, $n=d/2$ because n should be chosen as small as possible.

For the second type, we have the equation

$$r(d-1)=2(n-s)+sd,$$

or, equivalently,

$$(r-s)\,(d-1)+s=2n.$$

Thus, since $n\geqslant s\geqslant 1$, n takes the minimum value if $r=s+1$, i. e. $n=[(d+s-1)/2]$.

According to our analysis, P_V then has the following basis:

Case $d = 2t$:

$$X_0X_2 - X_1^2$$
$$X_0X_d - X_1X_{d-1},$$
$$X_1X_d - X_2X_{d-1},$$
$$X_2X_d^{d-3} - X_{d-1}^{d-2},$$
$$X_0^{t-2}X_1X_{d-1} - X_2^t,$$
$$X_0^{t-i-1}X_{d-1}^{2i} - X_2^{t-i}X_d^{2i-1}, \ i = 1, \dots, t-1.$$

Case $d = 2t + 1$:

$$X_0X_2 - X_1^2$$
$$X_0X_d - X_1X_{d-1},$$
$$X_1X_d - X_2X_{d-1},$$
$$X_2X_d^{d-3} - X_{d-1}^{d-2},$$
$$X_0^{t-i-1}X_{d-1}^{2i-1} - X_2^{t-i}X_d^{2i}, \ i = 0, \dots, t-1.$$

It is easily seen that this basis is a minimal one.

Example 3. Let V be a projective monomial curve given parametrically by the set

$$\{t^d, t^{d-1}s, t^{d-2}s^2, t^2s^{d-2}, ts^{d-1}, s^d\},$$

$d \geqslant 5$. Then $n_V = [(d-2)/2]$. It is easy to see that P_V has a basis consisting of the quadrics

$$X_0X_2 - X_1^2,$$
$$X_0X_{d-1} - X_1X_{d-2},$$
$$X_0X_d - X_2X_{d-2},$$
$$X_0X_d - X_1X_{d-1},$$
$$X_1X_d - X_2X_{d-1},$$
$$X_{d-2}X_d - X_{d-1}^2,$$

and binomials of the following types

$$X_0^{n-r-1}X_1X_{d-2}^r - X_2^{n-s}X_d^s;$$
$$X_0^{n-r}X_{d-2}^r - X_2^{n-s}X_d^s,$$

$$X_0^{n-r} X_{d-2}^r - X_2^{n-s-1} X_{d-1} X_d^s,$$

$$X_0^{n-r} X_{d-1}^r - X_1 X_2^{n-1},$$

$$X_1^{n-r} X_d^r - X_{d-2}^{n-1} X_{d-1}.$$

Processing as in Example 2, we can easily verify that only the following binomials come into consideration:

Case $d = 2t$: $X_0^{t-r-1} X_{d-2}^r - X_2^{t-r} X_d^{r-1}$, $r = 1, \ldots, t-1$.

Case $d = 2t+1$: $X_0^{t-1} X_{d-1} - X_1 X_2^{t-1}$,

$X_1 X_d^{t-1} - X_{d-2}^{t-1} X_{d-1}$,

$X_0^{t-r-1} X_1 X_{d-2}^r - X_2^{t-r+1} X_d^{r-1}$, $r = 1, \ldots, t-1$,

$X_0^{r-1} X_{d-2}^{t-r+1} - X_2^r X_{d-1} X_d^{t-r-1}$, $r = 1, \ldots, t-1$.

These binomials together with the above quadrics will form a minimal basis for P_V.

In particular, if $d = 2t$, the maximal degree of the elements of this basis is equal to $n_V = t - 1$. This shows that the bound $n_V + 1$ for this maximal degree is not always attained.

Moreover, if $d = 8$, we get an example of a projective monomial curve defined by binomials of degree $\leqslant 3$ but it is not an arithmetically Buchsbaum curve.

Received May 5, 1984

REFERENCES

[1] H. Bresinsky and B. Renschuch, *Basisbestimmung Veronesescher Projectionsideale mit allgemeiner Nullstelle* $(t_0^m, t_0^{m-r} t_1^r, t_0^{m-s} t_1^s, t_1^m)$, Math. Nachr. 96 (1980), 257 — 269.

[2] H. Bresinsky, P. Schenzel and W. Vogel, *On liaison, arithmetical Buchsbaum curves and monomial curves in* P^3, Preprint Series No .6 (1980/1981), Aarhus Univ., Dept. Math.

[3] J. Herzog, *Generators and relations of abelian semigroups and semigroup rings*, Manuscripta Math. 3 (1970), 175 — 193

[4] B. Renschuch, *Elementare und praktische Idealtheorie*, VEB Deutscher Verlag der Wissenschaften, Berlin 1976.

[5] J. Stückrad and W. Vogel, *On the number of equations defining an algebraic set of zeros in n-space*, Seminar D. Eisebud/ B. Singh/ W. Vogel Vol. 2, Teubner-Verlag, Leipzig 1982.

[7] N.V. Trung, *Projections of one-dimensional Veronese Varieties*, Math. Nachr., to appear.

INSTITUTE OF MATHEMATICS, P. O. BOX 631, BOHO, 10.000 HANOI, VIETNAM.