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ON MODIFIED CHAIN CONDITIONS

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1. INTRODUCTION

Throughout the paper we consider associative rings. Let M be a right A-module, where A is a ring. M is said to satisfy the weakly minimal condition for submodules if for every infinite descending chain $N_1 \supseteq N_2 \supseteq \ldots$ of submodules N_i of M there exist positive integers m, p such that $N_m A^p \subseteq N_i$ for all i or, equivalently, there is a positive integer q such that $N_q A^q \subseteq N_i$ for all i. Such a right A-module is called almost artinian (cf. [1], [4], [5]). A ring A is called almost right artinian if the right A-module A is almost artinian. Following [7], a ring is called a MHR-ring if it satisfies the minimal condition for principal right ideals. Now, a ring A is called an almost MHR-ring if A satisfies the weakly minimal condition for principal right ideals.

One of the most important basic results of almost right artinian (resp. almost MHR-) rings is the following: For any almost right artinian (resp. almost MHR-) ring A the Jacobson radical $J(A)$ of A is nilpotent (resp. nil) and the factor ring $A/J(A)$ is right artinian (resp. a MHR-ring) ([1, Theorem 1], [2, Theorem 3], resp.). In this connection it would be interesting to consider the situation in modules. That is the purpose of Section 2, where we prove that the same statement holds also for modules satisfying the weakly minimal condition for cyclic submodules (Theorem 2), and therefore for almost artinian modules. Furthermore, we give some module theoretic characterizations for right semisimple rings (Theorem 5, Corollary 6).

In Section 3 we describe the structure of two special classes of almost right artinian rings and related rings (Theorems 7, 8).

Let M be a right A-module. By $\Phi(M)$ we denote the Frattini submodule of M, i. e. $\Phi(M)$ is the intersection of all maximal submodules of M. If M contains no maximal sumodules, we set $\Phi(M) = M$. Then the subset

$$K(M) = \{x \mid x \in M, xA \subseteq \Phi(M)\}$$

is a well-determined submodule of M which is called the Kertész radical of M ([3], [6]). The following statements are immediate consequences of the above definition.

LEMMA 1. *Let M be a right A-module.*

(a) If $K(M) \neq M$, then $K(M)$ is the intersection of all maximal submodules M_i of M such that $MA \nsubseteq M_i$.

(b) If $M = N \oplus H$, then $K(M) = K(N) \oplus K(H)$.

(c) $K(M/K(M)) = (0)$.

Now we can prove the following

THEOREM 2. *For a right A-module M the following conditions are equivalent:*

(i) $K(M) = (O)$ and M satisfies the weakly minimal condition for cyclic submodules.

(ii) M is a direct sum of irreducible right A-modules.

In particular, if M satisfies the weakly minimal condition for cyclic submodules, then $M/K(M)$ satisfies the minimal condition for cyclic submodules.

Proof. A right A-module N is called irreducible if N is simple and $NA \neq (0)$, i.e. $NA = N$.

(i) $\Rightarrow$ (ii). By Lemma 1, the intersection of all such maximal submodules M_i of M with $MA \nsubseteq M_i$ is zero. First we prove that any non-zero submodule N of M contains a minimal submodule N_1. Let $0 \neq x \in N$. By (x) we denote the cyclic submodule of M generated by x. Since $K(M) = (0)$, $HA \neq (0)$ for every non-zero submodule H of M. If $(x)A$ is not minimal, we can find an $0 \neq x_1 \in (x)A$ with $(x)A \supset (x_1)$. If $(x_1)A^2$ is not minimal, we find an $0 \neq x_2 \in (x_1)A^2$ with $(x_1)A^2 \supset (x_2)$. Successively we get a strictly descending chain

$$(1) \qquad (x) \supset (x_1) \supset \dots \supset (x_i) \supset \dots$$

with $(x_i)A^{i+1} \supset (x_i)$. By assumption there is a positive integer q such that $(x_q)A^q \subseteq (x_i)$ for all i. Hence $(x_q)A^q A \subseteq (x_i)A \subseteq (x_i)$ for all i. Thus, (1) must break off at q, and $N_1 := (x_q) A^{q+1}$ is a minimal submodule of M in $(x) \subseteq N$.

Now, since $K(M) = (0)$, by Lemma 1 there is a maximal submodule M_1 of M such that $N_1 \nsubseteq M_1$, i. e. $N_1 \cap M_1 = (0)$. Hence $M = N_1 \oplus M_1$. Again by Lemma 1, $K(M) = K(N_1) \oplus K(M_1)$, therefore $K(M_1) = (0)$. Since $N_1 \subseteq (x)$, $(x) = N_1 \oplus (x) \cap M_1$. It is clear that $(x) \cap M_1$ is also a cyclic submodule of M_1: There is an $x_2 \in (x) \cap M_1$ such that $(x_2) = (x) \cap M_1$; (x_2) contains a minimal submodule N_2 if $(x_2) \neq (0)$. Since $K(M_1) = (0)$, $M_1 = N_2 \oplus M_2$. Then $(x_2) \cap M_2$ is a cyclic submodule of M_2. In this way we get finally a strictly descending chain

$$(2) \qquad (x_1) \supset (x_2) \supset \dots \supset (x_i) \supset \dots \qquad (x_1 := x)$$

with $(x_i)A^k \nsubseteq (x_j)$ for any $k = 1,2,\dots$ and $i < j$. From this (2) must be finite, i.e. there is a positive integer m such that (x_m) is a minimal submodule of M in (x). Hence $(x) = N_1 \oplus \dots \oplus N_m \oplus (x_m)$. Thus

$$M = \sum_{x \in M} (x) = \Sigma^{\oplus} M_i,$$

where each M_i is an irreducible right A-module.

(ii) $\Rightarrow$ (i) is straightforward.

The last statement is clear because $K(M/K(M)) = (0)$ by Lemma 1.

By $K_r(A)$ $(K_l(A))$ we denote the right (resp. left) Kertész radical of a ring A considered as a right (resp. left) A-module. Clearly $K_r(A)$ and $K_l(A)$ are ideals of A which are contained in the Jacobson radical $J(A)$ of A.

COROLLARY 3. *For a ring* A *the following conditions are equivalent:*

(i) *A is an almost MHR-ring with $K_r(A) = (0)$.*

(ii) *A is a direct sum of minimal right ideals R_i with $R_i A = R_i$.*

In particular, if A is an almost MHR-ring, then $A/K_r(A)$ is a MHR-ring.

Remarks. Let A be a MHR-ring with $J(A) = (0)$. Then the matrix ring

$$A^* = \begin{bmatrix} A & 0 \\ A & 0 \end{bmatrix}$$

is a MHR-ring with $K_r(A^*) = (0)$, $K_l(A^*) = J(A^*) = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}$. The ring A^* is right semisimple but not left semisimple (see definitions below, after the proof of Proposition 4).

Let A be an almost MHR-ring. Since $K_r(A) \subsetneq J(A)$, $A/J(A)$ is a MHR-ring, too. The statement that $J(A)$ is a nil ideal (cf. [2]) can be proved as follows: For $x \in J(A)$ we consider the descending chain $(x)_r \supseteq (x^2)_r \supseteq \dots (x^i)_r \supseteq \dots$ of principal right ideals $(x^i)_r$ of A. By assumption, there is a positive integer m such that $(x^m)_r A^m \subseteq (x^i)_r$ for all i. Since $(x^m)_r A^m = x^m A^m + x^m A \cdot A^m = x^m A^m$, $y := x^{2m} \in x^m A^m$. Hence $x^m A^m = (x^{2m})_r = (x^{2m+1})_r = \dots$ Then there is an integer h and an $a \in A$ with $y = x^{2m} . xh + x^{2m} . xa = y\ (xh + xa) = ys$ where $s := xh + xa \in J(A)$. As is well-known, for s as an element of the Jacobson radical ring $J(A)$ there exists an $t \in J(A)$ with $s - st + t = 0$. Thus $x^{2m} = y = y - y\ (s - st + t) = (y - ys) - (y - ys)\ t = 0$.

For any ring A, $K_r(A) + K_l(A) \subseteq J(A)$. But it is unknown whether $J(A) = K_r(A) + K_l(A)$ holds for every ring A. Using Corollary 3 we can prove the following

PROPOSITION 4. *For any almost MHR-ring A,* $J(A) = K_r(A) + K_l(A)$.

Proof. By Lemma 1 we have

(3) $\quad J(A)A \subseteq K_l(A)$.

If $K_r(A) = A$, the statement is clear. Suppose $A \neq K_r(A)$. By Corollary 3, $\overline{A} := A/K_r(A) = \overline{B} \oplus J(\overline{A})$, where $J(\overline{A})$ is the Jacobson radical of $\overline{A}$. It holds $J(\overline{A})\overline{A} = J(\overline{A})$ by Corollary 3, consequently $J(A)A + K_l(A) = J(A)$. Hence $J(A) = K_l(A) + K_r(A)$ by (3).

A right A-module M is called semisimple if M is a direct sum of irreducible right A-modules. Hence by Theorem 2, a right A-module M is semisimple if and only if $K(M) = (0)$ and M satisfies the weakly minimal condition for cyclic submodules. A ring A is called right semisimple if the right A-module A is semisimple. Following Tominaga [8], a right A-module M is called s-unital if $x \in xA$ for any $x \in M$. Hence every semisimple module is s-unital. A ring A is called right s-unital if the right A-module A is s-unital. A s-unital right A-module I is defined to be s-injective if for s-unital right A-modules M, N with a monomorphism α: $N \to M$ and a homomorphism β: $N \to I$ there exists a homomorphism β': $M \to I$ such that $\beta = \beta'\alpha$. Dually, one can define s-projective right A-modules.

THEOREM 5. *For a right s-unital ring A the following conditions are equivalent:*

(a) *A is right semisimple.*

(b) *Every s-unital right A-module is semisimple.*

(c) *Every s-unital right A-module is s-injective.*

(d) *Every s-unital right A-module is s-projective.*

(e) *Every s-unital simple right A-module is s-projective.*

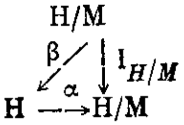
Proof. (a) $\Rightarrow$ (b). Let M be a (non-zero) s-unital right A-module and $0 \neq x \in M$. Then $xA \cong_A A/Ann(x)$, where $Ann(x) = \{a \mid a \in A,\ xa = 0\}$. Hence by (*a*), xA is a direct sum of s-unital simple right A-modules. Thus M is a direct sum of s-unital simle right A-modules, too.

(b) $\Rightarrow$ (a) is clear.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) are clear by using (a) $\Leftrightarrow$ (b).

(e) is a special case of (d).

(e) $\Rightarrow$ (a). Let S be the socle of the s-unital right A-module A. We want to show that $S = A$, i. e. (a) holds. Suppose the contrary, that $S \neq A$. Then there exists an $a \in A$ with $a \notin S$. By Zorn's Lemma there is a submodule M of the right A-module $H := aA + S$ which is maximal with respect to the conditions that $a \notin M$ and $M \supseteq S$. Obviously, M is a maximal submodule of H. Hence H/M is s-projective by (e). Then there exists a homomorphism $\beta : H/M \rightarrow H$ such that the diagramm



is commutative, i. e. $\alpha\beta = 1_{H/M}$, where α is the natural homomorphism of H onto H/M. From this $\beta \neq 0$ and $H = \mathrm{Im}\ (\beta) \oplus M$, a contradiction. Thus $A = S$.

A right and left s-unital ring is called s-unital. For a ring A one can easily see that $J(A) = (0)$ if and only if $K_r(A) = K_l(A) = (0)$. A ring A is called semisimple if A is right and left semisimple. It is not difficult to verify that for any ring A, if $K_r(A) = (0)$ then $K_l(A) = J(A)$ (if $K_l(A) = (0)$, $K_r(A) = J(A)$). From this the following conditions about a ring A are equivalent:

(a) A is semisimple.

(b) A is right semisimple and $K_l(A) = (0)$.

(c) A is left semisimple and $K_r(A) = (0)$.

COROLLARY 6. *For an s-unital ring A the following conditions are equivalent:*

(a) *A is semisimple.*

(b) *Every s-unital right A-module and every s-unital left A-module is semisimple.*

(c) *Every s-unital right A-module and every s-unital left A-module is s-injective.*

(d) *Every s-unital right A-module and every s-unital left A-module is s-projective.*

(e) *Every s-unital simple right A-module and every s-unital simple left A-module is s-projective.*

In [9, Theorem 3] Tominaga has given 9 ring theoretic conditions each of which is equivalent to the condition that A is semisimple in the above sense. Corollary 6 is well-known for rings with identity.

3. ALMOST ARTINIAN RINGS AND RELATED RINGS

From Theorem 2 and Corollary 3 we obtain that for any right A-module M the following conditions are equivalent:

(i) M is almost artinian and $K(M) = (0)$.

(ii) M is a direct sum of finitely many irreducible right A-modules.

For any almost artinian right A-module M, $M/K(M)$ is artinian. In particular, if A is an almost right artinian ring, then $A/K_r(A)$ is a right artinian ring containing a right identity.

Let A be an almost right artinian ring. In [2, Theorem 1] it was proved that A is almost right artinian if and only if A contains an idempotent e such that Ae is a right artinian ring and the left annihilator l(e) of e in A is nilpotent. From this it is easy to see that every ideal of a right artinian ring is almost right artinian. Hence a question naturally arises: Let R be an almost right artinian ring. Is there a suitable right artinian ring A containing a (right) ideal B with $B \cong R$? To this question H. Komatsu recently gave a negative answer by way of the following counter example which Professor H. Tominaga communicated to me in a letter dated December 25, 1983. Let R be a zero ring (i. e. $R^2 = (0)$) and R^+ is an infinite cyclic group. Then R is almost artinian but certainly not artinian. Suppose that R is a (right) ideal of a right artinian ring A. Then by using the additive structure of A we must have $RA = (0)$. This forces R to be an artinian ring, a contradiction.

Now we give an another example suggested by Komatsu's. Let B be an arbitrary right artinian ring and M be any left B-module. Then the matrix ring

$$A := \begin{pmatrix} B & M \\ 0 & 0 \end{pmatrix}$$

is an almost right artinian ring. Then, as we can easily see, for every special choice of M we have an almost right artinian ring A which cannot be isomorphic to a (right) ideal of a right artinian ring, for example if we take $M = \sum_{\text{infin.}} \oplus C(p^\infty)$, where $C(p^\infty)$ is a quasicyclic p-group.

Furthermore these examples show that it is impossible to «lift» properties of $A/J(A)$ to A in general. Hence, in order to get some more information about almost artinian rings we must impose some suitable additional conditions upon them. For example one can prove the following

THEOREM 7. *Let A be an almost right artinian ring such that every homomorphic image $A' \neq A$ of A is right artinian. Then A is contained in one of the following classes:*

(I) A is right artinian.

(II) $A^2 = (O)$, A^+ is isomorphic to the additive group of all rational numbers of the form

$$\frac{m}{p_1^{n_1} \cdots p_k^{n_k}}$$

where $p_1, \ldots, p_k$ are finitely many fixed prime numbers, m, $n_1, \ldots, n_k$ are integers.

(III)

$$A = \begin{bmatrix} S_m & \begin{bmatrix} S & O & \ldots & O \\ S & O & \ldots & O \\ \vdots & \vdots & & \vdots \\ S & O & \ldots & O \end{bmatrix}_{m \times m} \\ O & O \end{bmatrix},$$

where S_m is the total $m \times m$-matrix ring over an infinite skew field S.

Conversely, any ring A of (I), (II) or (III) is almost right artinian and A/I is right artinian for any ideal $I \neq (O)$ of A.

Proof. Let A be an almost right artinian ring such that for any nonzero ideal I of A, A/I is right artinian. If $A \notin$ (I), i.e. if A is not right artinian, then A is a non-prime ring with $J(A) \neq (O)$. By [10, Satz 3], $J(A)^2 = (O)$. If $A = J(A)$, then $A \in$ (II) by [10, Satz 5 (I)]. Suppose now that $A \neq J(A)$. Then $J(A)$ is a right and left $\bar{A}$-module, where $\bar{A} := A/J(A)$ is a semisimple artinian ring having an identity $\bar{e}$ which is lifted to an idempotent $e \neq 0$ in A. Hence

$$J(A) = \sum \oplus M_i \oplus H,$$

where each M_i is a unital simple right $\bar{A}$-module and $H\bar{A} = (0)$. Since $A = eAe + J(A)$, one can easily verify that $M := \sum \oplus M_i$ and H are ideals of A. From this either $H = (0)$ or $M = (0)$. Since $M_i A = M_i$, we can easily see that if $M \neq (0)$, then M is an artinian right A-module with the right artinian factor ring A/M, in contradiction with the assumption that A is not a right artinian ring. Hence $M = (0)$, $J(A) = H$, i.e. $J(A)A = (0)$, therefore

$$(4) \qquad A = eAe \oplus J(A), \qquad J(A)A = (0).$$

Regarding $J(A)$ as a left A-module, we get, by the same argument, that either $J(A)$ is a direct sum of unital left $\overline{A}$-modules or $\overline{A}J(A) = (0)$. By (4) we must obviously have $\overline{A}J(A) \neq (0)$. It is therefore clear that $J(A)$ must be a unital simple left $\overline{A}$-module, hence eAe is a simple artinian ring. From this fact we conclude $A \in$ (III).

The converse is clear.

Let A be an almost right artinian ring with a left identity e. Then

$$A \cong \begin{bmatrix} eAe & \mathrm{l}(e) \\ 0 & 0 \end{bmatrix}.$$

As we have already seen (before Theorem 7), A is in general not right artinian. In this last part of the paper we study rings A with a left identity such that for each non-zero ideal I of A, A/I is almost right artinian, but A is still not almost right artinian. We call such a ring an RA-ring.

THEOREM 8. *Let A be a non-prime RA-ring. Then A is contained in either of the following classes:*

(I) A has an identity, $J(A)^2 = (0)$ and $A/J(A)$ is a simple right artinian ring. If N is a non-zero ideal of A in $J(A)$, then N is a direct sum of infinitely many minimal right ideals of A and $J(A) = N \oplus N'$, where N' is a direct sum of finitely many minimal right ideals of A.

(II)

$$A \cong \begin{bmatrix} S_m & M \\ 0 & K_n \end{bmatrix}$$

where S_m and K_n are total matrix rings over skew fields S and K (m, n are positive integers), respectively, M is a unital left S_m-and unital right K_n-bimodule. If N is a non-zero bisubmodule of M, then N is a direct sum of infinitely many simple right K_n-modules and $M = N \oplus N'$, where N' is a direct sum of finitely many simple right K_n-modules.

Conversely, every ring in (I) and (II) is a non-prime RA-ring.

Before proving Theorem 8, we state the next

LEMMA 9. *The prime radical of any RA-ring is a zero ring.*

Poorf. Let A be a RA-ring and N be the prime radical of A. By [1, Proposition 7] (see also [5, Proposition 2] for a shorter proof) a module M with a submodule N is almost artinian if and only if both M/N and N are almost artinian. From this we have

(5) For any non-zero ideals B, C of A, $B \cap C \neq (0)$.

Suppose $N \neq (0)$. Then A is non-prime, i.e. there are non-zero ideals B, C of A with $BC = (0)$. Hence by (5), $D := B \cap C \neq (0)$ but $D^2 = (0)$, therefore $D \subseteq J(A)$. Hence $N = J(A)$ and $J(A)^k \neq (0)$, $J(A)^{k+1} = (0)$ for some positive integer k. Arguing by contradiction, assume that $k \geqslant 2$. Let e be a left identity of A. Then

$$A = eAe \oplus eA(1-e) \tag{6}$$

and with $J := J(A)$

$$J^k = eJ^k e \oplus eJ^k(1-e). \tag{7}$$

Since $eA(1-e) \subseteq J$ we get by (6) and (7) that $eJ^k e$ and $eJ^k(1-e)$ are ideals of A. Since $eJ^k(1-e)A = (0)$, $eJ^k(1-e)$ is an almost right A-module, therefore $eJ^k(1-e) = (0)$. Hence

$$J^k = eJ^k e. \tag{8}$$

Considering the factor rings A/J^2 and A/J^k, we have

$$J = x_1 A + \ldots + x_u A + H + J^2 \qquad (x_1, \ldots, x_u \in J) \tag{9}$$

and

$$J^{k-1} = y_1 A + \ldots + y_t A + K + J^k \qquad (y_1, \ldots, y_t \in J^{k-1}), \tag{10}$$

where $HA \subseteq J^2$, $KA \subseteq J^k$. Since $J^k = J \,.\, J^{k-1}$, (8), (9) and (10) yield

$$J^k = \sum_{i,j} (x_i A)(y_j A) + \sum_j H(y_j A).$$

Since $H(y_j A) = H(ey_j A) = (He)(y_j A) \subseteq J^2 \,.\, J^{k-1} = J^{k+1} = (0)$,

$$J^k = \sum_{i,j} (x_i A)(y_j A). \tag{11}$$

Now, for $x = x_i a \,.\, y_j b$ $(a, b \in A)$ we have $x = x_i y$ with $y := a y_j b \in J^{k-1}$. By (10), $y = y_1 a_1 + \ldots + y_t a_t + z + s$ $(a_j \in A,\ z \in K,\ s \in J^k)$. Hence $x = x_i y = x_i y_1 a_1 + \ldots + x_i y_t a_t$, because $x_i z = x_i z \,.\, e = x_i \,.\, ze \in J \,.\, J^k = (0)$ and $x_i s \in J \,.\, J^k = (0)$. Comparing this with (11) we get that J^k, as a right A-module, is finitely generated. On the other hand, J^k $(= eJ^k e)$ is a unital right A/J-module, therefore $J^k = \Sigma \oplus R_n$ where each R_n is a unital simple right A/J-module. Since J^k is finitely generated, $J^k = R_1 \oplus \ldots \oplus R_m$. This implies that A is an almost right artinian ring, a contradiction. Thus $J^2 = (0)$.

Proof of Theorem 8. Let A be a non-prime RA-ring. Then $A \neq J \neq (0)$ and by Lemma 9, $J^2 = (0)$. Since A/J is almost right artinian, it is right artinian by

Corollary 3. Let e be a left identity of A. Then $A = eAe \oplus eA(1\text{-}e)$ with $eA(1\text{-}e) \subseteq J$. It is clear that eJe and $eA(1\text{-}e)$ are ideals of A. Since $eA(1\text{-}e)A = (0)$, $eA(1\text{-}e)$ must be zero. We have $J = eJe$, in particular, A has an identity e. Now, by the same argument as that used for proving [10, Satz 5] we get either $A \in (I)$ or $\in$ (II).

The converse is clear.

Remark. The ring A in Theorem 8 is not almost right artinian, but A/I is right artinian for each non-zero ideal I of A.

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