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NOTE ON WAGNER-PLATEN'S REPRESENTATION OF SOLUTIONS OF A GENERAL FILTERING STOCHASTIC DIFFERENTIAL EQUATION

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We will show in the following that the solution of a general stochastic differential equation which was always dealt with in the nonlinear filtering theory can be represented as a stochastic generalization of the Taylor formula which is due to W. Wagner and E. Platen (see [2] or [3]).

This representation is useful for an approximate solution for filtering problems.

Let $(\Omega, \mathfrak{F}, P)$ be a probability space.

Consider a stochastic dynamical system described by a stochastic differential equation of the form

$$(1)\quad x_t = x_{t_0} + \int_{t_0}^{t} a(u, x_u)\, du + \int_{t_0}^{t} b(u, x_u)\, dW_u + \int_{t_0}^{t} c(u, x_u)\, dY_u$$

where

$t \in [t_0, T]$, $\mathfrak{F}_t$ is an increasing family of sub-σ fields of $\mathfrak{F}$, $x_t = \{x_t^i\}_{i=1}^m$ and $a(t, x) = \{a^i(t, x)\}_{i=1}^m$ are R^m valued vectors, $a(t, x)$ is — measurable and $\mathfrak{F}_t$ — adapted, $b(t, x) = \{b^{ij}(t, x)\}_{1 \leqslant j \leqslant n}^{1 \leqslant i \leqslant m}$ is a $m \times n$ — matrix and each $b^{ij}(t, x)$ is — measurable, and $\mathfrak{F}_t$ — adapted, $c(t, x) = \{c^{ij}(t, x)\}_{1 \leqslant j \leqslant \nu}^{1 \leqslant i \leqslant m}$ is a $m \times \nu$ — matrix and each $b^{ij}(t, x)$ is — measurable, and $\mathfrak{F}_t$ — adapted,

$W_t = \{W_t^j\}_{j=1}^n$ is a R^n — valued standard $\mathcal{F}_t$ — Wiener process, with independent components,

$Y_t = \{N_t^k - t\}_{k=1}^{\nu}$ where each N_t^k $(k=1,\dots,\nu)$ is a real valued standard Poisson process (i.e. a Poisson process of intensity 1) with independent components.

Moreover, we suppose that W_t^j $(1 \leqslant j \leqslant n)$ and N_t^k $(k=1,\dots,\nu)$ are mutually independent.

Suppose also that the following conditions on the existence and uniqueness of the solution of (1) are assured (see [4] for instance):

(A) $\mathcal{F}_t^{x t_0, W, y} \subseteq \mathcal{G}_t$, $\mathcal{G}_t \perp\!\!\!\perp \sigma[W_v - W_u;\ Y_q - Y_p;\ {t \leqslant u \leqslant v \leqslant T \atop t \leqslant p \leqslant q \leqslant T}]$

where

$$\mathcal{G}_t^{x t_0, W, y} = \sigma[x_{t_0}, W_s, Y_s;\ t_0 \leqslant s \leqslant t] \vee [\text{all } P - \text{null sets}].$$

(B)
$$\| a(t,x) - a(t,y) \|^2 + \| b(t,x) - b(t,y) \|^2 + \int_{t_0}^{t} \| c(t,x) - c(t,y) \|^2 \, dy_t \leq K \| x - y \|^2$$

with some positive constant K.

(C)
$$\| a(t,x) \|^2 + \| b(t,x) \|^2 + \int_{t_0}^{t} \| c(t,x) \|^2 \, d Y_t \leq K \left[1 + \int_{t_0}^{T} \| x(s) \|^2 \Gamma(ds) \right]$$

where Γ is a Borel measure on $[t_0, T]$.

On the Filtering Theory, W_t expresses a white noise while Y_t corresponds to a « point process noise » (See [1]).

It is well-known that $W_t^{(j)}$ is a martingale $(1 \leqslant j \leqslant n)$ and so is $Y_t^k (1 \leqslant k \leqslant \nu)$. Moreover, their quadratic variations equal to t:

$$\langle W^j \rangle_t = t \qquad (1 \leqslant j \leqslant n),$$
$$\langle Y^k \rangle_t = t \qquad (1 \leqslant k \leqslant \nu).$$

Now for $s, t \in [t_0, T]$, $s \leqslant t$, (1) can be written as

$$(2)\quad x_t = x_s + \int_s^t a(u, x_u)\, du + \sum_{j=1}^{n} \int_s^t b^j(u, x_u)\, dW_u^j + \sum_{k=1}^{\nu} c^k(u, x_u)\, d Y_u^k$$

where
$$b^j(t,x) = \left\{ b^{ij}(t,x) \right\}_{i=1}^{m}, \qquad 1 \leqslant j \leqslant n$$
$$c^k(t,x) = \left\{ c^{ik}(t,x) \right\}_{i=1}^{m}, \qquad 1 \leqslant k \leqslant \nu.$$

Set

$$\tilde{b}^{il}(t,x) = \begin{cases} b^{il}(t,x) & \text{if} \quad 1 \leqslant l \leqslant n, \\ c^{i,l-n}(t,x) & \text{if} \quad n+1 \leqslant l \leqslant n+\nu. \end{cases}$$

Denote by $\tilde{b}(t, x)$, a $m \times (n+\nu)$ — matrix of elements $\tilde{b}^{il}(t, x)$:

$$\tilde{b} = (\tilde{b}^{il}) = \begin{pmatrix} b^{11} & \dots & b^{1n} & c^{11} & \dots & c^{1\nu} \\ b^{21} & \dots & b^{2n} & c^{21} & \dots & c^{2\nu} \\ \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ b^{m1} & \dots & b^{mn} & c^{m1} & \dots & c^{m\nu} \end{pmatrix}.$$

Let $\tilde{W}^l_t$ $(1 \leqslant l \leqslant n+\nu)$ be a martingale defined by

$$\tilde{W}^l_t = \begin{cases} W^l_t & \text{if} \quad 1 \leqslant l \leqslant n \\ Y^{l-n}_t & \text{if} \ n+1 \leqslant l \leqslant n+\nu. \end{cases}$$

Equation (2) then becomes:

$$(3) \qquad x_t = x_s + \int_s^t a(u,x_u)\, du + \sum_{l=1}^{n+\nu} \int_s^t \tilde{b}^l(u,x_u)\, d\tilde{W}^l_u,$$

where $\tilde{b}^l(t, x) = \left\{ \tilde{b}^{il}(t, x) \right\}_{l=1}^{m}$, $1 \leqslant l \leqslant n+\nu$.

It is obvious that $x_t = \left\{ x^i_t \right\}_{i=1}^m$ is a m-dimensional continuous local semimartingale of the form

$$x_t = x_s + M_t + A_t$$

where $M_t = \sum_{l=1}^{n+\nu} \int_s^t \tilde{b}^l(u,x_u)\, d\tilde{W}^l_u$ is a R^m-valued continuous local martingale,

$A_t = \int_s^t a(u,x_u)$ du is a R^m- valued process of bounded variation.

Let $F: [t_0, T] \times R^m \to R^m$ be a C^2- class function. Then by Ito formula, F (t, x_t) is also a m-dimensional continuous local semimartingale:

$$(4) \quad F(t, x_t) - F(s, x_s) = \int_s^t \frac{\partial F}{\partial S}(u, x_u)\, du + \sum_{i=1}^m \int_s^t \frac{\partial F}{\partial x^i}(u, x_u)\, d M^i_u +$$

$$+ \sum_{i=1}^m \frac{\partial F}{\partial x^i}(u, x_u)\ dA^i_u + \frac{1}{2} \sum_{ij=1}^m \int_s^t \frac{\partial^2 F}{\partial x^i \partial x^j}(u,x_u)\, d\,\langle M^i, M^j \rangle_u .$$

It holds

$$\sum_{i=1}^{m}\int_s^t \frac{\partial F}{\partial x^i}(u,x_u)\, d M^i_u = \sum_{i=1}^{m}\sum_{l=1}^{n+v}\int_s^t \frac{\partial F}{\partial x^i}(u,x_u)\,\tilde{b}^{il}(u,x_u)\, d\tilde{W}^l_u .$$

Noticing that $\tilde{W}^l$ ($l = 1, 2, \ldots n+v$) are independent

and $\langle W^l , W^{l'} \rangle_t = \begin{cases} t & \text{if } l = l' \\ 0 & \text{other wise} \end{cases}$

we have

$$\langle M^i , M^j \rangle_t = \langle \sum_{l=1}^{n+v}\int_s^t \tilde{b}^{il}(u,x_u)\, d\tilde{W}^l , \sum_{l=1}^{n+v}\int_s^t \tilde{b}^{jl}(u,x_u)\, d\tilde{W}^l \rangle =$$

$$= \sum_{l=1}^{n+v}\int_s^t \tilde{b}^{il}(u,x_u)\,\tilde{b}^{jl}(u,x_u)\, du$$

Then (4) takes the following form

$$(5)\ F(t,x_t) = F(s,x_s) + \int_s^t \Big[\frac{\partial F}{\partial S}(u,x_u) + \sum_{i=1}^{m} a^i(u,x_u)\,\frac{\partial F}{\partial x^i}(u,x_u) +$$

$$+ \frac{1}{2}\sum_{i,j=1}^{m}\sum_{l=1}^{n+v}\frac{\partial^2 F}{\partial x^i \partial x^j}\,\tilde{b}^{il}(u,x_u)\,\tilde{b}^{jl}(u,x_u)\Big]\, du +$$

$$+ \sum_{i=1}^{m}\sum_{l=1}^{n+v}\int_s^t \frac{\partial F}{\partial x^i}(u,x_u)\,\tilde{b}^{il}(u,x_u)\, d\tilde{W}^l_u .$$

Using of the operation L^k ($k = 0,1,\ldots n+v$) introduced by W. Wagner and *E.* Platen (cf. [2] or [3]) we have

$$L^k F = \begin{cases} \dfrac{\partial F}{\partial t} + \sum\limits_{i=1}^{m} a^i \dfrac{\partial F}{\partial x^i} + \dfrac{1}{2}\sum\limits_{i,j=1}^{m}\sum\limits_{l=1}^{n+v}\tilde{b}^{il}\tilde{b}^{jl}\,\dfrac{\partial^2 F}{\partial x^i \partial x^j} & \text{if } k = 0 \\ \sum\limits_{l=1}^{m}\tilde{b}^{il}\,\dfrac{\partial F}{\partial x^i} & \text{if } 1 \leqslant k \leqslant n+v \end{cases}$$

We denote by $\tilde{W}^0_t = t$ for convenience.

Then (5) can be rewritten as

$$(6) \qquad F(t,x_t) = F(s, x_s) + \sum_{k=0}^{n+v} \int_s^t (L^k F)(u,x_u)\, d\tilde{W}_u^k .$$

Now we are in position to get the following

PROPOSITION. *In preserving all assumptions and notations of Theorem 1 in [2] with the additional assumptions mentioned in the first part of the note, we have the Wagner — Platen's representation for the solution of the equation (3):*

$$(7) \qquad x_t = x_s + \sum_{\alpha \in A} F_\alpha(s,x_s)\, I_\alpha(s,t) + \sum_{\alpha \in B(A)} I_\alpha(F_\alpha, s, t)$$

where

$\alpha = (j_1, \dots, j_k)$ is a row vector of finite length $(k < \infty)$ and components $j_i \in \{0,1,\dots,n\}$.

A is any subset of $M = \{(j_1, \dots, j_k) ;\ k = 1,2,\dots\}$,

$B(A) = \{ \alpha \in M \setminus A : \alpha$ deleted the first component $\in A\}$,

$$E_\alpha = \left(F^i_\alpha\right)_{i=1}^m , \quad F^i_\alpha = \begin{cases} x^i & \text{for } \alpha = V, \text{ vector of no component.} \\ a^i & \text{for } \alpha = (0), \\ b^{ij} & \text{for } \alpha = (j),\ j = 1,2,\dots,n, \\ L^{j1} \dots L^{jk-1} F_{(jk)} & \text{for } \alpha = (j_1, \dots, jk) \text{ with } k \geq 2. \end{cases}$$

$$I_\alpha(g, s, t) = \int_S^t \int_S^{u_k} \cdots \int_S^{u_2} g(u_1, x_{u_1})\, d\tilde{W}_{u_1}^{j_1} \dots d\tilde{W}_{u_k}^{jk}$$

$I_\alpha(s,t) = I_\alpha(1, s, t)$.

Notice finally that when $\tilde{W}_t = Y_t$ which has bounded variation, $\int^t \phi_s\, d\tilde{W}_s$ is therefore a stochastic Stieltjes integral. But for the same treatment for both cases of $\tilde{W}_t$, we required that $\int_{t_0}^t \phi_s\, dY_s$ can be regarded as an integral which is constructed in Ito's Theory. For this purpose we have to impose on ϕ_t the additional condition: ϕ_t is $\mathcal{F}_t$ — predictable.

It means for the case that $a(t,x_t)$ and $b(t,x_t)$ are in addition $\mathcal{G}_t$ — predictable. Then the integrals such as

$$\int_S^t \int_S^{u_k} \cdots \int_S^{u_2} F_\alpha\, d\tilde{W}_{u_1}^{j_1} \dots d\tilde{W}_{u_k}^{j_k},$$

are treated in computation in a same way when the $\widetilde{W}_t^i$'s are either Wiener processes or Poisson martingales.

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