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ON THE PURSUIT PROCESS IN DIFFERENTIAL GAMES

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1. INTRODUCTION

In this paper we are concerned with differential games which arise in a pursuit process of one control object by another control object. The problem can be described as follows.

Let x, y denote the phase vectors of two control objects whose motions are described by the ordinary differential equations

$$\dot{x} = f(x, u)\,;\; x(o) = x_0\,, \tag{1.1}$$

$$\dot{y} = g(y, v)\,;\; y(o) = y_0\,, \tag{1.2}$$

where u and v are the controls. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, where x_1, y_1 are the geometrical positions and x_2, y_2 the velocities of the objects.

In the pursuit process, one control object, x, is the pursuer, the other, y, is the evader. The pursuit process is completed when the two objects coincide geometrically, that is when

$$x_1 = y_1\,. \tag{1.3}$$

To simplify the notations it is convenient to transform the pursuit process into a differential game. Namely, denoting by R the direct sum of the phase spaces of the objects, we can rewrite the equations (1 . 1), (1 . 2) into a single differential equation

$$\dot{z} = F(z, u, v), \tag{1.4}$$

where $z = (x, y) \in R$. Condition (1 . 3) defines in the space R a certain subset M. The game is now given by its phase vector space R, the equation (1. 4) and the set M on which the game must be completed. There are, two problems that can be considered: the pursuit game and the evasion game. In the pursuit game, one has to find the value of u at each instant of time in order to complete the game, whereas in the evasion game, the problem is to find the of v at each instant of time in order to prevent the game from terminating.

Differential games of pursuit have been extensively studied in recent years. In [1 — 11] various sufficcient conditions for completing the pursuit have been developed, under the assumptions that the controls u and v must satisfy constraints of the form

$$u \in P, \quad v \in Q,$$

or

$$\int^{+\infty} \| u (t) \|^2 dt \leqslant \rho^2 \quad ; \quad \int^{+\infty} \| v(t) \|^2 dt \leqslant \sigma^2 .$$

In the present paper we shall develop sufficient conditions undes constraints of different types imposed on the controls.

2. THE MAIN RESULT

We denote the phase vector by z and assume that the motion is described by the ordinary differential equation :

$$\dot{z} = Az - Bu + Cv \; ; \; z(0) = z_0 \; . \tag{2.1}$$

Here $z \in R^n$, $u \in R^p$ is the pursuit control and $v \in R^q$ is the evasion control; A, B, C are matrices of orders $n \times n$, $n \times p$, $n \times q$ respectively; the controls $u(t)$, $v(t)$ are measurable functions satisfying

$$\int_0^{+\infty} \| u(t) \|^2 \; dt \leqslant \rho^2, \; v(t) \in Q, \tag{2.2}$$

where $\rho > 0$ and Q is a compact convex subset of R^q. Let M be a subset of R^n of the form $M = M_1 + M_2$, where M_1 is a subspace of R^n and M_2 a subset of the orthogonal complement to M_1 in R^n. Let π denote the orthogonal projection trom R^n onto L.

We shall say that the pursuit process in the differential game (2.1) — (2.2) is completed after time t_1, if for any measurable function $v = v(t)$, $0 \leqslant t \leqslant t_1$, $v(t) \in Q$, there exists a measurable function $u = u(t)$, $0 \leqslant t \leqslant t_1$,

$$\int_0^{t_1} \| u(t) \|^2 \, dt \leqslant \rho^2 ,$$

such that the solution $z = z(t)$, $0 \leqslant t \leqslant t_1$ of the equation

$$\dot{z} = Az(t) - Bu(t) + Cv(t), z(o) = z_0,$$

satisfies $z(t_1) \in M$.

Let $\theta(t)$ be a non negative non decreasing and piecewise continuous function defined for all $t \geqslant 0$, and let $r(t)$ be a non negative, increasing anp continuously differentiable function such that $r(t) \leqslant t$ for every $t \geqslant 0$. Denote

$$\Omega(t) = \begin{cases} [\theta(t), r(t)] & \text{if } \theta(t) \leqslant r(t) \\ \phi & \text{otherwise.} \end{cases}$$

We shall be interested in computing the value $u(t)$ of the pursuit control at each time t when the values $v(s)$ of the evasion control are known for all $s \in \Omega(t)$. In other words, we shall be interested in finding the function

$$u(t) = u(v(s) : s \in \Omega(t)).$$

For any given $T > 0$, let

$$\Delta_1(T) = \{ 0 \leqslant t \leqslant T : \Omega(t) \neq \phi \} ; \Delta_2(T) = [0, T] \setminus \Delta_1(T) ;$$

$$\Delta_3(T) = \bigcup_{t \in \Delta_1(T)} \Omega(t) ; \Delta_4(T) = [0, T] \setminus \Delta_3(T).$$

can be verified that the sets $\Delta_i(T)$ are measurable and that

$$\Delta_3(T) = \bigcup_{t \in \Delta_1(T)} r(t) \tag{2.3}$$

Finally, let

$$H(T) = \int_{\Delta_4(T)} \pi \, e^{(T-s)A} \, CQds.$$

e now in a position to formulate our basic hypotheses.

hesis 1. There exists a number $T_1 > 0$ such that $M_2 \stackrel{*}{-} H(T_1) \neq \phi$, where $\stackrel{*}{-}$ geometrical difference in the sence of L. S. Pontryagin [1].

esis 2. There exists a matrix Φ_{T_1} of piecewise continuous functions

) of x, y

$$\Phi_{T_1}(x, y) = \begin{pmatrix} \varphi^{11}_{T_1}(x, y) \ldots\ldots \varphi^{1r}_{T_1}(x,y) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \varphi^{r1}_{T_1}(x, y) \ldots\ldots \varphi^{rr}_{T_1}(x, y) \end{pmatrix},$$

which is defined on the intervals $0 \leqslant x \leqslant T_1$, $0 \leqslant y \leqslant T_1$ and satisfies

1. $\Phi_{T_1}(t, s) = \tilde{o}$ if $s \notin \Omega(t)$ or $t \in \Delta_2(T_1)$,

2. There exist continuous $(p \times q)$ — matrix functions $F_{T_1}(t)$ and piecewise continuous $(p \times q)$ — matrix functions $G_{T_1}(t, s)$ such that

2a. $G_{T_1}(t, s) = \tilde{\theta}$ if $s \notin \Omega(t)$ or $t \in \Delta_2(T_1)$,

2b. $\pi e^{(T_1 - t)A} BG_{T_1}(t, s) = \varphi_{T_1}(t, s)\ \pi e^{(T_1 - s)A} C,$

2c. $\pi e^{(T_1 - t)A} BF_{T_1}(t) = \left(E - \int\limits_0^{T_1} \varphi_{T_1}(s, r(t))\ ds \right) \pi e^{(T_1 - r(t))A} C,$

where $dim\ L = r$, $\tilde{o}$ is the zero square matrix of order r, E is the unit square matrix of order r and $\tilde{\theta}$ is the zero $(p \times q)$ — matrix.

Let $u^*(t)$, $t \in \Delta_2(T_1)$ be an arbitrary pursuit control such that

$$\int\limits_{\Delta_2(T_1)} \| u^*(t) \|^2\, dt \leqslant \rho^2.$$

Define

$$\tilde{\rho}^2 = \rho^2 - \int\limits_{\Delta_2(T_1)} \| u^*(t) \|^2\ dt,$$

$$\chi^2(T_1) = \sup_{\substack{v(s) \in Q \\ 0 \leqslant s \leqslant T_1}} \int\limits_{\Delta_1(T_1)} \| F_{T_1}(t)\, v(r(t))\, \dot{r}(t) + \int\limits_{\theta(t)}^{r(t)} G_{T_1}(t, s)\, v(s)\, ds \|^2\ dt.$$

Hypothesis 3: $\chi(T_1) \leqslant \tilde{\rho}.$

Hypothesis 4:

$$\pi e^{T_1 A}\left(z_0 - \int_{\Delta_2(T_1)} e^{-sA} Bu^*(s)\, ds \right) \in G(T_1) + M_2 \stackrel{*}{-} H(T_1)) , \tag{2.4}$$

where

$$G(T_1) = \left\{ \int_{\Delta_1(T_1)} \pi e^{(T_1 - s)A} B\, \omega(s)\, ds : \int_{\Delta_1(T_1)} \| \omega(s) \|^2 ds \leqslant (\tilde{\rho} - \chi(T_1))^2 \right\} .$$

The main result is the following

THEOREM 1

Under hypotheses 1 — 4, the pursuit process in the differential game (2.1) — (2.2) is completed after time T_1.

Proof.

From (2.4), it follows that there exist vectors $g \in G(T_1)$ and $m \in M_2 \stackrel{*}{-} H(T_1)$ such that

$$\pi e^{T_1 A}\left(z_0 - \int_{\Delta_2(T_1)} e^{-sA} Bu^*(s)\, ds \right) = g + m .$$

Since $g \in G(T_1)$, there exists a measurable function $\tilde{\omega}(s)$, $s \in \Delta_1(T_1)$ satisfying

a)
$$\int_{\Delta_1(T_1)} \| \tilde{\omega}(s) \|^2 ds \leqslant (\tilde{\rho} - \chi(T_1))^2 ,$$

b)
$$\pi e^{T_1 A}\left(z_0 - \int_{\Delta_2(T_1)} e^{-sA} Bu^*(s)\, ds \right) = \int_{\Delta_1(T_1)} \pi e^{(T_1 - s)A} B\tilde{\omega}(s)\, ds + m .$$

Assume now that $\overline{v}(t)$, $0 \leqslant t \leqslant T_1$, $\overline{v}(t) \in Q$ is an arbitrary evasion control. Then there exists a vector $m_2 \in M_2$ such that

$$\pi e^{T_1 A}\left(z_0 - \int_{\Delta_2(T_1)} e^{-sA} Bu^*(s)\, ds \right) =$$

$$= \int_{\Delta_1(T_1)} \pi e^{(T_1 - s)A} B\tilde{u}(s)\, ds + m_2 - \int_{\Delta_4(T_1)} \pi e^{(T_1 - t)A} C\overline{v}(t)\, dt . \tag{2. 5}$$

The pursuit control $\bar{u}(t)$, $0 \leqslant t \leqslant T_1$ is defined as follows

$$\bar{u}(t) = \begin{cases} u^*(t), & \text{if } t \in \Delta_2(T_1), \\ F_{T_1}(t)\,\bar{v}(r(t))\,\dot{r}(t) + \int\limits_{\theta(t)}^{r(t)} G_{T_1}(t,s)\,\bar{v}(s)\,ds + \tilde{\omega}(t), & \text{if } t \in \Delta_1(T_1). \end{cases}$$

By Minkowski's inequality we have

$$\sqrt{\int\limits_{\Delta_1(T_1)} \| \tilde{\omega}(t) + F_{T_1}(t)\,\bar{v}(r(t))\,\dot{r}(t) + \int\limits_{\theta(t)}^{r(t)} G_{T_1}(t,s)\,\bar{v}(s)\,ds \|^2\,dt} \leqslant$$

$$\leqslant \sqrt{\int\limits_{\Delta_1(T_1)} \| \tilde{\omega}(t) \|^2\,dt} + \sqrt{\int\limits_{\Delta_1(T_1)} \| F_{T_1}(t)\,\bar{v}(r(t))\,\dot{r}(t) + \int\limits_{\theta(t)}^{r(t)} G_{T_1}(t,s)\,\bar{v}(s)\,ds \|^2\,dt} \leqslant$$

$$\leqslant \tilde{\rho} - \chi(T_1) + \chi(T_1) = \tilde{\rho}.$$

Consequently,

$$\int\limits_0^{T_1} \| \bar{u}(t) \|^2\,dt = \int\limits_{\Delta_1(T_1)} \| \tilde{\omega}(t) + F_{T_1}(t)\,\bar{v}(r(t))\,\dot{r}(t) + \int\limits_{\theta(t)}^{r(t)} G_{T_1}(t,s)\,\bar{v}(s)\,ds \|^2\,dt +$$

$$+ \int\limits_{\Delta_2(T_1)} \| \overset{*}{u}(t) \|^2\,dt \leqslant \int\limits_{\Delta_2(T_1)} \| \overset{*}{u}(t) \|^2\,dt + \tilde{\rho}^2 = \rho^2,$$

i. e $\bar{u}(t)$, $0 \leqslant t \leqslant T_1$ is an admissible pursuit control. By Cauchy's **formula for** the solution of equation (2.1) we have

$$Z(T_1) = e^{T_1 A} Z_0 - \int\limits_0^{T_1} e^{(T_1 - t)A} B\bar{u}(t)\,dt + \int\limits_0^{T_1} e^{(T_1 - t)A} C\bar{v}(t)\,dt.$$

Therefore

$$\pi Z(T_1) = \pi e^{T_1 A} Z_0 - \int\limits_{\Delta_2(T_1)} \pi e^{(T_1 - t)A} Bu^*(t)\,dt - \int\limits_{\Delta_1(T_1)} \pi e^{(T_1 - t)A} B\bar{u}(t)\,dt +$$

$$+ \int\limits_{\Delta_3(T_1)} \pi e^{(T_1 - s)A} C\bar{v}(s)\,ds + \int\limits_{\Delta_4(T_1)} \pi e^{(T_1 - s)A} C\bar{v}(s)\,ds =$$

$$= \pi e^{T_1 A} Z_o - \int_{\Delta_2(T_1)} \pi e^{(T_1 - t)A} Bu^*(t)dt - \int_{\Delta_1(T_1)} \pi e^{(T_1 - t)A} B\omega(t)\, dt -$$

$$- \int_{\Delta_1(T_1)} \pi e^{(T_1 - t)A} BF_{T_1}(t)\overline{v}(r(t))\dot{r}(t)dt - \int_{\Delta_1(T_1)} \left(\int_{\theta(t)} \pi e^{(T_1 - t)A} BG_{T_1}(t,s)\, \bar{v}(s)ds \right) dt +$$

$$+ \int_{\Delta_3(T_1)} \pi e^{(T_1 - s)A}\, C\overline{v}(s)ds + \int_{\Delta_4(T_1)} \pi e^{(T_1 - s)A}\, C\overline{v}(s)ds =$$

$$= \Bigg\{ \pi e^{T_1 A} \Big(Z_o - \int_{\Delta_2(T_1)} e^{-sA} Bu^*(s)ds \Big) -$$

$$- \int_{\Delta_1(T_1)} \pi e^{(T_1 - t)A} B\tilde{\omega}(t)dt + \int_{\Delta_4(T_1)} \pi e^{(T_1 - t)A}\, C\overline{v}(t)dt \Bigg\} -$$

$$- \int_{\Delta_1(T_1)} \pi e^{(T_1 - t)A} BF_{T_1}(t)\overline{v}(r(t))\, \dot{r}(t)dt + \int_{\Delta_3(T_1)} \pi e^{(T_1 - t)A}\, C\overline{v}(t)\, dt -$$

$$- \int_{\Delta_1(T_1)} \left(\int_{\theta(t)}^{r(t)} \pi e^{(T_1 - t)A} BG_{T_1}(t, s)\overline{v}(s)\, ds \right) dt. \qquad (2.6)$$

From (2.5), (2.6) we deduce

$$\pi z(T_1) = m_2 + \int_{\Delta_3(T_1)} \pi e^{(T_1 - S)A}\, C\,\bar{v}(S)\, ds - \int_{\Delta_1(T_1)} \pi e^{(T_1 - S)A}\, BF_{T_1}(t)\, \bar{v}(r(t))\, \dot{r}(t)\, dt -$$

$$- \int_{\Delta_1(T_1)} \left(\int_{\theta(t)}^{r(t)} \pi e^{(T_1 - t)A}\, BG_{T_1}(t, s)\, \bar{v}(s)\, ds \right) dt. \qquad (2.7)$$

Further, we have

$$\int_{\Delta_1(T_1)} \pi e^{(T_1 - t)A}\, BF_{T_1}(t)\, \bar{v}(r(t))\, \dot{r}(t)\, dt =$$

$$= \int\limits_{\Delta_1(T_1)} \left(E - \int\limits_0^{T_1} \Phi_{T_1}(s, r(t))\, ds \right) \pi e^{(T_1 - r(t))A}\, C \bar{v}(r(t)) \dot{r}(t)\, dt =$$

$$= \int\limits_{\Delta_1(T_1)} \pi e^{(T_1 - r(t))A}\, C \bar{v}(r(t)) \dot{r}(t)\, dt -$$

$$- \int\limits_{\Delta_1(T_1)} \left(\int\limits_0^{T_1} \Phi_{T_1}(s, r(t))\, ds \right) \pi e^{(T_1 - r(t))A}\, C \bar{v}(r(t)) \dot{r}(t)\, dt. \tag{2.8}$$

But in view of (2.3) we can write:

$$\int\limits_{\Delta_1(T_1)} \pi e^{(T_1 - r(t))A}\, C \bar{v}(r(t)) \dot{r}(t)\, dt = \int\limits_{\Delta_3(T_1)} \pi e^{(T_1 - s)A}\, C \bar{v}(s)\, ds,$$

$$\int\limits_{\Delta_1(T_1)} \left(\int\limits_0^{T_1} \Phi_{T_1}(s, r(t))\, ds \right) \pi e^{(T_1 - r(t))A}\, C \bar{v}(r(t)) \dot{r}(t)\, dt =$$

$$= \int\limits_{\Delta_3(T_1)} \left(\int\limits_0^{T_1} \Phi_{T_1}(t, s)\, dt \right) \cdot \pi e^{(T_1 - s)A}\, C \bar{v}(s)\, ds .$$

Therefore, we obtain from (2.7) (2.8):

$$\pi Z(T_1) = m_2 - \int\limits_{\Delta_1(T_1)} \left(\int\limits_{\theta(t)}^{r(t)} \pi e^{(T_1 - t)A} \cdot BG_{T_1}(t, s) \bar{v}(s)\, ds \right) dt +$$

$$+ \int\limits_{\Delta_3(T_1)} \left(\int\limits_0^{T_1} \Phi_{T_1}(t, s)\, dt \right) \pi e^{(T_1 - s)A}\, C \bar{v}(s)\, ds . \tag{2.9}$$

Noting that $\Phi_{T_1}(t,s) = \tilde{0}$ if $t \in \Delta_2(T_1)$ or $s \notin \Omega(t)$, we have

$$\int\limits_{\Delta_1(T_1)} \int\limits_{\theta(t)}^{r(t)} \Phi_{T_1}(t,s)\, \pi e^{(T_1 - s)A}\, C \bar{v}(s)\, ds\, dt =$$

$$= \int\limits_{\Delta_3(T_1)} \left(\int\limits_0^{T_1} \Phi_{T_1}(t,s)\, dt \right) \pi e^{(T_1 - s)A} \; C\, \bar{v}(s)\, ds =$$

$$= \int\limits_0^{T_1} \int\limits_0^{T_1} \Phi_{T_1}(t,s)\, \pi e^{(T_1 - s)A} \; C\, \bar{v}(s)\, ds\, dt \, .$$

This implies, in view of (2.9), $\pi z(T_1) = \mathbf{m}_2$, i.e. $z(T_1) \in M$. The proof is complete.

3. SPECIAL CASES

In this section the main result stated in Theorem 1 will be specialized to.

I. Consider first the case where

$$\theta(t) = r(t) = t \quad \forall\, t \geqslant 0.$$

Then

$\Delta_1(T) = \Delta_3(T) = [0, T]$; $\Delta_2(T) = \Delta_4(T) = \phi$ for any $T \geqslant 0$.

As an immediate consequence of Theorem 1 we obtain

COROLLARY 1: *Let T_1 be a positive number T_1 such that*

a) *There exists a continuous $(p \times q)$-matrix function $F(t)$ satisfying*

$$\pi e^{tA} BF(t) = \pi e^{tA}\, C \text{ for any } 0 \leqslant t \leqslant T_1 .$$

b) $$\chi^2(T_1) = \sup_{\substack{v(s) \in Q \\ 0 \leqslant s \leqslant T_1}} \int\limits_0^{T_1} \| F(t)\, v(t) \|^2 \, dt \leqslant \rho^2 .$$

c) $\pi e^{T_1 A} z_0 \in M_2 + G(T_1)$, where

$$G(T_1) = \left\{ \int\limits_0^{T_1} \pi e^{(T_1 - s)A} B\omega(s) ds : \int\limits_0^{T_1} \| \omega(s) \|^2 \, ds \leqslant (\rho - \chi(T_1))^2 \right\}$$

Then the pursuit process in the differential game (2.1)—(2.2) is completed after time T_1.

II. Assume there exists an increasing, continuously differentiable function $I(t)$ on the interval $0 \leqslant t < +\infty$ such that:

1. $I(o) = 0$.

2. $\dot{I}(t) \geqslant 1$ for any $0 \leqslant t < +\infty$.

3. There exists a continuous $(p \times q)$-matrix function $F(t)$ satisfying

$$\pi e^{I(t)A} C = \pi e^{tA} BF(t) \quad \text{for any } t \geqslant 0.$$

Assume, furthermore, that

$$\Delta_1(I(t)) = [I(t) - t\,;\ I(t)]\,;\ \Delta_2(I(t)) = [0\,;\ I(t) - t]\,;$$

$$\Delta_3(I(t)) = [0, I(t)]\,;\ \Delta_4(I(t)) = \phi \quad \text{for any } t \geqslant 0.$$

COROLLARY 2. *Let t_1 be a positive number such that*

a) $$\chi^2(t_1) = \sup_{\substack{v(s) \in Q \\ 0 \leqslant s \leqslant I(t)}} \int_0^{t_1} \| F(t)\, v(I(t_1) - I(t))\, \dot{I}(t) \|^2\, dt \leqslant \rho^2,$$

where $t_0 = I(t_1) - t_1$; $u^*(t)$, $0 \leqslant t \leqslant t_0$, is an arbitrary pursuit control satisfying

$$\int_0^{t_0} \| u^*(t) \|^2\, dt \leqslant \rho^2,$$

and

$$\tilde{\rho}^2 = \rho^2 - \int_0^{t_0} \| u^*(t) \|^2 dt.$$

b) $$\pi e^{I(t_1)A} \left(z_0 - \int_0^{t_0} e^{-sA} Bu^*(s) ds \right) \in M_2 + (G(t_1), \text{ where}$$

$$G(t_1) = \left\{ \int_0^{t_1} \pi e^{(t_1 - s)A} B\omega(s) ds : \int_0^{t_1} \| \omega(s) \|^2\, ds \leqslant (\tilde{\rho} - \chi(t_1))^2 \right.$$

Then the pursuit process in the differential game (2.1) — (2.2) is completed after time $I(t_1) = t_0 + t_1$.

Note that an analogous result for differential games with integral constraints on controls has been obtained earlier by A. Ya. Azimov [5].

III. Assume that $\nu \geqslant 0$ is given and $g(t)$ is a nonnegative, increasing, continuously differentiable function defined on the interval $\nu \leqslant t < +\infty$ and satisfying $g(t) \leqslant t$ for any $t \geqslant \nu$.

Consider the case where $\theta(t) \equiv r(t) \equiv g(t)$ for any $t \geqslant \nu$. Then for any $T > \nu$:

$$\Delta_1(T) = [\nu, T]\,;\ \Delta_2(T) = [0, \nu]\,;$$

$\Delta_3(T) = [g(v), g(T)]$; $\Delta_4(T) = [0, g(v)] \cup [g(T), T]$.

From Theorem 1 we have

COROLLARY 3. Let t_1 be a positive number satisfying:

a) There exists a continuous $(p \times q)$ — matrix function $F(t)$ such that

$$\pi e^{(t_1 - t)A} BF(t) = \pi e^{(t_1 - r(t))A} C \text{ for any } t \geqslant v.$$

b) $$\chi^2(t_1) = \sup_{\substack{v(s) \in Q \\ 0 \leqslant s \leqslant t_1}} \int_v^{t_1} \| F(t) v(r(t)) \dot{r}(t) \|^2 \, dt \leqslant \tilde{\rho}^2,$$

where

$$\tilde{\rho}^2 = \rho^2 - \int_0^v \| u^*(t) \|^2 \, dt,$$

and $u^*(t)$, $0 \leqslant t \leqslant v$ is an arbitrary pursuit control, i. e.

$$\int_0^v \| u^*(t) \|^2 \, dt \leqslant \rho^2 .$$

c) $$\pi e^{t_1 A} \left(z_0 - \int_0^v e^{-sA} Bu^*(s) ds \right) \in G(t_1) + \left(M_2 \overset{*}{-} H(t_1) \right), \text{ where}$$

$$G(t_1) = \left\{ \pi e^{(t_1 - s)A} B\omega(s) ds : \int_v^{t_1} \| \omega(s) \|^2 \, ds \leqslant \left(\tilde{\rho} - \chi(t_1) \right)^2 \right\},$$

$$H(t_1) = \int_0^{g(v)} \pi e^{(t_1 - s)A} CQ ds + \int_{g(t_1)}^{t_1} \pi e^{(t_1 - s)A} CQ \, ds .$$

Then the pursuit process in differential game (2.1) — (2.2) is completed after time t_1

Note that analogous results for differential games with geometrical and integral constraints on controls have been obtained by S. P. Berzan in [6] and Phan Huy Khai in [12].

4. APPLICATION

In this section we apply the above results to solve concrete pursuit game with different types of constraints imposed on the controls.

I. Assume that the motions of the points $x \in R^n$, $y \in R^n$ are described by the differential equations.

$$\dot{x} + \alpha x = u ; \qquad x(0) = x_0 , \tag{4.1}$$

$$\dot{y} + \beta y = v ; \qquad y(0) = y_0 , \tag{4.2}$$

where $\alpha > 0$, $\beta > 0$. The controls u(t), v(t) are measurable functions satisfying

$$\int_0^{+\infty} \| u(t) \|^2 \, dt \leqslant \rho^2 ; \quad \| v(t) \| \leqslant \sigma. \tag{4.3}$$

We shall say that the pursuit process in the differential game (4.1) — (4.3) is completed after the time t_1 if $\| x(t_1) - y(t_1) \| \leqslant \varepsilon$, where $\varepsilon > 0$ is given in advance. Let us calculate then value u(t) of the control parameter u at each instant of time t, assuming that the value $v\left(\frac{t}{2}\right)$ is known. That is, let us compute:

$$u(t) = u \left(v\left(\frac{t}{2}\right)\right).$$

For any $t \geqslant 0$, $T \geqslant 0$ we have

$$\theta(t) = r(t) = \frac{1}{2} t \ ; \Delta_1 \ (T) = [0, T]; \ \Delta_2 \ (T) = \varnothing ;$$

$$\Delta_3 \ (T) = \left[0, \frac{T}{2}\right]; \ \Delta_4 (T) = \left(\frac{T}{2}, \ T\right).$$

Let $z = (z_1 , \ z_2)^T = (x, y)^T \in R^{2n}$,

$$A = \begin{pmatrix} -\alpha E & \tilde{0} \\ 0 & -\beta E \end{pmatrix} ; B = \begin{pmatrix} -E \\ 0 \end{pmatrix} ; C = \begin{pmatrix} \tilde{0} \\ E \end{pmatrix},$$

where $(z_1 , z_2)^T$ is the transpose of the vector (z_1 , z_2), E is the unit matrix of order n, $\tilde{0}$ is the zero matrix of order n. The equations (4.1), (4.2) can be rewritten as

$$\dot{z} = Az - Bu + Cv ; z(0) = z_0 = (z_1^0, z_2^0)^T = (x_0 , y_0)^T .$$

Then:

$$M = \left\{ z = (z_1 , z_2)^T \in R^{2n} : \| z_1 \ z_2 \| \leqslant \varepsilon \right\}.$$

$$L = \left\{ z = (z_1 , z_2)^T \in R^{2n} : z_1 = - z_2 \right\}.$$

Denote orthogonal projection of the space R^{2n} onto its subspace L by π. Let $\Pi \doteq (E, -E)$. Then for a suitable system of coordinate of L, the matrix of π can be written as Π.

A simple computation shows that

$$\Pi\, e^{(T-t)A}\, B = e^{-\alpha(T-t)}\, E;\quad \Pi\, e^{\left(T-\frac{t}{2}\right)A}\, C = e^{-\beta\left(T-\frac{t}{2}\right)} E.$$

Setting

$$F_T(t) = e^{(\alpha-\beta)\,T} \cdot e^{\left(\frac{\beta}{2}-\alpha\right)t}\, E \text{ for } 0 \leqslant t \leqslant T,$$

we have

$$\Pi\, e^{(T-t)A}\, BF_T\,(t) = \Pi\, e^{\left(T-\frac{t}{2}\right)}\, C \text{ for any } 0 \leqslant t \leqslant T.$$

Let $v(t)$, $0 \leqslant t \leqslant T$ be an arbitrary evasion control. Then

$$\int_0^T \left\| F_T(t)\, v\left(\frac{1}{2}\, t\right) \cdot \frac{1}{2} \right\|^2 dt \leqslant \frac{\sigma^2}{2} \int_0^{\frac{T}{2}} \left(e^{(\alpha-\beta)T} \cdot e^{2\left(\frac{\beta}{2}-\alpha\right)t} \right)^2 dt.$$

Assuming now that $\frac{\beta}{2} < \alpha \leqslant \beta$, we can write

$$\int_0^T \left\| F_T\,(t)\, v\left(\frac{1}{2}\, t\right) \frac{1}{2} \right\|^2 dt \leqslant \frac{\sigma^2}{2} \int_0^{\frac{T}{2}} e^{2(\beta-2\alpha)t}\, dt \leqslant \frac{1}{4} \cdot \frac{\sigma^2}{2\alpha-\beta}$$

Since

$$\lim_{T\to+\infty} \left\| \Pi\, e^{TA}\, z_0 \right\| = \lim_{T\to+\infty} \left\| e^{-\alpha T}\, z_0 \right\| \text{ for any } z_0 \in R^{2n},$$

we deduce

$$H(T) = \int_{\frac{T}{2}}^{T} \Pi\, e^{(T-s)A}\, CQ\, ds.$$

It is a simple matter to verify that H (T) is a ball of radius

$$\sigma \int_{\frac{T}{2}}^{T} e^{-\beta(T-s)}\, ds < \frac{\sigma}{\beta}.$$

Therefore $\varepsilon S \overset{*}{-} H(T) \neq \emptyset$ if $\sigma \leqslant \varepsilon\beta$, where S is the unit ball in L. From Corollary 3 we then obtain

PROPOSITION 1: *Under the assumptions* $\alpha > 0$, $\beta > 0$, $\frac{\beta}{2} < \varkappa \leqslant \beta$, $\sigma \leqslant \varepsilon\beta$ *and* $\sigma^2 \leqslant 4(2\alpha - \beta)^2$, *the pursuit process in the differential game* (4.1) — (4.3) *is completed after finite time for any position* $z^o = (z_1^o, z_2^o)^T$.

II. Assume now that the motions of the points $x \in R^n$, $y \in R^n$ are described the differential equations:

$$\ddot{x} + a_1 \dot{x} + a_2 x = u; \quad x(0) = x_o, \dot{x}(0) = \dot{x}_o, \tag{4.4}$$

$$\ddot{y} + b_1 \dot{y} + b_2 y = \sigma v; \quad y(0) = y_o, \dot{y}(0) = \dot{y}_o, \tag{4.5}$$

where a_1, a_2, b_1, b_2 are real numbers. The controls $u(t)$, $v(t)$ are measurable functions satisfying

$$\int_0^{+\infty} \| u(t) \|^2 dt \leqslant \rho^2 \ ; \| v(t) \| \leqslant 1 \ . \tag{4.6}$$

We shall say that the pursuit process in the differential game (4.4) — (4.6) is completed after time t_1 if $x(t_1) = y(t_1)$. Let us compute the value $u(t)$ of the parameter u at each instant of time t, assuming that the value $v(t)$ is known. That is, let us compute $u(t) = u(v(t))$.

Here we have

$$\theta(t) = r(t) = t\ ; \ \Delta_1(T) = \Delta_3(T) = [0,T]\ ; \ \Delta_2(T) = \Delta_4(T) = \phi\ .$$

Let us denote $z = (z_1, z_2, z_3, z_4)^T = (x, \dot{x}, y, \dot{y})^T \in R^{4n}$,

$$A = \begin{pmatrix} \tilde{0} & E & \tilde{0} & \tilde{0} \\ -a_2 E & -a_1 E & \tilde{0} & \tilde{0} \\ \tilde{0} & \tilde{0} & \tilde{0} & E \\ \tilde{0} & \tilde{0} & -b_2 E & -b_1 E \end{pmatrix}; B = \begin{pmatrix} \tilde{0} \\ -E \\ 0 \\ 0 \end{pmatrix}; C = \begin{pmatrix} \tilde{0} \\ 0 \\ 0 \\ \sigma E \end{pmatrix},$$

where $(z_1, z_2, z_3, z_4)^T$ is the transpose of (z_1, z_2, z_3, z_4), E is the unit matrix of order n, $\tilde{0}$ is the zero matrix of order n. Rewrite the equations (4.4), (4.5) as

$$\dot{z} = Az - Bu + Cv\ ; z(o) = z_0 = (z_1^o, z_2^o, z_3^o, z_4^o)^T = (x_o, \dot{x}_o, y_o, \dot{y}_o)^T .$$

We have then

$$M = \left\{ z = (z_1, z_2, z_3, z_4)^T \in R^{4n} : z_1 = z_3 \right\},$$
$$L = \left\{ z = (z_1, z_2, z_3, z_4)^T \in R^{4n} : z_1 = -z_3,\ z_2 = z_4 = 0 \right\}.$$

Note that M is subspace of R^{4n} and L is its orthogonal complement in R^{4n}. The orthogonal projection of the space R^{4n} onto L is denoted by π. Let $\Pi = (E, \widetilde{O}, -E, \widetilde{O})$. Then for a suitable system of coordinate of L, the matrix of π can be written as Π.

Let $\varphi_1(t)$ and $\varphi_2(t)$ be the solutions of the homogeneous equation

$$\ddot{\zeta} + a_1 \dot{\zeta} + a_2 \zeta = 0 \ , \tag{4.7}$$

with initial conditions $\varphi_1(o) = 1$, $\dot{\varphi}_1(o) = 0$; $\varphi_2(o) = 0$, $\dot{\varphi}_2(o) = 1$ and $\psi_1(t)$, $\psi_2(t)$ be the solutions of the homogeneous equation

$$\ddot{\eta} = b_1 \eta + b_2 \eta = 0 \ , \tag{4.8}$$

with initial conditions $\psi_1(o) = 1$, $\dot{\psi}_1(o) = 0$; $\psi_2(o) = 0$, $\dot{\psi}_2(o) = 1$.

Putting

$$F(t) = \begin{cases} \sigma \dfrac{\psi_2(t)}{\varphi_2(t)} E & , \quad \text{if} \quad t > 0 \\ E & , \quad \text{if} \quad t = 0, \end{cases}$$

we have

$$\chi^2(t) = \sup_{\| v \| \leqslant 1} \int_0^t \| F(r)\, v(r) \|^2 \, dr \leqslant \sigma^2 \int_0^t \frac{\psi_2^2(r)}{\varphi_2^2(r)} \, dr \ .$$

Hypothesis 5a. The equations (4.7), (4.8) have real characteristic roots λ_1, λ_2 and γ_1, γ_2 satisfying $\lambda_1 < \lambda_2$ $\gamma_1 < \gamma_2$; $\gamma_2 < \lambda_2 \leqslant 0$, $\delta^2 \sigma^2 < 2 \rho^2 (\lambda_2 - \gamma_2)$, where

$$\delta = max \left(1 ; \ \frac{\lambda_2 - \lambda_1}{\gamma_2 - \gamma_1} \right).$$

Hypothesis 5b. The equations (4.7), (4.8) have double characteristic roots λ and γ satisfying $\gamma < \lambda \leqslant 0$ and $\sigma^2 < 2\rho^2(\lambda - \gamma)$.

Hypothesis 5c. The equation (4.7) has a double characteristic roots λ and the equation (4.8) has conjugate complex characteristic roots $\alpha \pm i\beta$ satisfying $\alpha < \lambda \leqslant 0$ and $\sigma^2 < 2\rho^2(\lambda - \alpha)$.

Hypothesis 5d. The equaitons (3.7), (4.8) have conjugate complex characteristic roots $\alpha \pm i\beta$, $\varepsilon \pm i\omega$ satisfying $\varepsilon < \alpha \leqslant 0$, $\omega = k\beta$ and $\sigma^2 < 2\rho^2(\alpha - \varepsilon)$, where k-natural number.

Hypothesis 5e. The equation (4.7) has a double characteristic root λ and the equation (4.8) has real charracteristic roots $\gamma_1 < \gamma_2$ satisfying $\gamma_2 < \lambda \leqslant 0$ and $\sigma^2 < 2\rho^2(\lambda - \gamma_2)$.

Hypothesis 5f. The equation (4.7) has real characterristic roots $\lambda_1 < \lambda_2$ and the equation (4.8) has conjugate complex characteristic roots $\alpha \pm i\beta$ satisfying $\alpha < \lambda_2 \leqslant 0$ and $(\lambda_2 - \lambda_1)^2\, \delta^2\, \sigma^2 < 2\beta^2\rho^2(\lambda^2 - \alpha)$, where

$$\delta = \sup_{0<r<\frac{2\pi}{\beta}} \frac{\sin \beta r}{1 - e^{(\lambda_1 - \lambda_2)r}}.$$

As an immediate consequence of Corollary 1 we have

PROPOSITION 2. *Assume that one of the hypotheses 5a, 5b, 5c, 5d, 5e, 5f is fulfilled. Then the pursuit process in the differential game (4.1)—(4.3) is completed after a finite time for any position* $z^0 = (z_1^0, z_2^0, z_3^0, z_4^0)^T$.

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