ACTA MATHEMATICA VIETNAMICA
Vol. 9 No. 1 (1984)

WEAKLY ADMISSIBLE TRANSLATES OF PROBABILITY MEASURES ON LOCALLY CONVEX SPACE

DANG HUNG THANG
University of Hanoi

I. INTRODUCTION

In recent years, there has been growing research in fields connected with probability measures on linear topological spaces. Of particular interest is the study concerned with absolute continuity of probability measures. This is of fundamental importance not only in the development of probability theory on linear topological spaces but also for applications in Statistics of stochastic processes (see e.g. [6]).

In this paper, we introduce the notion of weakly absolute continuity of measures on linear topological spaces (see Definition 2.1). This definition generalizes the notion of absolute continuity. Our main goal is to examine the set of weakly admissible translates of probability measures on locally convex space and the dichotomy theorem for stable measures.

Let H_μ be the set of weakly admissible translates of a measure μ on a locally convex space E. An advantage of our notion is that, in contrast to the set A_μ of admissible translates of μ, the set H_μ has many interesting properties. For examples, it will be seen in Section 2 that

i) H_μ is a measurable linear subspace of E,

ii) H_μ is isomorphic to the dual of the topological vector space of μ-measurable linear functions.

We focus on symmetric stable measure in Section 3. In the case where the index of μ is greater than 1, H_μ becomes a Banach space of μ-measure 0 and it is dense in the support of μ. Specially, if μ is a centred Gaussian measure, the set H_μ turns out to coincide with the set A_μ, and this set can be characterized as the intersection of all measurable linear subspaces of μ-measure one. For the case of stable measures with discrete spectrum, the set A_μ and H_μ can be described through the spectrum of μ.

It is well known that two Gaussian measures are either equivalent or orthogonal [4]. This phenomenon is called the dichotomy theorem for Gaussian measures. Until now, the conjecture that two stable measures are either equivalent or orthogonal has not yet been proved. We shall prove in Section 4 that two stable measures are either weakly equivalent or orthogonal.

II. THE SET OF WEAKLY ADMISSIBLE TRANSLATES

Let E be a locally convex space and E' be its dual. $\mathcal{B}$ denotes the Borel 6-algebra generated by the open sets of E.

2.1. DEFINITION. *Let μ and ν be two probability measures on E. Then we say that μ is absolutely continuous in the weak sense with respect to ν, and we write $\mu \overset{w}{\ll} \nu$, if for every sequence $(y_n) \subset E'$ such that $(\cdot, y_n) \to 0$ in ν-measure, we have $(\cdot, y_n) \to 0$ in μ-measure.*

It can be checked easily that $\mu \ll \nu$ implies $\mu \overset{w}{\ll} \nu$. As we will see later, the converse is false. If $\mu \overset{w}{\ll} \nu$ and $\nu \overset{w}{\ll} \mu$ we say that μ is weakly equivalent to ν and write $\mu \overset{w}{\sim} \nu$.

If a is an element of E, we denote by μ_a the measure translated by a, i.e. the measure given by

$$\mu_a(B) = \mu(B - a) \text{ for every Borel set } B.$$

2.2. DEFINITION. *An element a of E is said to be an admissible (resp. weakly admissible) translate of μ if $\mu_a \ll \mu$ (resp. $\mu_a \overset{w}{\ll} \mu$). If $\mu_a \perp \mu$ then a is said to be an orthogonal translate of μ.*

Throughout this paper, A_μ (resp. H_μ) will denote the set of admissible (resp. weakly admissible) translates of μ and S_μ the set of orthogonal translates of μ.

The characteristic function of μ is denoted by $\widehat{\mu}(y)$.

2.3. PROPOSITION. *$a \in H_\mu$ if and only if*

$$(2-1)\begin{cases}\text{For every sequence } (y_n) \subset E' \text{ such that } (., y_n) \to 0 \text{ in } \mu\text{-measure,}\\ \text{we have}\\ \qquad \lim (a, y_n) = 0.\end{cases}$$

Proof. Suppose that a satisfies the condition (2 — I). Let $(y_n) \subset E'$ be such that $(., y_n) \to 0$ in μ-measure. Then $\widehat{\mu}(ty_n) \to 1$ for all real numbers t. Since $\widehat{\mu}_a(ty_n) = \exp\{it(a, y_n)\}\widehat{\mu}(ty_n)$, we have $\widehat{\mu}_a(ty_n) \to 1$ for all t. Thus $(., y_n) \to 0$ in μ_a-measure. This proves $\mu_a \overset{w}{\ll} \mu$, i. e. $a \in H_\mu$. Conversely, suppose that a does not satisfy the condition (2 — I). Then there exists a sequence $(y_n) \subset E'$ such that $(., y_n) \to 0$ in μ-measure but $(a, y_n) \not\to 0$. If $(., y_n)$ also converges to 0 in μ_a-measure then it follows that $\exp\{it(a, y_n)\} \to 1$ for all t. From this, we have $(a, y_n) \to 0$, which is impossible. Consequenly, $(., y_n)$ does not converges to 0 in μ_a-measure. This proves $a \notin H_\mu$.

2.4 COROLLARY. *H_μ is a linear subspace of E.*

2.5 PROPOSITION. *Let μ be a symmetric measure. If $a \in H_\mu$ then $\mu \overset{w}{\sim} \mu_a$.*

Proof. Let $(y_n) \subset E'$ be a sequence in E' such that $(., y_n) \to 0$ in μ_a-measure. Then $|\widehat{\mu}(ty_n)|^2 \to 1$ for all t. Since μ is symmetric, it follows that $\widehat{\mu}(ty_n) \to 1$ for all t. Hence $(., y_n) \to 0$ in μ-measure. This proves $\mu \overset{w}{\ll} \mu_a$. Since $a \in H_\mu$, we have $\mu \overset{w}{\sim} \mu_a$.

Let G_μ be the intersection of all measurable linear subspaces of μ-measure one.

2.6. THEOREM. *We have* $A_\mu \subset S^c_\mu \subset G_\mu \subset H_\mu$

Proof. Obviously, we have $A_\mu \subset S^c_\mu$. To prove $S^c_\mu \subset G_\mu$ we assume that $a \notin G_\mu$. Then there exists a measurable linear G of μ-measure one such that $a \notin G$. Therefore $(G - a) \cap G = \phi$ anp $\mu_a(G) = \mu(G - a) = 0$. Hence $\mu_a \perp \mu$ i. e. $a \in S_\mu$. It remains only to prove $G_\mu \subset H_\mu$. Assume $a \notin H_\mu$. By Proposition 2.3

$$(2-2)\begin{cases}\text{There existe a sequence } (y_n) \subset E' \text{ such that } (., y_n) \to 0 \text{ in}\\ \mu\text{-measure but } |(a, y_n)| > \varepsilon \text{ for all n and } \varepsilon > 0\end{cases}$$

For each n we can find k_n such that

$$\mu\{x \in E : |(x, y_{k_n})| > 2^{-n}\} < 2^{-n}$$

Set $\quad G = \{x \in E : \Sigma |(x, y_{k_n})| < \infty\}.$

Clearly, G is a measurable linear subspace of μ-measure one. In virtue of (2-2) we have $a \notin G$. Hence $a \notin G_\mu$. The theorem is proved.

2.7. COROLLARY. *Every translate of* μ *is either weakly admissible or orthogonal.*

Now we are going to give a better description of H_μ.

2.8. DEFINITION. 1) *A probability measure* μ *on* E *is called a Lusin measure if* $\forall \varepsilon > 0$ *there exists a weakly compact and absolutely convex set* K *such that* $\mu(K) > 1 - \varepsilon$.

2) *A measurable linear subspace* V *of* E *is called a* μ — *Lusin space of* E, *if* $\forall \varepsilon > 0$ *there exists a weakly compact and absolutely convex set* K *such that* $K \subset V$ *and* $\mu(K) > 1 - \varepsilon$.

Denote by G_μ^L the intersection of all μ-Lusin space of E.

2.9. THEOREM. *If* μ *is a Lusin measure the* $H_\mu = G_\mu^L$.

Proof. Suppose that G is a μ-Lusin space and $a \notin G$. Then there exist weakly compact and absolutely convex sets K_n such that

$$K_n \subset G,\ K_n + K_n + \subset K_{n+1},\ \mu(K_n) > 1 - 2^{-n}.$$

Set $L = \bigcup_{n=1}^{\infty} K_n$. By Haln-Banach theorem, we can choose $y_n \in E'$ such that $(a, y_n) > 1$ and $|(x, y_n)| \leq 1$ when $x \in K_n$. It is readily seen that $(x, y_n) \to 0$ for every $x \in L$. since $\mu(L) = 1$ we have $(x, y_n) \to 0$ in μ-measure. From Proposition 2.3 and the fact that $(a, y_n) > 1$ for all n, it follows that $a \notin H_\mu$. Hence $H_\mu \subset G_\mu^L$.

Conversely, assume that $a \notin H_\mu$. From the proof of Proposition 2.6 we see that there exists a sequence $(y_n) \subset E'$ such that

$$\mu\{x \in E : |(x, y_n)| > 2^{-n}\} < 2^{-n}$$

and $|(a, y_n)| > \varepsilon$ for all n and some $\varepsilon > 0$.

Set $N(x) = \Sigma \mid (x, y_n) \mid$ and $G = \{x \in E : N(x) < \infty\}$. Clearly, $\mu(G) = 1$. We shall prove that G is a μ-Lusin space. Let $\varepsilon > 0$ be given and choose $t > 0$ such that $\mu\{N \leqslant t\} \geqslant 1 - \frac{\varepsilon}{2}$. Observe that the set $\{N \leqslant t\}$ is closed and absolutely convex. Since μ is a Lusin measure, there exists a weakly compact and absolutely convex set K such that $\mu(K) > 1 - \frac{\varepsilon}{2}$. Set $\widehat{K} = K \cap \{N \leqslant t\}$. It can be checked that $\widehat{K} \subset \{N \leqslant t\} \subset G$ and $\mu(\widehat{K}) > 1 - \varepsilon$. We also have that $\widehat{K}$ is a weakly compact and absolutely convex set, hence G is a μ-Lusin space, as desired. Since $a \notin G$ we have $a \notin G^L_\mu$. The theorem is proved.

Let $L_o(\mu)$ denote the set of all μ-measurable functions defined on E. Then $L_o(\mu)$ forms a metrizable vector topological space if it is equipped with the topology of convergence in μ-measure. Denote by $\mathcal{L}_\mu$ the closure of $\widetilde{E'} = \{(\cdot, y), y \in E'\}$ in $L_o(\mu)$. $\mathcal{L}_\mu$ is a metrizable vector topological space, but it is not necessarily locally convex. Every element of $\mathcal{L}_\mu$ is called a μ-measurable linear function. Let $\mathcal{L}'_\mu$ denote the dual of $\mathcal{L}_\mu$, and $\mathcal{L}'_\mu$ be equipped with the strong topology.

2.10. THEOREM. *If μ is a Lusin measure then H_μ is isomorphic to $\mathcal{L}'_\mu$.*

We need several lemmas.

2.11. LEMMA. *The characteristic function of a Lusin measure μ is continuous on E', if E' is equipped with the Markey topology.*

Proof Given $\varepsilon > 0$. Put $\delta = \frac{\varepsilon^2}{2}$. By assumption, there exists an absolutely convex and $\sigma(E, E')$ — compact set K such that $\mu(K) > 1 - \frac{\delta}{4}$. We have

$$\mid 1 - \widehat{\mu}(y) \mid \leqslant \int_E \mid 1 - exp\{i(x,y)\} \mid^2 d\mu(x) = 4\int_K \sin^2 \frac{(x,y)}{2}\, d\mu(x) +$$

$$+ 4\int_{K^c} sin^2 \frac{(x,y)}{2}\, d\mu(x) \leqslant \int_K \mid (x,y) \mid^2 d\mu(x) + \delta.$$

Put $U = \{y \in E' : \sup_{x \in K} \mid (x,y) \mid \leqslant \sqrt{\delta}\}$, we have that $y \in U$ implies $\mid 1 - \widehat{\mu}(y) \mid < \varepsilon$.

Since U is a neighbourhood of the Mackey topology, the lemma is proved.

It is trivial to show the following.

2.12. LEMMA. *If μ is a Lusin measure, then every closed subspace of E of measure one is a μ — Lusin space.*

The following lemma is well known.

2.13. LEMMA. *Let (X_α) be a net of random variables. For $X_\alpha \to 0$ in probability it is necessary and sufficient that the corresponding net of their characteristic functions converge to 1.*

Proof of Theorem 2.10. Let I: $E' \to \mathcal{L}_\mu$ be the linear map which takes $y \in E'$ into the corresponding equivalence class $\tilde{y}$ containing $(., y)$. We assert that I is continuous on E' in the Mackey topology M. To see this let (y_α) be a net in E' such that $y_\alpha \to 0$ in the topology M. By Lemma 2.11 $\widehat{\mu}(ty_\alpha) \to 0$ for each $t \in \mathbb{R}$. By Lemma 2.13 $\tilde{y}_\alpha \to 0$ in $\mathcal{L}_\mu$. Consequently, I is continuous at O and hence on E' in the topology M. Since the topology M is compatible with the duality (E', E) it follows that the transpose I^* of I is a continuous operator from $\mathcal{L}'_\mu$ into E when $\mathcal{L}'_\mu$ and E are equipped with the strong topology. Since the strong topology on E is finer than the initial topology on E, it follows that I^* is also continuous when E is equipped with the initial topology. Since IE' is dense in $\mathcal{L}_\mu$. I^* is injective. Now we have to show that $H_\mu = I^*(\mathcal{L}'_\mu)$. At first, we prove $I^*(\mathcal{L}'_\mu) \subset H_\mu$. Let $\xi \in \mathcal{L}'_\mu$ and let $(y_n) \subset E'$ such that $(., y_n) \to 0$ in μ — measure. Then we have $\lim (I^*\xi, y_n) = \lim(\xi, Iy_n) = 0$. By Proposition 2.3, we have $I^*\xi \in H_\mu$. To prove the opposite inclusion, we assume $a \in H_\mu$. Define a map $\tilde{a}$: $IE' \to R$ by $\tilde{a}(Iy) = (a, y)$. Then $\tilde{a}$ is a well defined map. Indeed, suppose that $Iy_1 = Iy_2$ i. e. $(x, y_1) = (x, y_2)$ for μ — almost all x. Then the set $G = \{x \in E: (x, y_1) = (x, y_2)\}$ is the closed subspace of μ — measure one hence, by Lemma 2.12, it is a μ — Lusin space. By Theorem 2.9, we have $a \in G$ i.e. $(a, y_1) = (a, y_2)$. Clearly, $\tilde{a}$ is linear. By Proposition 2.3, $\tilde{a}$ is also continuous. Since IE' is dense in $\mathcal{L}_\mu$, $\tilde{a}$ admits a unique extension to $\mathcal{L}_\mu$ i.e. $\tilde{a} \in \mathcal{L}'_\mu$. On the other hand, for each $y \in E'$ we have $(I^*(\tilde{a}), y) = (\tilde{a}, Iy) = (a, y)$. Hence $a = I^*(\tilde{a})$. This proves $H_\mu \subset I^*(\mathcal{L}'_\mu)$ and thereby completes the proof of Theorem 2.10.

From now on, for convenience for $y \in E'$ we denote Iy by $\tilde{y}$, and for $a \in H_\mu$ we denote $I^{*-1}(a)$ by $\tilde{a}$. We always have

$$(a, y) = (\tilde{a}, \tilde{y}) \quad \text{for } a \in H_\mu,\ y \in E'$$

2.14. THEOREM. *Let* μ *be a Lusin measure. In addition, suppose that* $\mathcal{L}_\mu$ *is locally convex. Then the closure of* H_μ *in* E *is equal to the intersection* M_μ *of all closed subspaces of* E *of* μ*-measure one.*

Proof. From Theorem 2.9 and Lemma 2.12 it follows that $\overline{H}_\mu \subset M_\mu$. To prove the opposite inclusion, we assume $a \notin \overline{H}_\mu$. By Hahn-Banach theorem, we can find $y \in E'$ such that $(a, y) = 1$ and $(x, y) = 0$ for all $x \in H_\mu$. Hence we have $(\xi, Iy) = (I^*, \xi, y) = 0$ for all $\xi \in L_\mu$. From the assumption that L_μ is locally convex we have $Iy = 0$ i.e. $(x, y) = 0$ for μ-almost all x.Set $M = \{x \in E : (x, y) = 0\}$. Since $(a, y) = 1$, we have $a \notin M$. Clearly, M is a closed subspace of μ-measure one. Hence $a \notin M_\mu$. The theorem is proved.

Remark. As we will see later, it may occur that no weakly admissible translate exists, except the zero translate. However, if $\mathcal{L}_\mu$ is locally convex then H_μ is a sufficiently large set.

III. THE SET OF WEAKLY ADMISSIBLE TRANSLATES OF STABLE MEASURES

1.3. DEFINITION. *A probability measure* μ *on* E *is said to be a stable measure of index* p $(0 < p \leqslant 2)$ *if for every* $y \in E'$, $(., y)$ *is a stable random variable of index* p *in the probability space* $(E, \mathcal{B}, \mu)$.

The class of all symmetric stable measures of index p on E is denoted by $R_p(E)$. In this section we are interested only in Lusin measures. It can be conjectured that if $\mu \in R_p(E)$ then μ is a Lusin measure.

3.2. THEOREM. *If* $\mu \in R_p(E)$ *where* $1 \leqslant p \leqslant 2$, *then* H_μ *is a Banach space. In the case* $p = 2$, H_μ *is a Hilbert space.*

Proof. Let $\mu \in R_p(E)$. Then the characteristic function $F_y(t)$ of $(., y)$ is given by

$$F_y(t) = \exp\left\{- d_y|t|^p\right\}.$$

where $d_y \geqslant 0$ and it depends only on the equivalence class $\tilde{y}$. We define $\| \tilde{y} \|_\mu = d_y^{1/p}$. From Schilder [8] if $1 \leqslant p \leqslant 2$ then $\tilde{E}' = \{\tilde{y}, y \in E\}$ is a normed linear space under the norm $\| \tilde{y} \|_\mu$. By Lemma 2.13 we have $\tilde{y}_n \to 0$ in $\mathcal{L}_\mu$ if and only if $\| \tilde{y}_n \|_\mu \to 0$. Hence $\mathcal{L}_\mu$ is the completion of $\tilde{E}'$ under the norm $\|.\|_\mu$, thus $\mathcal{L}_\mu$ becomes a Banach space. Moreover, it is easily seen that if $r < p$ then

$\int_E |(x, y)|^r d\mu(x) = c_{p,r}\|\tilde{y}\|_\mu^r$ for all $y \in E'$, where the constant $C_{p,r}$ depends only on p and r.

In particular, if $p = 2$ i.e. μ is a centred Gaussian measure, $\mathcal{L}_\mu$ is a Hilbert space. By Theorem 2.10 $H_\mu = I(L'_\mu)$. If $a \in H_\mu$ we define the norm of a by $\|a\|_{H_\mu}$ $\|\tilde{a}\|_{\mathcal{L}_\mu}$. Clearly, H_μ is a Banach space under the norm $\|\cdot\|_{H_\mu}$ and we always have

$$|(a, y)| \leqslant \|a\|_{H_\mu} \|\tilde{y}\|_\mu \quad \text{for } a \in H_\mu,\ y \in E'.$$

Note that this norm topology is finer than the topology on H_μ induced by the topology on E.

By definition, the support supp μ of a probability measure μ is the smallest closed set which carries the total mass one. From Theorem 2.14 and the fact that the support of a symmetric stable measure is a closed subspace of E [7] we obtain the following

3.3. THEOREM. *If $\mu \in R_p(E)$ $(1 \leqslant p \leqslant 2)$ then*

$$\operatorname{supp} \mu = H_\mu$$

The following theorem shows that H_μ is a negligible set with respect to μ.

3.4. THEOREM. *If $\mu \in R_p(E)$ $(1 < p \leqslant 2)$ then*

$$\mu(H_\mu) = 0 \text{ when } \dim H_\mu = \infty,$$
$$\mu(H_\mu) = 1 \text{ when } \dim H_\mu < \infty.$$

Proof. If dim $H_\mu < \infty$ then H_μ is closed. From Theorem 3.3 $H_\mu = \operatorname{supp} \mu$ thus $\mu(H_\mu) = 1$. Now suppose that dim $H_\mu = \infty$. Then the vector subspace $\tilde{E}'$ of L_μ is obviously of infinite dimension. From the Dvoretsky — Rogers theorem [3] there exists a sequence (y_n) in E' such that

$$\sum_{n=1}^{\infty} \|\tilde{y}_n\|_\mu = \infty$$

$$\sum_{n=1}^{\infty} |(\xi, \tilde{y}_n)| < 8 \qquad \text{for every } \xi \in \mathcal{L}'_\mu.$$

Set $N(x) = \Sigma\, |(x, y_n)|$. We have $H_\mu \subset \{N < \infty\}$. Indeed, if $a \in H_\mu$ then

$$\sum_{n=1}^{\infty} |(a, y_n)| = \Sigma\, |(\tilde{a}, \tilde{y}_n)| < \infty$$

Clearly, the function $N(x)$ is a seminorm and

$$\int_E N(x)d\mu(x) = \Sigma \int_E |(x, y_n)|\, d\mu(x) = C_{p,1}\Sigma\, \|\tilde{y}_n\| = \infty$$

We know form the zero-one law in [2] that $\mu\{N < \infty\} = 0$ or 1. Assume $\mu\{N < \infty\} = 1$. From the A.de Acosta's theorem [1] we have

$$\int_E N(x)d\mu(x) < \infty.$$

This is a contradiction. Therefore $\mu\{N < \infty = 0$ so we have $\mu(H_\mu) = 0$

The proof is complete.

3.5. DEFINITION. *Let E be a Banach space. A measure $\mu \in R_p(E)$ is called a stable measure of index p with discrete spectrum if there exists a sequence $(x_n) \subset E$ such that*

$$\widehat{\mu}(y) = exp\{-\sum_{n=1}^{\infty} |(x_n, y)|^p\}.$$

The sequence (x_n) is called the spectrum of μ. To avoid reduction to lower dimensional cases we always assume that

$$x_n \notin span\{x_m ; m \neq n\}$$

3.6. THEOREM. *Let μ be a stable measure of index p with discrete spectrum $(x_n) \subset E$. Then we have*

$$H_\mu = \{x \in E : x = \sum_{n=1}^{\infty} b_n x_n, (b_n) \in l_q\}$$

$$A_\mu = \{x \in E : x = \sum_{n=1}^{\infty} a_n x_n, (a_n) \in l_2\}.$$

where $p^{-1} + q^{-1} = 1$.

Proof. By the Hahn-Banach theorem there exists a sequence $(y_n) \subset E'$ such that

$$(x_n, y_m) = \begin{cases} 1 \text{ when } m = n \\ 0 \text{ when } m \neq n \end{cases}$$

It has been shown in [9] that

i) $x = \Sigma(x, y_n) x_n$ for μ-almost all x.

ii) $\mathcal{L}_\mu$ and l_p are isometric and the isometry is defined by the map $s : \mathcal{L}_\mu \to l_p$ which takes $f \in \mathcal{L}_\mu$ into the sequence $\{f(x_n)\} \in l_p$.

Put $A = S.I$ where I is the map taking $y \in E'$ into $\widetilde{y} \in \mathcal{L}_\mu$. We have

$$Ay = \{(x_n, y)\}$$

We shall show that

$$A^*b = \Sigma b_n x_n \qquad \text{for } b = (b_n) \in l_q.$$

Indeed, since $\Sigma \| x_n \|^p < \infty$, the series $\Sigma b_n x_n$ converges in E to an element $h \in E$. For each $y \in E'$ we have $(h,y) = \Sigma h_n(x_n, y) = (b, Ay) = (A^* b, y)$. So $A^*b = h = \Sigma b_n x_n$.

From Theorem 2.10 we have

$$H_\mu = I^*(\mathcal{L}'_\mu) = I^* s^* l_q = A^* l_q = \{ x \in E : x = \Sigma b_n x_n \ (b_n) \in l_q \}$$

Next, put $A = \{ x \in E : x = \Sigma a_n x_n,\ (a_n) \in l_2 \}$.

It has been shown in [9] that

$$A_\mu = \{ x \in E : \Sigma \mid (x, y_n) \mid^2 < \infty \}.$$

We now show that $A = A_\mu$. If $a \in A$ then $a_n = (a, y_n)$ so $a \in A_\mu$.
Conversely, let $a \in A_\mu$. Set $G = \{ x \in E : x = \Sigma (x, y_n) x_n \}$. By i) G is a measurable linear subspace of μ — measure one. By Theorem 2.6 we have $a \in G$. Hence $a = \Sigma (a, y_n) x_n$ with $\Sigma \mid (a, y_n) \mid^2 < \infty$, i.e. $a \in A$. The theorem is proved.

Noting that a centred Gaussian measure is a stable measure of index 2 with discrete spectrum, we obtain from Theorem 2.5 and Theorem 3.6 :

3.7. COROLLARY *If μ is a centred Gaussian measure on E, then H_μ is exactly the set A_μ and it can be characterized as the intersection of all measurable linear subspaces of μ — measure one.*

3.8. DEFINITION *A stochastic process $\{ X_t , 0 \leqslant t \leqslant 1 \}$ is said to be a stable motion of index p if*

i) $\{X_t\}$ has stationary and independent increment.

ii) $E[\exp\{uiX_t\}] = \exp\{- t \mid u \mid^p\}$.

From Zinn [10] and Theorem 2.10 we obtain the following.

3.9. THEOREM *Let $\{ X_t , 0 \leqslant t \leqslant 1 \}$ be the stable motion of index p. Then we have*

$$H_\mu = \begin{cases} \{ x(t) = \int_0^t g(s)ds,\ g \in L_q[0,1] \} & \text{if } 1 \leqslant p \leqslant 2 \\ \{ x(t) \equiv 0 \} & \text{if } 0 < p < 1. \end{cases}$$

$A_\mu = \{ x(t) \equiv 0 \}$ *if* $0 < p < 2$

where μ is the stable measure on $L_2[0,1]$ induced by the process X_t.

IV. EQUIVALENCE AND ORTHOGONALITY OF STABLE MEASURES

4.1. DEFINITION. *Let $\mathcal{H}$ be a family of probability measures on E. We say that the dichotomy (resp. weak dichotomy) theorem is true for $\mathcal{H}$ if any two measures in $\mathcal{H}$ are either equivalent (resp. weakly equivalent) or orthogonal.*

It is well known that the dichotomy theorem is true for Gaussian measures [4]. Kakutani [5] proved the dichotomy theorem for product measures on R^∞. The dichotomy theorem for stable measures still remains unknown. We shall prove here the weak dichotomy theorem for stable measures on locally convex vector spaces.

4.2. THEOREM. *Two stable measures on E are either weakly equivalent or orthogonal.*

Proof. Assume that μ and ν are two stable measures such that μ is not weakly equivalent to ν. Then there exists (for example) a sequence $(y_n) \subset E'$ such that $(., y_n) \to 0$ in μ-measure but $(., y_n)$ does not converge to 0 in ν-measure. Thus there exists a subsequence $(y_{n'})$ such that

$$\left| \widehat{\nu}(ty_{n'}) - 1 \right| > \varepsilon \tag{4—1}$$

for all n' and some $t \in R$, $\varepsilon > 0$.
We also have a subsequence $(y_{n'_k})$ of the sequence $(y_{n'})$ such that $(x, y_{n'_k})$ converges to 0 for μ-almost all x. Set

$$G = \{x \in E: \lim (x, y_{n'_k}) = 0\}$$

Clearly, G is a measurable linear subspace of μ-measure one. In virtue of the zero-one law [2] $\nu(G) = 0$ or 1. If $\nu(G) = 1$ then $(x, y_{n'_k})$ converges to 0 in ν-measure so $\widehat{\nu}(ty_{n'_k}) \to 1$ which contradict (4—1). Hence $\nu(G) = 0$. From this it follows that $\mu \perp \nu$.

In the next theorem we find a sufficient condition for two stable measures to be either equivalent or orthogonal.

4.3. THEOREM. *Let $\mu \in R_p(E)$ such that*

i) There exists a sequence $(y_n) \subset E'$ such that $\{(x, y_n)\}$ is a sequence of independent random variables in the probability space $(E, \mathcal{B}, \mu)$

ii) For each $y \in E'$ we have the expansion

$$(x, y) = \Sigma a_n (x, y) \qquad \text{in } \mu\text{-measure}$$

Then, if ν is a stable measure such that $\{(x, y_n)\}$ is also a sequence of independent random variables in the probability space $(E, \mathcal{B}, \nu)$, ν is either equivalent or orthogonal to μ.

Proof. By Theorem 4.2 we can assume $\mu \overset{w}{\sim} \nu$. Then for each $y \in E'$, we also have

$$(x, y) = \Sigma\, a_n\, (x, y_n) \qquad \text{in } \nu\text{-measure.}$$

Let $p_n(t)$ and $q_n(t)$ be the densities of (x, y_n) on the probability spaces $(E, \mathcal{B}, \mu)$ and $(E, \mathcal{B}, \nu)$ respectively. Put

$$p(x) = \prod_{n=1}^{\infty} \frac{p_n[(x, y_n)]}{q_n[(x, y_n)]}.$$

From assumption and the Kolmogorov zero-one law, it follows that

$$\nu\{0 < P(x) < \infty\} = 1 \text{ or } 0.$$

Assume first that $\nu\{0 < P(x) < \infty\} = 1$. Consider the measure λ given by

$$\lambda(B) = \int_B P(x)\, d\nu(x), \qquad B \in \mathcal{B}.$$

The characteristic function of λ is given by

$$\widehat{\lambda}(y) = \int_E \exp\{i(x, y)\}\, P(x)\, d\nu(x) = \int_E \exp\{i\, \Sigma\, a_n (x, y_n)\}$$

$$\prod_{n=1}^{\infty} \frac{p_n[(x, y_n)]}{q_n[(x, y_n)]}\, d\nu(x) = \prod_{n=1}^{\infty} \int_E \exp\left\{i\, a_n (x, y_n)\right\} \frac{p_n[(x, y_n)]}{q_n[(x, y_n)]}\, d\nu(x) =$$

$$= \prod_{n=1}^{\infty} \int_R \exp\left\{i\, a_n\, t\right\} p_n(t)\, dt = \exp\left\{-\Sigma\, |a_n|^p\right\} = \widehat{\mu}(y).$$

The measures λ and μ have the same characteristic function so $\lambda = \mu$.

This proves $\mu \ll \nu$. Since the Radon-Nikodym derivative $\frac{d\mu}{d\nu} = P(x)$ is positive for ν-almost all x we have $\nu \ll \mu$. Hence $\mu \sim \nu$.

Now suppose that $\nu\{0 < P(x) < \infty\} = 0$ and μ is not orthogonal to ν. Define a map $S : E \to R^{\infty}$ by $S(x) = \{(x, y_n)\}_{n=1}^{\infty}$. From the assumption, $S\mu$ and $S\nu$ are not orthogonal. By Kakutani's theorem [5], $S\mu$ and $S\nu$ are equivalent and

$$S\nu\left\{0 < \prod_{n=1}^{\infty} \frac{p_n(t_n)}{q_n(t_n)} < \infty\right\} = 1$$

From this

$$\nu\{0 < P(x) < \infty\} = 1.$$

A contradiction. Hence $\mu \perp \nu$ and the theorem is proved.

4.4. COROLLARY. *Let μ be a symmetric stable measure satisfying the condition i) and ii) in Theorem 4.3. Then μ is either equivalent or orthogonal to μ_a for each $a \in E$.*

Proof. Suppose that $\{(x, y_n)\}$ is a sequence of independent random variables on the probability space $(E, \mathcal{B}, \mu)$. Let $F(t_1, t_2 ..., t_n)$ be the characteristic function of the random vector $\{(x, y_k)\}_{k=1}^{n}$ and $f_k(t)$ be the characteristic function of the random variable (x, y_k) on the probability space $(E, \mathcal{B}, \mu_a)$. Then we have

$$F(t_1, t_2, ..., t_n) = \widehat{\mu}_a\left(\sum_{k=1}^{n} t_k y_k\right) = \exp\left\{i\left(a, \sum_{k=1}^{n} t_k y_k\right)\right\} \widehat{\mu}\left(\sum_{k=1}^{n} t_k y_k\right) =$$

$$= \exp\left\{i \sum_{k=1}^{n} (a, t_k y_k)\right\} \widehat{\mu}(t_1 y_1) ... \widehat{\mu}(t_n y_n) = \prod_{k=1}^{n} \exp\{i(a, t_k y_k)\} \widehat{\mu}(t_k y_k) =$$

$$= \prod_{k=1}^{n} f_k(t_k).$$

So $\{(x, y_k)\}_{k=1}^{\infty}$ is a sequence of independent random variables on the probability space $(E, \mathcal{B}, \mu_a)$. From Theorem 4.3, Corollary 4.4 follows.

Since a stable measure with discrete spectrum satisfies the condition i) and ii) in Theorem 4.3, we have

4.5 COROLLARY. *If μ is a stable measure with discrete spectrum then for every $a \in E$ either $\mu_a \sim \mu$ or $\mu_a \perp \mu$.*

The following theorem gives a sufficient condition for two stable measures to be orthogonal.

4.6 THEOREM. *Let μ and ν be two stable measures. If $H_\mu \neq H_\nu$ then $\mu \perp \nu$.*

Proof. From Proposition 2.3, it is easy to see that if $\mu \overset{w}{\sim} \nu$ then $H_\mu = H_\nu$. From this if $H_\mu \neq H_\nu$ then μ and ν are not weakly equivalent. Hence by Theorem 4.2 $\mu \perp \nu$.

This is not a necessary condition. Indeed, if μ is a non-Gaussian stable measure with discrete spectrum. then by Theorem 3.6 A_μ is a proper subset of H_μ. Hence we can choose $a \in H_\mu$ with $a \notin A_\mu$. We have $\mu \perp \mu_a$ but $\mu \overset{w}{\sim} \mu_a$, so $H_\mu = H_{\mu_a}$.

4.7. THEOREM. *Let E be a Banach space and let μ and ν be two stable measures of index p and p' respectively with discrete spectrum. If $p \neq p'$ then $\mu \perp \nu$.*

Proof. It is easy to prove that if $\mu \overset{w}{\sim} \nu$ then $\mathcal{L}_\mu$ is isomorphic to $\mathcal{L}_\nu$. Since $\mathcal{L}_\mu$ is isometric to l_p, $\mathcal{L}_\nu$ is isometric to l_p and $p \neq p'$ it follows that μ and ν are not weakly equivalent. Hence by Theorem 4.2 $\mu \perp \nu$.

4.8. THEOREM. *Suppose that $\mu \in R_p(E)$ and $\nu \in R_{p'}(E)$ $(1 \leqslant p, p' \leqslant 2)$. Then $\mu \overset{w}{\sim} \nu$ if and only if $H_\mu = H_\nu$ as sets and their norms are equivalent.*

Proof. Assume that $H_\mu = H_\nu$ as sets and their norms are equivalent. If μ is not weakly equivalent to ν then there exists (for example) a sequence $(y_n) \subset E'$ such that $\| \tilde{y}_n \|_\mu \to 0$ but $\| \tilde{y}_n \|_\nu \geqslant 1$, for all n.By Hahn-Banach Theorem and Theorem 2.10 we can find a sequence $(a_n) \subset H_\nu$ such that $\| a_n \|_{H_\nu} = 1$ and $(\tilde{a}_n, \tilde{y}_n) = \| \tilde{y}_n \|_\nu$. Since $(a_n) \subset H_\mu$ we have

$$1 \leqslant \| \tilde{y}_n \|_\nu = | (\tilde{a}_n, \tilde{y}_n) | = | (a_n, y_n) | \leqslant \| a_n \|_{H_\mu} \| \tilde{y}_n \|_\mu \leqslant c \| a_n \|_{H_\nu} \| \tilde{y}_n \|_\mu = c \| \tilde{y}_n \|_\mu \to 0.$$

This is impossible. Hence $\mu \overset{w}{\sim} \nu$.

The converse follws easily from Definition 2.1 and Proposition 2.3.

Received December, 15, 1983

REFERENCES

[1] A.de Acosta, *Stable measures and seminorms*, Ann. Proba., 5,3 (1975), 865 — 875.

[2] R.Dudley and M.Kanter, *Zero-one laws for stable measures*, Proc. Amer. Math. Soc., 45 (1974), 245 — 282.

[3] A.Dvoretsky and C.A.Rogers. *Absolute and unconditional convergence in normed linear spaces*, Proc. Nat Acad. Sci. U.S.A. 36 (1950), 192 — 197.

[4] J. Fedman, *Equivalençe and perpendicularity of Gaussian process*. Pacific J.Math., 8 (1958) 699 — 708.

[5] S.Kakutani, *On equivalence of infinite product measures*, Ann. Math., 49 (1948), 214 — 227.

[6] R.S. Liptser and A.N. Shiryaev, *Statistics of Stochastic processes* (in Russian), Moscow 1974.

[7] B.S.Rajput, *On the support of certain symmetric stable probability measures on TVS*, Proc. Amer. Math. Soc., 63 (1977), 306 — 312.

[8] M.Schilder, *Some structure theorems for the symmetric stable measures*, Ann. Math. Statist., 41 (1970), 412 — 428.

[9] Dang. H. Thang and Ng.D. Tiên, *On symmetric stable measures with discrete spectral measure on Banach space*, Springer-Verlag Lecture Notes in Math., 828 (1980), 286 — 301.

[10] J.Zinn, *Admissible translates of stable measure*, Studia Math., 54, (1976), 245 — 257.