ACTA MATHEMATICA VIETNAMICA
Volume 8 Number 2 (1983)

LOCALLY LIPSCHITZIAN SET–VALUED MAPS AND GENERALIZED EXTREMAL PROBLEMS WITH INCLUSION CONSTRAINTS

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INTRODUCTION

During the last decade the theory of set-valued maps has risen to become one of the most useful tools for the study of many important optimization problems.

In [1] Boltyanskii considered extremal problems for discrete systems given by set-valued maps and established an optimality criterion of the type of the maximum principle when the set-valued maps are supposed to have smooth local selections. Later, Pham Huu Sach [4a], studying generalized extremal problems with inclusion contraints where the set-valued maps on the right hand side have smooth support functions, obtained a support principle which yields just the maximum principle when applied to discrete systems.

The primary purpose of the present paper is to extend these results to a class of locally lipschitzian set-valued maps and to study some closely related problems.

The method to be used is to characterize a set-valued map by some associated real-valued functions, in such a manner that, instead of studying the set-valued map directly on can consider only its « characteristic » functions. The advantage of this approach is that it allows the theory of generalized derivative of single-valued maps to be used for set-valued maps as well. This turns out to be very convenient in many circumstances. In particular, in that way we are able to establish the support principle for generalized extremal problems under much weaker assumptions than in [1] and [4a], namely assuming only that the set-valued map under consideration is locally lipschitzian while the constraint set is not necessarily convex. In addition we shall obtain for set-valued maps an «Interior Mapping Theorem». Propositions of this kind have attracted much attention from researchers in recent years; for the essential results on this subject, see [7] and [4a]. Finally the method will permit us to elucidate the relationship between the Clarke derivative of a set-valued map as defined by Aubin in [7] and the adjoint of this set-valued map as introduced by Pshenichnyi in [6].

1. DEFINITIONS AND PRELIMINARY PROPERTIES

Let X, Y be two Hilbert spaces, $Z = X \times Y$ their cartesian product, X^*, Y^*, Z^* the dual spaces to X, Y, Z. A set-valued may $F : X \to 2^Y$ is said to be *locally lipschitzian* at a point $x \in X$ if there exists a real positive number α and a neighbourhood U of x such that

$$F(x_1) \subset F(x_2) + \alpha \| x_1 - x_2 \|_X \cdot B_Y(O, 1) \qquad (1.1)$$

for every x_1, $x_2 \in U$, where $\|\cdot\|_X$ denotes the norm in X and $B_X(O, \delta)$ the ball in X of radius σ and center O (when no confusion is possible we shall delete the subscript X in this notation; also we shall write sometimes $B^*_X(O, \delta)$ instead of $B_{X^*}(O, \delta)$).

Denote by Γ the graph of F:

$$\Gamma = \{(x,y) \in Z \mid Y \in F(x)\},$$

and let

$$d_F(z) - d(z; \Gamma) = \inf\{\|z - \zeta\| : \zeta \in \Gamma\} \ \forall\ z \in Z,$$
$$f(z) = f(x,y) = d(x; F(x)) = \inf\{\|y - v\| : v \in F(x)\} \ \forall\ z \in Z,$$
$$C^F(y^*,x) = \sup\{< y^*, v >: v \in F(x)\} \ \forall\ x \in X,\ \forall\ Y^* \in Y^*.$$

Throughout the sequel, unless otherwise specified, we shall always assume that the set-valued map F under consideration is locally lipschitzian at every point $x \in X$, and that for every $x \in X$ the set $F(x)$ is nonempty, closed and convex. It can easily be proved that for such a map the set

$$\{y^* \in Y^* \mid C^F(y^*, x) < \infty\}$$

is a nonempty convex cone not depending upon x. We shall denote this cone by Y^*_F.

From the above definition we can readily derive the following simple properties.

PROPERTY 1.1

(i) $z \in \Gamma \Leftrightarrow d_F(z) = 0 \Leftrightarrow f(z) = 0$;

(ii) $d_F(z) \leqslant f(z) \ \forall \ z \in Z$;

(iii) *Let* α *be the Lipschitz constant of* f *on some neighbourhood of a point* $x_o \in \Gamma$. *Then there is a neighbourhood* U *of* z_o *such that for every* $z \in U$:

$$j(z) \leqslant \alpha d_E(z).$$

Proof:

(i) and (ii) are trivial. To prove (iii) consider a neighbourhood U_1 of z_o such that f is lipschitzian with Lipschitz constant α on U_1. Let β be a real positive number satisfying $z_o + B_Z(0, 3\beta) \subset U_1$. Then for every $z \in U = z_o + B_Z(0, \beta)$ and every $\varepsilon > \beta$ we can find a point $z_1 \in \Gamma$ such that $\| z - z_1 \| \leqslant d(z, \Gamma) + \varepsilon$. Noting that $d(z, \Gamma) \leqslant \| z - z_o \| \leqslant \beta$, we have $\| z_1 - z_o \| \leqslant 2\beta + \varepsilon \leqslant 3\beta$, i.e. $z_1 \in U_1$, and hence,

$$f(z) - f(z_1) \leqslant \alpha \| z - z_1 \| \leqslant \alpha d(z, \Gamma) + \alpha\varepsilon.$$

Since $j(z_1) = 0$ and ε is arbitrary, we conclude $f(z) \leqslant \alpha d_E(z)$, as was to be shown.

PROPERTY 1.2

A set-vallued map F *is locally lipschitzian at a point* $x_o \in X$ *if and only if for some neighbourhood* U *of* x_o *the function* f *is lipschitzian with respect to* (x, y) *on* $U \times Y$.

Proof:

If the set-valued map F is locally lipschitzian we have from (1.1):

$$d(y; F(x_1)) \geqslant d(y; F(x_2) + \alpha \| x_1 - x_2 \| . B(0, 1))$$

for every $y \in Y$, $x_1, x_2 \in U$. It is easily seen that

$$d(y; \ F(x_2) + \alpha \| x_1 - x_2 \| \ . B(0, \ 1)) \geqslant d(y; \ F(x_2)) - \alpha \| x_1 - x_2 \| ,$$

and hence,

$$d(y; F(x_1)) \geqslant d(y: F(x_2)) - \alpha \| x_1 - x_2 \|$$

for every $y \in Y$, $x_1, x_2 \in U$. This means that the function f(x, y) is locally lipschitzian with respect to x (with Lipschitz constant α) for every fixed $y \in Y$.

On the other hand, the function $d(y; F(x))$ is Lipschitz continuous with respect to y (with Lipschitz constant 1). It follows that the function $f(x, y)$ is lipschizian with respect to (x, y) on $U \times Y$.

Conversely, if for some neighbourhood U of $x_o \in X$ the function $f(x, y)$ is lipschitzian on $U \times Y$, we can find a number α such that

$$f(y, x_1) \leqslant f(y, x_2) + \alpha \| x_1 - x_2 \|$$

for every $x_1, x_2 \in U$, $y \in Y$. Then for every $y \in F(x_2)$ we have $d(y; F(x_1)) \leqslant \leqslant \alpha \| x_1 - x_2 \|$. Thus $F(x_2) \subset F(x_1) + \alpha \| x_1 - x_2 \|$. (0, 1)
for every $x_1, x_2 \in U$, which implies that F is locally lipschitzian.

PROPERTY 1.3

A set-valued map F is convex if and only if the function f is convex.
The proof is immediate.

PROPERTY 1.4

For every $z \in Z$ we have

$$f(z) = f(x, y) = -\min_{\substack{y^* \in Y^*_F \\ \| y^* \| \leqslant 1}} \{C^F(y^*, x) - \langle y^*, y \rangle\} \tag{1.2}$$

If $z \notin \Gamma$ there exists a unique element y^* satisfying

$$f(x, y) = - C^F(y^*, x) + \langle y^*, y \rangle$$

and then $\| y^* \| = 1$.

Proof :

The proof is straightforward from the following Lemma which in turn is an immediate consequence of the separation theorem.

LEMMA 1.1

If B is a convex set in the Hilbert space y, then

$$d(0; B) = - \min_{y \in B_1^\infty} \sup_{b \in B} \langle y^*, b \rangle, \tag{1.3}$$

and if $d(O; B) > 0$, there exists a unique element $y^* \in B_1^\infty$ satisfying

$$d(O; B) = - \sup_{b \in B} \langle y^*, b \rangle,$$

where B_1^∞ denotes the set $\{ y^* \in Y^* \mid \sup_{b \in B} \langle y^*, b \rangle < \infty ; \| y^* \| \leqslant 1 \}$.

PROPERTY 1.5

A set-valued map F is locally lipschitzian if and only if the support function $C^F(y^*, x)$ is locally lipschitzian with respect to x uniformly for y^* in $B^*_F = Y^*_F \cap B^*_Y(0,1)$ (i.e. with a Lipschitz constant not depending on $y^* \in B^*_F$).

This follows immediately from Property 1.4

PROPERTY 1.6

A set-valued map F is convex if and only if the support function $C^F(y^*, x)$ is concave in x.

The proof is trivial.

2. SUPPORT FUNCTION ; GENERALIZED GRADIENT AND NORMAL CONE

In what follows $\partial h(x)$ will stand for the generalized gradient of a locally lipschitzian function h at the point x; $h^0(x, .)$ the generalized directional derivative of h at x; $T(z, \Gamma)$ the tangent cone to the set Γ at the point $z \in \Gamma$; $N(z, \Gamma)$ the normal cone of Γ at z, as were defined by Clarke (see [2a, 2b]).

A locally lipschitzian function $f(x)$ is said to be *Lipschitz-regular* at x (in Clark's sense) if for every $v \in X$ the directional derivative $f'(x; v)$ exists and coincides with the generalized directional derivative $f^0(x; v)$.

The following proposition shows a relation between the graph of F and the function f.

PROPOSITION 2.1

If $z_0 \in \Gamma$, *then*

(a) $T(z_0, \Gamma) = \{ \zeta \in Z \mid f^0(z_0, \zeta) = 0 \}$

(b) $N(z_0, \Gamma) = \overline{\bigcup_{t \geqslant 0} t \partial f(z_0)}$

where the bar indicates the closure.

Proof :

Since $T(z_0, \Gamma) = \{ \zeta \in Z \mid d(z_0, \zeta) = 0 \}$ and $N(z_0, \Gamma) = \overline{\bigcup_{t \geqslant 0} t \partial d(z_0)}$ (see [2a, 2b]), this Proposition is an immediate consequence of the following

LEMMA 2.1

If $z_0 \in \Gamma$, *then we have*

(i) $d^0(z_0, \zeta) \leqslant f^0(z_0; \zeta) \leqslant \alpha d^0(z_0, \zeta)$

for every $\zeta \in Z$, *and*

(*ii*) $\partial d(z_0) \subseteq \partial f(z_0) \subseteq \alpha \partial d(z_0)$

where α *is a Lipschitz-constant of* f *in a neighborhood of* z_0.

Proof of Lemma 2.1

Given $\zeta \in Z$, for every $z = (x, y) \in Z$ and every $\varepsilon > 0$ we can find $Y_1 \in F(x)$ such that

$$f(x,y) \geqslant \| y - y_1 \| - \varepsilon.$$

It is easy to see that $z^1 = (x, y_1) \in \Gamma$ and satisfies the following conditions

(i) $\| z - z^1 \| \leqslant f(z) + \varepsilon$

(ii) $f(z + \zeta) - f(z) \leqslant f(z^1 + \zeta) + \varepsilon$.

Hence, for every two sequences $\{z_n\} \subset Z$, $\{\varepsilon_n\} \subset R$ we can find a sequence $\{z'_n\} \subset \Gamma$ such that

(iii) $\|z^1_n - z_n\| \leqslant f(z_n) + \delta^2_n$,

and

(iv) $f(z_n + \delta_n \zeta) - f(z_n) \leqslant f(z^1_n + \delta_n \zeta) + \delta^2_n$.

If $z_n \to z_0$ and $\delta_n \to 0$ (as $n \to \infty$), then $z^1_n \to z_o$, since $f(z_n) \underset{(n \to \infty)}{\to} f(z_o) = 0$. From (iv) we have

$$\limsup_{\substack{\delta_n \to 0 \\ z_n \to z_o}} \frac{f(z_n + \delta_n \zeta) - f(z_n)}{\delta_n} \leqslant \limsup_{\substack{\delta_n \to 0 \\ z'_n \to z_n \\ z'_n \in \Gamma}} \frac{f(z^1_n + \delta_n \zeta)}{\delta_n}.$$

and hence

$$f^o(z_o, \zeta) \leqslant \limsup_{\substack{\delta \to 0 \\ z \to z_o \\ z \in \Gamma}} \frac{f(z + \delta\zeta)}{\delta}.$$

The converse inequality being trivial we obtain

$$f^{o}(z_o\,;\,\zeta) = \limsup_{\substack{\delta\to 0\\ z\to z_o\\ z\in\Gamma}} \frac{f(z+\delta\zeta)}{\delta}.$$

An analogous equality for the function $d(z)$ was established in [9] for the case of finite-dimensional spaces. By an argument similar to the previous one we can see that it still holds in the general case.

Lemma 2.1 now follows from applying Property 1.1.

The next lemma will play a crucial role in the rest of the paper.

LEMMA 2.2

Let Ω be a compact space, and let $g: X \times \Omega \to R$ be such that:

(a) $g(x,.)$ *is l.s.c. in* ω;

(b) $g(., \omega)$ *is locally lipschitzian in* x, *uniformly for* ω *in* Ω;

(c) $\delta_x g(x,\omega)$ *is u.s.c. in* (x,ω).

If $\varphi(v) = \min\{g(x,\omega);\ \omega\in\Omega\}$, *then;*

(i) φ *is locally lipschitzian;*

(ii) $\partial\varphi(x) \subseteq co\{x^*\in X^* \mid x^*\in\partial_x g(x,\omega);\ \omega);\ \omega\in I(x)\}$,

where $I(x) = \{\omega\in\Omega \mid g(x,\omega) = \varphi(x)\}$.

Remark 2.1

A similar result was obtained by Clarke for the case where X is finite-dimensional and g $(., \omega)$ satisfies a condition of regularity with respect to x (see [2a], Theorem 2.1).

Proof of Lemma 2.2

First note that since g is l.s.c. in ω and Ω is compact, the function, φ is well defined. For the same reasons I $(x) \neq \phi$. The proof of (i) is immediate from (b). To prove (ii) we need the following result of Thibault [5], 2—2):

Let $h: X \to R$ be a locally lipschitzian function, H be a subset of X such that X/H is a Haar-nul set and at every $x\in H$ the function h is Gâteaux differentiable and has Gâteaux differential $\nabla h(x)$. Then we have

(1) $h^{\circ}(\overline{x}, v) = max\{\langle x^*, v\rangle \mid x^* \in L_H(h, \overline{x})\}$ *for every* $v \in X$,

(2) $\partial h(\overline{x}) = \{\overline{co}\, L_H(h, (\overline{x})\}$,

where $L_H(h, \overline{x}) = \{\lim_{n\to\infty} \nabla h(x_n) \mid x_n \in H, x_n \underset{(n\to\infty)}{\longrightarrow} \overline{x})\}$ *and the «limit» of* $\{\nabla h(x_n)\}$ *is in the weak* topology.*

Now, from Christensen's Theorem [8] applied to the locally lipschitzian function φ it follows that there exists a subset $M \subset X$ such that φ is Gâteaux differentiable on M and $X \setminus M$ is a Haar-null set.

For every $x_n \in M, \tilde{\omega} \in I(x_n), v \in X$ we have

$$\langle \nabla\varphi(x_n), v\rangle = \lim_{\delta\to 0} \frac{\varphi(x_n + \delta v) - \varphi(x_n)}{\delta}$$

$$\leqslant \limsup_{\delta\to 0} \frac{g(x_n + \delta x, \tilde{\omega}) - g(x_n, \tilde{\omega})}{\delta}$$

hence

$$\langle \nabla\varphi(x_n), v\rangle \leqslant g_x^{\circ}(x_n, \tilde{\omega}, v),$$

i.e

$$\nabla\varphi(x_n) \in \partial_x g(x_n, \tilde{\omega}) \tag{2.1}$$

It is easily seen that the set-valued map $x \mapsto I(x)$ is u.s.c. and from condition (c) the set-valued map $x \mapsto G(x)$ defined by

$$G(x) = \{x^* \mid x^* \in \partial_x g(x, \tilde{\omega}); \tilde{\omega} \in I(x)\}$$

is u.s.c. as well.

From (2.1) we have

$$L_M(\varphi, x) \subseteq G(x).$$

Part (ii) of Lemma 2.2 now follows from the second part of the mentioned result of Thibault and the compactness of the set $G(x)$ (in the weak* topology).

Remark 2.2

If the function $[-g(\cdot, \omega)]$ is Lipschitz regular with respect to x, then $[-\varphi]$ is Lipschitz regular, and we have the equality in (ii).

Property 1.5 shows that the support function $C^F(y^*, x)$ is locally lipschitzian with respect to x, so that we can consider the generalized subdifferential of C^F with respect to x, i.e. the set $\partial_x C^E(y^*, x)$. One might ask aboet the upper semicontinuity of the set-valued map $(y^*, x) \mapsto \partial_x C^F(y^*, x)$. In many special cases this property can be established without difficulty. but the situation is more complicated in the general case. For brevity of presentation we shall make the blanket assumption that the set-valued map $\partial_x C^F(y^*, x)$ is u.s.c. in (y^*, x) and that $B_E^* = Y_E^* \cap B_Y^*(0, 1)$ is closed (and hence, is compact in the weak* topology).

From Property 1.4 and Lemma 2.2 we can deduce the following proposition which states one of the most important properties of support functions.

PROPOSITION 2.2

Under the stated assumption we have

$$\partial f(z) \subseteq co\{(x^*, y^*) \mid x^* \in -\partial_x C^F(y^*, x), y^* \in I(z)\},$$

where $I(z) = \{ y^* \in B_F^* \mid \langle y^*, y\rangle - C^F(y^*, x) = f(z)\}$.

If $f(z) > 0$, *then* $I(z)$ *consists of one single element* y^*, *with* $\| y^* \| = 1$, *and the symbol «co» can be deleted.*

Remark 2.3

If $z \in \Gamma$, the $I(z) = \{y^* \in B_F^* \mid C^F(y^*, x) = \langle y^*, y\rangle\}$, i.e. $I(z)$ is the set of normal vectors to the set $F(x)$ at the point $y \in F(x)$. More exactly we have.

$$I(z) = \{y^* \in N(y\,;\, F(x)) \mid \| y^* \| \leqslant 1\}.$$

As a consequence of Propositions 2.1 and 2.2 we get the following theorem about the relationship between the normal cone $N(z\,;\,\Gamma)$ and the support function.

THEOREM 2.1.

If $z \in \Gamma$, *then*

$$N(z, \Gamma) \subseteq \overline{CO}\, \{ (x^*, y^*) \mid x^* \in -\partial_x C^F(y^*,x)\,;\, y^* \in N(y\,;\, F(x)) \}.$$

Remark 2.4

If the function $-C^F(y^*, x)$ is Lipschitz regular with respect to x (which is the case, for example, when F is convex) we can show that the set

$$G(x) = \{ (x^*, y^*) \mid x^* \in -\partial_x C^F(y^*, x), y^* \in I(x) \}$$

is convex, so that the symbol «co» can be dropped. On the other hand the set $\{ (x^*, y^*) \mid x^* \in -\partial_x C^F(y^*, x). y^* \in N(y\,;\, E(x)) \}$ is closed, and from Remark 2.2 we have

$$N(z, \Gamma) = \{ (x^*, y^*) \mid x^* \in -\partial_x C^F(y^*, x), y^* \in N(y, F(x)) \}.$$

Following Aubin [7] let us define Clarke's derivative of a set-valued F at a point $z_o = (x_o, y_o) \in graph\ F$ as the set-valued *map* $DF_{z_o}(.)$ from X to Y whose graph is the Clarke's tangent cone to the graph of F at z_o. In other words, $v \in DF_{z_o}(u)$ if and only if $(u, v) \in T(z_o, \Gamma)$. Further, following Pshenichnyi [6] let us associate to the set-valued map F from X to Y at the point $z_o = (z_o, y_o)$ the adjoint set-valued *map* F^* from Y^* to X^* defined by

$$F^*(y^*) = -\partial_x C^F(y^*, x_o),$$

for every $y^* \in N(y_o, F(x_o))$.

Theorem 2. 1 relates the derivative to the adjoint of the set-valued map, and Remark 2 .4 says that when the map F is Lipschitz regular (in the sense that the function $-C^F(y^*, x)$ is Lipschitz regular) then the adjoint is exactly dual to the derivative (in the sense that the graph of the adjoint is dual to the graph of the derivative). In the special case when F is convex, we obtain the results in [6].

THEOREM 2.2 (Interior Mapping Theorem)

If Y is finite-dimensional, and if for every $y^* \in Y_F^*$ ... $\|y^*\| = 1$ we have

$$0 \notin \partial_x C^F(y^*, \overline{x}) \tag{2.2}$$

then for every positive real number δ

$$F(x) \subset int\ F(\overline{x} + \overline{B}(0, \delta)).$$

Proof :

Assume the contrary, that there exists an element $y_o \in F(\overline{x})$ such that $y_o \notin int\ F(\overline{x} + \overline{B}(0, \delta))$.. Then we can find a sequence $\{y_n\}$ such that $y_n \to y_o$ and $y_n \notin F(x + \overline{B}(0, \delta))$ for every n. Taking $\varepsilon_n = \| y_n - y_o \| > 0$ and setting $\varphi_n(x) = f(\overline{x} + x, y_n)$ we have $\varphi_n(x) \geqslant 0$ for every $x \in \overline{B}(0, \delta)$, and $\varphi_n(0) = f(\overline{x}, y_n) \leq \| y_n - y_o \| = \varepsilon_n$. From Ekeland's Variational Principle [3] we can find an element $v_n \in \overline{B}(0, \delta)$ such that $\| v_n - 0 \| \leq \sqrt{\varepsilon_n}$, and that

$$0 \in \partial\varphi(V_n) + \sqrt{\varepsilon_n}\, B_X^*(0, 1) \tag{2.3}$$

Since $\varphi_n(v_n) = f(x + v_n, y_n) > 0$ from Proposition 2 . 2 we can assert that there exists an unique element $y_n^* \in Y_F^*$ such that $\| y_n^* \| = 1$ *and*

$$\partial\varphi(v_n) \subset -\partial_x C^F(y_n^*, v_n + \overline{x}) \tag{2 . 4}$$

Combining (2 . 3) and (2 . 4) gives

$$0 \in -\partial_x C^F(y_n^*, v_n + \bar{x}) + \sqrt{\varepsilon_n}\, B^*(0, 1) \tag{2.5}$$

Since $\| y_n^* \| = 1$ for every n, we can assume by taking a subsequence if necessary, that the sequence $\{y_n^*\}$ converges to some element $y_0^* \in Y_F^*$ with $\| y_0^* \| = 1$.

Letting $n \to \infty$ in (2.5) yields

$$0 \in \partial_x C^F(y_0^*, \bar{x}),$$

which contradicts (2.2). The proof is complete.

Remark 2. 5.

In order the Theorem be true in the general case where Y is infinite-dimensional we have to require a stronger condition than (2. 2), namely that there exist a positive real number ε and a neighbourhood U of x such that

$$0 \notin \partial_x C^F(y^*, x) + B^*(0, \varepsilon),$$

for every $x \in U$ and every $y^* \in Y_F^*$, $\| y^* \| = 1$.

3. GENERALIZED EXTREMAL PROBLEM

We shall say that a convex cone K in a Hilbert space E is nontrivial if K is not a subspace. Given a nontrivial convex cone K in E, a point $x \in X$ is said to be K-*optimal* for a single-valued map $S : X \to E$ on the set $M \subset X$ if for every $x \in M$ satisfying $s(x) - s(\bar{x}) \in K$ we have $s(x) - s(\bar{x}) \in K$.

The following theorem generalizes a result of [4a]

THEOREM 3. 1. (Support Principle)

Let Y be a finite-dimensional space, K a nontrivial closed convex cone in a finite-dimensional space E, $s : X \to E$ a locally lipschitzian single-valued map, and C a closed subset of X. If $\bar{x}$ is a K-optimal point for $s(.)$ on the set

$$M = \{ x \in C \mid 0 \in F(x) \},$$

then there exist vector $y^* \in Y_F^*$, $k^* \in K^*$ not all zero such that

(i) $0 \in \partial_x \langle k^*, s(\bar{x}) \rangle - \partial_x C^F(y^*, \bar{x}) + N(\bar{x}, C)$

(ii) $\sup \{ \langle y^*, v \rangle \mid v \in F(\bar{x}) \} = 0$

Proof :

Since K is nontrivial we can choose an element $k_o \in K$ with $\| k_o \| = 1$ such that $k_o \notin -K$. Eor every $\varepsilon > 0$ let $K_\varepsilon = \varepsilon k_o + K$. It is obvious that K_ε is a closed convex set, $K_\varepsilon \subset K$, and

$$K_\varepsilon^\infty = \{ k^* \in E^* \mid \sup_{k \in K_\varepsilon} \langle k^*, k \rangle < \infty \}$$

$$= \{ k^* \in E^* \mid \sup_{k \in K} \langle k^*, k \rangle = 0 \} = K^*$$

Setting $s_\varepsilon(x) = d(s(x) - s(\bar{x})\ ;\ K_\varepsilon)$, we have by Lemma 1. 1

$$s_\varepsilon (x) = -\min_{k^* \in K_1^*} \{ \langle k^*, s(\bar{x}) - s(x) + \varepsilon k_o \rangle \},$$

where $K_1^* = \{ k^* \in K^* \mid \| k^* \| \leqslant 1 \}$

Since $\partial_x \langle k^*, s(x) \rangle$ is u. s. c. in (k^*, x), from Lemma 2. 2 we get

$$\partial s_\varepsilon(x) \subseteq co \{ \partial_x \langle k^*, s(x) \rangle \mid k^* \in I_s(x) \},$$

where $I_s(x) = \{ k^* \in K_1^* \mid s_\varepsilon(x) = \langle k^*, s(x) - s(\bar{x}) - \varepsilon k_o \rangle \}$.

If $S_\varepsilon (x) > 0$ the set $I_s(x)$ consists of just one element $\{ k_\varepsilon^* \}$ with $\| k_\varepsilon^* \| = 1$ and in that case

$$\partial s_\varepsilon (x) \subseteq \partial_x \ \langle k_\varepsilon^*, s(x) \rangle \tag{3. 1}$$

Let $h_\varepsilon(x) = max \ \{ f(x, 0);\ s_\varepsilon(x) \}$. We have $h_\varepsilon(x) \geqslant 0$ for every $x \in C$ and $h_\varepsilon(\bar{x}) = s_\varepsilon(\bar{x}) \leqslant \varepsilon \| k_o \| = \varepsilon$ hence

$$h_\varepsilon(\bar{x}) \leqslant \inf_{x \in C} h_\varepsilon (x) + \varepsilon.$$

By Ekeland's Variational Principle [3] we can find a point $x_\varepsilon \in C$ such that

(a) $\quad \| x_\varepsilon - \bar{x} \| \leqslant \sqrt{\varepsilon}$ (3. 2)

(b) $\quad h_\varepsilon(x) + \sqrt{\varepsilon} \ \| x - x_\varepsilon \| > h_\varepsilon(x_\varepsilon),$

for every $x \in C$, $x \neq x_\varepsilon$.

From (b) we derive

$$0 \in \partial h_\varepsilon(x_\varepsilon) + \sqrt{\varepsilon} \ B_X^*(0, 1) + (\alpha + 1)\, \partial d_C(x_\varepsilon), \tag{3. 3}$$

wheve α is a Lipschitz-constant of $h_\varepsilon(.)$ on some neighbourhood of the point $\bar{x}$.

Note that $f(x_\varepsilon, 0)$ and $s_\varepsilon(x_\varepsilon)$ are not all zero. Indeed, if $f(x_\varepsilon, 0) = 0$, then $0 \in F(x_\varepsilon)$, and hence $x_\varepsilon \in M$. If $s_\varepsilon(x_\varepsilon) = 0$ we have $s(x_\varepsilon) - s(\bar{x}) \in K_\varepsilon \subseteq K$

and fromthé K-opimality of $\bar{x}$ we conclude $s(\bar{x}) - s(x_\varepsilon) \in K$, and hence $0 = (s(x_\varepsilon) - s(\bar{x})) + (s(\bar{x}) - s(x_\varepsilon)) \in K_\varepsilon + K \subset K_\varepsilon$, which contradicts the fact that $k_0 \notin - K$. From Lemma 2.2, Proposition 2.2. and (3.1) we can assert that for every

$$x_\varepsilon^* \in \partial h_\varepsilon(x_\varepsilon) \text{ there exist } y_\varepsilon^* \in Y_F^*,\ k_\varepsilon^* \in K^*$$

such that

$$(\alpha) \quad \| y_\varepsilon^* \| + \| k_\varepsilon^* \| = 1,$$

$$(\beta) \quad x_\varepsilon^* \in \partial_x \langle k_\varepsilon^*, s(x_\varepsilon) \rangle - \partial_x C^F(y_\varepsilon^*, x_\varepsilon), \tag{3.4}$$

$$(\gamma) \quad C^F(y_\varepsilon^*, x_\varepsilon) = - \| y_\varepsilon^* \| \cdot f(x_\varepsilon, 0).$$

Combining (3.4) and (3.3) gives

$$(\alpha) \quad \| y_\varepsilon^* \| + \| k_\varepsilon^* \| = 1$$

$$(\beta) \quad 0 \in \partial_x \langle k_\varepsilon^*, s(x_\varepsilon) \rangle - \partial_x C^F(y_\varepsilon^*, x_\varepsilon) + \sqrt{\varepsilon}\, B_X^*(0, 1) + (\alpha+1)\, \partial d_C(x_\varepsilon)$$

$$(\gamma) \quad C^F(y_\varepsilon^*, x_\varepsilon) = -\|y_\varepsilon^*\| \cdot f(x_\varepsilon, 0).$$

By taking a subsequence if necessary, we can assume that $y_\varepsilon^* \to y^*$, $k_\varepsilon^* \to k^*$, (as $\varepsilon \to 0$). On the other hand we have $x_\varepsilon \underset{(\varepsilon \to 0)}{\longrightarrow} \bar{x}$, and by letting $\varepsilon \to 0$ the inclusion (i) immediately follows from (α), (β). To prove (ii) observe that $C^F(y^*, x)$ is lipschitzian in x uniformly for $y^* \in B_F^*$, i.e.

$C^F(y_\varepsilon^*, \bar{x}) \leqslant \alpha \| x_\varepsilon - \bar{x} \| + C^F(y_\varepsilon^*, x_\varepsilon)$. On the other hand $C^F(y^*, \bar{x})$ is l.s.c. in y^*. Therefore

$$C^F(y^*, \bar{x}) \leqslant \lim_{\varepsilon \to 0} C^F(y_\varepsilon^*, x) \leqslant \lim_{\varepsilon \to 0} (-\|y_\varepsilon^*\| f(x_\varepsilon, 0)) = 0.$$

The converse inequality is plain, since $0 \in F(\bar{x})$, so that we have (ii). The proof is thus complete.

The author wishes to thank Dr. Pham Hưu Sach for his advice and support.

Received September 4, 1982

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