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ASYMPTOTIC REGULARITY AND THE STRONG CONVERGENCE OF THE PROXIMAL POINT ALGORITHM

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INTRODUCTION

Given a real Hilbert space with an inner product $\langle .,. \rangle$, a multi-valued mapping A from H into itself is said to be a monotone operator if $\langle x - x', y - y' \rangle \geqslant 0$ whenever $y \in Ax$ and $y' \in Ax'$. A is called maximal monotone if it is monotone and the graph $G(A) = \{ (x, y) \in H \times H : y \in Ax \}$ is not properly contained in the graph of any other monotone operator.

It is well-known that many problems from convex programming, variational inequalities, partial differential equations and other fields can be converted into a problem of finding a solution to an equation $0 \in Ax$ with A being a maximal monotone operator. A fundamental method for solving such equations is the proximal point algorithm which is based on the fact that the « proximal mapping » P, defined as $P = (I + A)^{-1}$ where I is the identity mapping of H, is a single-valued and nonexpansive mapping from all of H into itself, i. e., $\| Px - Px' \| \leqslant \| x - x' \|$ for every x and x' in H, and $Px = x$ if and only if $0 \in Ax$. The proximal point algorithm generates for any initial point x_1 a sequence of iterates $\{x_n\}$ by an approximate rule $x_{n+1} \approx Px_n$, or more exactly $\| x_{n+1} - Px_n \| \leqslant \varepsilon_n$ where $\{\varepsilon_n\}$ is a sequence of positive numbers satisfying $\Sigma \varepsilon_n < \infty$.

The question arises as to whether the sequence $\{x_n\}$ converges to a fixed point of P whenever such a point exists. The question can be formulated in the following general form. Firstly recall that (see, for instance, T. Rockafellar [11] the proximal mapping $P = (I + A)^{-1}$ is not only nonexpansive but satisfies a stronger condition, namely

$$\| Px - Py \|^2 = \| x - y \|^2 - \| (I - P)x - (I - P)y \|^2$$

for x, y in H, or equivalently, that P is of the form $P = \frac{1}{2}(I + T)$ where T is some nonexpansive mapping. The question now is whether the proximal point algorithm, applied to a mapping of the form $P = I + (1 - \lambda)T$ with an arbitrary nonexpansive mapping T and with $0 < \lambda < 1$ will converge to a fixed point of P. Such a question can naturally be posed in any normed vector space Problems of this kind have attracted the attention of several authors (see for instance F. E. Browder, W. V. Petryshyn [4], Z. Opial [9], T. Rockafellar [11] and H. Schaefer [13]).

In this paper we are concerned with the strong convergence of the proximal point algorithm. In the original case we give a sufficient condition which is much weaker than the one given by T. Rockafellar in [11] and which does not require the uniqueness of solution of the equation $0 \in Ax$. Our result can be extended to the problem in an arbitrary normed vector space. It is worthwhile to notice that we need no assumption about completeness or convexity on the space. Our proof is based on a new result on the asymptotic regularity of nonexpansive mappings which has its own interest, being an extension to an arbitrary normed vector space of classical results obtained in uniformly convex Banach spaces by M. A. Krasnoselskii [6], H. Schaefer [13] and F. E. Browder-W. V. Petryshyn [4]. In addition to these results, another sufficient condition for the strong convergence of the algorithm is obtained by using a rather deep theorem of R. Robert [10] and E. Zarantonello [14] on the generic single-valuedness of maximal monotone operators.

The paper consists of two sections. In the first section we prove some results on the asymptotic regularity of nonexpansive mapping. In the second we present various sufficient conditions for the strong convergence of the proximal point algorithm.

1. ASYMPTOTIC REGULARITY OF NONEXPANSIVE MAPPINGS

In what follows D will always denote a subset of a normed vector space X and $I : D \to D$ the identity mapping of D. A mapping $P : D \to D$ is called nonexpansive if $\| Px - Py \| \leqslant \| x - y \|$ for every x and y in D. P is said to be *asymptotically regular* if for every point x in D, $\| P^{n+1}x - P^n x \| \to 0$ as $n \to \infty$.

THEOREM 1. *Let D be a given bounded convex subset of a normed vector space X and let $T : D \to D$ be a nonexpansive mapping. Then for all λ satisfying $0 < \lambda < 1$, the mapping $P : D \to D$ such that $P = \lambda I + (1 - \lambda) T$ is asymptotically regular and nonexpansive.*

Proof. It is clear that P is a nonexpansive self-mapping of D, so taking an arbitrary point x_1 in D and $x_{n+1} = Px_n$ for $n = 1, 2, \ldots$ one has only to prove that $\| x_{n+1} - x_n \| \to 0$ as $n \to \infty$.

Setting $a_n = x_{n+1} - x_n$ and $b_n = Tx_{n+1} - Tx_n$, we have

$$a_{n+1} = \lambda a_n + (1 - \lambda)\, b_n \tag{1}$$

The sequence of nonnegative numbers $\{ \| a_n \| \}$ is nonincreasing, since

$$\| a_{n+1} \| = \| Px_{n+1} - Px_n \| \leqslant \| x_{n+1} - x_n \| = \| a_n \|,$$

therefore $\| a_n \| \searrow r \geqslant 0$ and we must prove that $r = 0$.

Suppose the contrary, that $r > 0$. We choose an integer $k > (1 - \lambda)^{-1}$ such that k^{-1}. diam $D < r/2$ and a positive number ε such that $\varepsilon . (1 - \lambda^k) < r/2$. Since $\| a_n \| \searrow r$ there exists an integer m such that for $n \geqslant m$ we have $r \leqslant \| a_n \| \leqslant r + \varepsilon$. For notational simplicity and without any loss of generality we can assume that $m = 1$.

Now it follows from (1) by simple computations that

$$a_{k+1} = \lambda^k\, a_1 + (1 - \lambda) \sum_{i=1}^{k} \lambda^{k-i} b_i$$

Therefore $a_{k+1} \in B = co\, \{a_1, b_1, b_2, \ldots, b_k\}$.

Taking $x = k^{-1} \cdot \sum_{i=1}^{k} b_i$ and $y = \dfrac{1}{1-\lambda^k} \cdot a_{k+1} - \dfrac{\lambda^k}{1-\lambda^k} \cdot x$

it is easily verified that $y \in B$ and $a_{k+1} = \lambda^k x + (1 - \lambda^k)\, y$
Hence we have

$$r \leqslant \| a_{k+1} \| \leqslant \lambda^k . \| x \| + (1 - \lambda^k) . \| y \| \leqslant \lambda^k \| x \| + (1 - \lambda^k)\, (r + \varepsilon)$$

then

$$\| x \| \geqslant r - \varepsilon\, (1 - \lambda^k) > r - r/2 = r/2.$$

On the other hand,

$$\| x \| = k^{-1} \cdot \sum_{i=1}^{k} b_i = k^{-1} \cdot \| Tx_{k+1} - Tx_1 \| \leqslant k^{-1} \cdot diam\, M$$

or $r/2 < k^{-1} diam\ D$ which contradicts our choice of k. Hence $r = 0$ and the proof is complete.

Remark 1. Theorem 1 is an extension to normed vector spaces of well-known results of M. A. Krasnoselskii [6], H. Schaefer [13] and F. E. Browder — W. V. Petryshyn [4] in uniformmly convex Banach spaces. The result can be extended further to the class of multivalued nonexpansive mappings in the sense defined by S. B. Nadler in [8]. The interested reader is referred to D. B. Khang [15] for other developments on the notion of asymptotic regularity.

2. STRONG CONVERGENCE OF THE PROXIMAL POINT ALGORITHM

Let D be a convex subset of a normed vector space X and $T : D \to D$ be a nonexpansive mapping of D. For an arbitrary λ with $0 < \lambda < 1$, define a mapping $P : D \to D$ by setting $P = \lambda I + (1 - \lambda) T$. It is easy to verify that the fixed point set of P coincides with the fixed point set of T. Therefore, instead of computing fixed points of T one may compute fixed points of P, where P has the advantage of being asymptotically regular.

The proximal point algorithm, in this general case, follows the scheme of successive approximation. That is, starting from any point z_1 in D, we choose in the n-th iteration an arbitrary point z_{n+1} in D satisfying $\| z_{n+1} - Pz_n \| \leqslant \varepsilon_n$ where $\{\varepsilon_n\}$ is a given sequence of positive numbers with $\Sigma \varepsilon_n < \infty$.

In this section we give some sufficient conditions for the algorithm to converge strongly to a fixed point of P.

LEMMA 1. *Let D be a subset of a normed vector space X, $P : D \to D$ an asymptotically regular and nonexpansive mapping and $\{\varepsilon_n\}$ a sequence of positive numbers satisfying $\sum_{1}^{\infty} \varepsilon_n < \infty$. Then for every point z_1 in D and every sequence $\{z_n\}$ in D satisfying $\| z_{n+1} - Pz_n \| < \varepsilon_n$ for $n \geqslant 1$, the sequence $\{\| z_{n+1} - z_n \|\}$ converges to zero as $n \to \infty$.*

Proof. Given an arbitrary positive number ε, there exists an integer **N** such that $\sum_{n=N}^{\infty} \varepsilon_n < \varepsilon$. We construct a sequence $\{x_n\}$ in D by setting $x_o = z_N$ and

$x_{n+1} = Px_n$. Then taking $u_n = z_{n+1} - Pz_n$ and $v_n = z_{N+n} - x_n$, we have $\| u_n \| \leqslant \varepsilon_n$ and $\| v_n \| = \| z_{N+n} - x_n \| = \| u_{N+n-1} + Pz_{N+n-1} - Px_{n-1} \| \leqslant$
$\leqslant \| u_{N+n-1} \| + \| z_{N+n-1} - x_{n-1} \| = \| u_{N+n-1} \| + \| v_{n-1} \|$

Upon easy computation we get for all n

$$\| v_n \| \leqslant \sum_{k=N}^{N+n-1} \| u_k \| \leqslant \sum_{k=N}^{\infty} \| u_k \| \leqslant \sum_{k=N}^{\infty} \varepsilon_k < \varepsilon.$$

Hence

$\| z_{N+n+1} - z_{N+n} \| \leqslant \| z_{N+n+1} - x_{n+1} \| + \| z_{N+n} - x_n \| + \| x_{n+1} - x_n \|$
$\leqslant \| v_{n+1} \| + \| v_n \| + \| x_{n+1} - x_n \| < 2\varepsilon + \| x_{n+1} - x_n \|$

Since P is asymptotically regular one can choose n_o such that $\| x_{n+1} - x_n \| <$ $< \varepsilon$ for $n \geqslant n_o$. Then for $n \geqslant N + n_o$ we have $\| z_{n+1} - z_n \| < 3\varepsilon$, which completes the proof.

THEOREM 2. *Let D be a convex set in a normed vector space X and T be a nonexpansive mapping from D into itself. For $0 < \lambda < 1$ define $P = \lambda I +$ $+ (1 - \lambda) T$ and $Q = I - P$ and suppose that*

(i) P has at least one fixed point (or, equivatently, $Q^{-1}(0) \neq \emptyset$);

(ii) Q^{-1} is upper semi-continuous at 0.

Then the algorithm applied to P converges strongly to a fixed point of P, i. e. for every point z_1 in D and every sequence $\{z_n\}$ in D satisfying

$$\| z_{n+1} - Pz_n \| \leqslant \varepsilon_n \text{ with } \sum_{1}^{\infty} \varepsilon_n < \infty,$$

$\{z_n\}$ converges strongly to a fixed point z of $\bar{P}$.

Proof. We can assume in addition that D is bounded since for every sequence $\{z_n\}$ we can restrict ourselves to the convex and bounded subset D' of D defined by

$$D' = \left\{ x \in D : \| x - \bar{z} \| \leqslant \| z_o - \bar{z} \| + \sum_{1}^{\infty} \varepsilon_n \right\}$$

where $\bar{z}$ is an arbitrary fixed point of P. Notice here that P maps D' into itself.

Then by Theorem 1, P is asymptotically regular and it follows from Lemma 1 that $\| z_{n+1} - z_n \| \to 0$ as $n \to \infty$.

Now let F be the (strong) closure of the set $\{ z_1, z_2, \dots \}$ in X. We show that $F \cap Q^{-1}(0)$ is nonempty.

Suppose the contrary, that $F \cap Q^{-1}(o) = \emptyset$. Then since F is closed and Q^{-1} is upper semi-continuous at 0 there exists a neighbourhood W of 0 in X such that $F \cap Q^{-1}(W) = \emptyset$. On the otherhand, we have

$$\| Qz_n \| = \| z_n - Pz_n \| \leqslant \| z_n - z_{n+1} \| + \varepsilon_n \to 0 \text{ as } n \to \infty.$$

Therefore for n large enough we have $Qz_n \in W$, or $Q^{-1}(Qz_n) \cap F = \emptyset$ which is a contradiction since $z_n \in Q^{-1}(Qz_n) \cap E$.

Hence there exists a point $\bar{z} \in Q^{-1}(o) \cap F$ which is also a fixed point of P. It follows from the definition of F that either $\bar{z} = z_{n_0}$ for some n_0 and the algorithm stops after a finite number of iterations, or there exists a subsequence $\{z_{n_k}\}$ strongly converging to $\bar{z}$. But we have

$$\| z_{n+1} - \bar{z} \| \leqslant \| z_{n+1} - Pz_n \| + \| Pz_n - P\bar{z} \| \leqslant \varepsilon_n + \| z_n - \bar{z} \|,$$

so $\| z_n - \bar{z} \| \to 0$ and the proof of Theorem 2 is complete.

The following finite-dimensional case of the theorem seems to be of interest since it cannot be reduced — at least by simple way — to the well-known results on the weak convergence of the algorithm (see Z. Opial [9]).

COROLLARY 1. *Let D be closed convex set in a finite dimensional Banach space (which may not be euclidean) and $T: D \to D$ a nonexpansive mapping. For $0 < \lambda < 1$ define $P = \lambda I + (1 - \lambda) T$ and assume that P has a fixed point. Then the proximal point algorithm applied to P converges to a fixed point of P.*

Proof. As in the proof of Theorem 2 we can assume D to be bounded, so since X is finite dimensional, D is compact. The mapping Q^{-1}, where Q is defined by $Q = I - P$, is a closed mapping defined on a compact domain, since dom Q^{-1} = range $Q = (1 - \lambda)(I - P)(D)$. Therefore Q^{-1} is upper semi-continuous (see for example C. Berge [1], Ch. VI., Th. 1.7) and Theorem 2 applies.

In the rest of the paper we shall consider the important special case where X is a Hilbert space and P is the proximal mapping of a maximal monotone (multi-valued) operator A from X into itself, i. e. $P = (I + A)^{-1}$. Let us recall that P is then of the form $P = \frac{1}{2}(I + T)$ where $T: X \to X$ is a nonexpansive (single-valued) mapping defined on all of X and the mapping Q defined by $Q = I - P = \frac{1}{2}(I - T)$ can be written in the form $Q = (I + A^{-1})^{-1}$ and then is both nonexpansive and maximal monotone (see for example T. Rockafellar [11]). The fixed point set of P is clearly the set of solutions of the equation $0 \in Ax$. Hence Theorem 2 implies

COROLLARY 2. *Let X be a Hilbert space and A a maximal monotone (multi-valued) operator in X. Suppose that the set of solutions of the equation $0 \in Ax$ is compact and nonempty, and the mapping A^{-1} in upper semi-continuous at 0. Then the proximal point algorithm applied to the mapping 0 converges strongly to a solution of the equation $0 \in Ax$.*

Proof. As mentioned above, the mapping P can be written in the form $P = \frac{1}{2}(I+T)$ with T being some nonexpansive mapping. Moreover $Q^{-1} = I + A^{-1}$ is upper semi-continuous at 0 since A^{-1} is upper semi-continuous at 0 and $A^{-1}(o)$ is compact. Corollary 2 then follows immediately from Theorem 2.

In order to give another sufficient condition for the strong convergence of the proximal point algorithm we recall some well-known definitions (see for example P. C. Duong – H. Tuy [12]).

Given a Hilbert space X, a multi-valued mapping A from X into itself is called *locally surjective* at an interior point x_0 of the domain $D(A) = \{x \in X : Ax \neq \phi\}$ if A maps the neighbourhoods of x_0 onto neighbourhoods of Ax_0. A is called *locally surjective* if it is locally surjective at every interior point of $D(A)$. Finally A is said to be surjective if $A(D(A)) = X$.

Our last theorem can now be stated as follows

THEOREM 3. *Let X be a Hilbert space and A be a maximal monotone (multi-valued) operator in X. Define $P = (I + A)^{-1}$ and $Q = I \dot{-} P$ and suppose that*

(i) A is surjective

(ii) Q is locally surjective

Then the proximal point algorithm applied to P converges strongly to a solution of the equation $0 \in Ax$.

We will need the following lemma of the theory of maximal monotone operators

LEMMA 2. *(E. Zarantonello [14], R. Robert [10] Let X be a Hilbert space and A a maximal monotone operator in X such that int $D(A) \neq \phi$. Then there exists a subset G_δ, dense in int $D(A)$, at every point of which A is single-valued.*

Proof of Theorem 3. Let $\{z_n\}$ be a sequence of proximal points and $S_n = \{x \in X : \| x - z_n \| < \delta_n\}$ $(n \geqslant 1)$ where $\{\delta_n\}$ is an arbitrary sequence of positive numbers converging to zero.

Since Q is locally surjective at z_n, there is an open set B_n such that $Q(S_n) \supset B_n \supset Qz_n$. But Q^{-1} is a maximal monotone operator defined on all of

X (for $Q^{-1} = I - A^{-1}$ and A is surjective) so Lemma 2 implies the existence of $x_n \in B_n$ such that $Q^{-1}x_n$ consist of a single point. Since $B_n \subset Q(S_n)$, there exists a point $z'_n \in S_n$ such that $Q^{-1}(Qz_n) = Q^{-1}x_n$ consists of a single point, that is $Q^{-1}(Qz_n) = \{z_n\}$.

As in the proof of Theorem 2 we have only to consider P on a bounded set. Then Theorem 1 and Lemma 2 imply $\| Qz_n \| \to 0$ as $n \to \infty$. But $\| Qz_n - Qz'_n \| \leqslant \leqslant \| z_n - z'_n \| < \delta_n$ since $z'_n \in S_n$, therefore $\| Qz'_n \| \to 0$ as $n \to \infty$.

Finally let $\bar{z}$ be a fixed point of P and V an arbitrary neighbourhood of $\bar{z}$. Since Q is locally surjective at $\bar{z}$, there exists a neighbourhood W of $0 = Q(\bar{z})$ such that $V \cap Q^{-1}(w) \neq \phi$ for every $w \in W$. Then for n large enough $Qz'_n \in W$, so $V \cap Q^{-1}(Qz'_n) \neq \phi$ or $z'_n \in V$ since $Q^{-1}(Qz'_n) = \{z'_n\}$. Therefore $z'_n \to z$ and hence $z_n \to \bar{z}$.

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