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ON THE CONTINUITY OF MULTIVALUED MAPPINGS AND THE STABILITY OF FIXED POINTS

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1. INTRODUCTION

Suppose that X and Y are Hausdorff topological spaces, $F: X \to 2^Y$ is a multivalued mapping. F is said to be upper semicontinuous (u. s. c) at $\bar{x} \in X$ if whenever $V \subset Y$ is open and $F(\bar{x}) \subset V$ one has a neighborhood $U(\bar{x}) \subset X$ with $F(\bar{x}) \subset V$ for $x \in U(x)$. F is said to be lower semicontinuous (l. s. c) at $(\bar{x}) \in X$ if whenever V is open and $F(\bar{x}) \cap V \neq \emptyset$ one has $F(\bar{x}) \cap V \neq \emptyset$, for $x \in U(\bar{x})$, some neighborhood of $\bar{x}$. F is said to he u.s.c (or l. s.c) on a set $C \subset X$ if F is u. s. c (or l. s. c) at each point of C when F is considered as a mapping between C and Y. F is said to be continuous if F is u. s. c and l. s. c. These notions are essentially given in Berge [1]. In general, there are mappings which are u.s.c. but not l. s. c and *vice versa.*

It is natural to ask which u. s. c multivalued mapping is l. s. c and *vice versa*.

In what follows, we shall show that if X is a barrel space and Y is a topological locally convex Hausdorff space and $\{F_\nu, \nu \in I\}$ is a family of u.s.c. convex multivalued mappings with $\bigcup_{\nu \in I} F_\nu(x)$ bounded for each $x \in X$, then the family $\{F_\nu, \nu \in I\}$ is lower semi—equicontinuous in a sense to be defined (see Definition 2. 4)

Further, let $C \subset Y$ be a closed convex subset, $F : X \times C \to 2^C$ a u. s. c convex compact multivalued mapping with nonempty closed values, then the multivalued mapping $\Gamma : X \to 2^C$ defined by $\Gamma(x) = \{y \in C / y \in F(x, y)\}$ is convex and continuous. In addition, let $(\Omega, \mathcal{Q}, \mu)$ be a complete measurable space, Y a separable Frechet space, $F : \Omega \times X \times Y \to 2^Y$ be a multivalued mapping with nonempty closed convex values. Suppose further that F is measurable in $t \in \Omega$ and convex u. s. c in $(x, y) \in X \times Y$, with $\overline{F(t, X, Y)}$ compact, for any fixed $t \in \Omega$. Then the multivalued mapping $\Lambda : \Omega \times X \to 2^Y$ defined by $\Lambda(t, x) =$ $= \{y \in Y / y \in F(t, x, y)\}$ is measurable in t and convex continuous in x.

Finally, applying the above results we shall prove some properties on the continuity of solutions of some variational inequalities.

2. CONTINUITY OF MULTIVALUED MAPPINGS

Let X, Y be topological Hausdorff spaces and I the index set. Consider the multivalued mapping $F : X \to 2^Y$ with nonempty values. Using Theorem 2 in ([1], Chapter VI, 1) one can easily prove the following lemmas :

LEMMA 2. 1. *F is u. s. c if and only if for any closed subset $A \subset Y$, the set $B = \{x \in X / F(x) \cap A \neq \emptyset\}$ is closed in X.*

LEMMA 2. 2. *F is l. s. c if and only if for any closed subset $A \subset Y$ the set $B = \{x \in X / F(x) \subset A\}$ is closed in X.*

Now suppose that X and Y are real topological linear spaces.

DEFINITION 2. 3. *a) F is said to be convex if*

$$\lambda F(x) + (1 - \lambda) F(y) \subset F(\lambda x + (1 - \lambda) y)$$

for any $x, y \in X$, $\lambda \in [0,1]$.

b) F is said to be concave if

$$F(\lambda x + (1 - \lambda) y) \subset \lambda F(x) + (1 - \lambda) F(x)$$

fhr all $x, y \in X$, $\lambda \in [0,1]$.

Let $\{F_\nu, \nu \in I\}$ be a familily of muttivalued mappings from X into 2^Y with nonempty values. We introduce the following definitions :

DEFINITION 2.4. a) *The family $\{F_v, v \in I\}$ is said to be lower semi-equicontinuous (l. s. e. c.) at $\bar{x} \in X$ if for any neighborhood V of the origin in Y the set*

$$U = \bigcap_{v \in I} \{x \in X / F_v(x) \cap (F_v(\bar{x}) + V) \neq \emptyset\}$$

is a neighborhood of $\bar{x}$. If $\{F_v, v \in I\}$ is l. s. e c. at every point of X we say that it is l. s. e. c.

b) *The family $\{F_v, v \in I\}$ is said to be upper semi-equicontinuous (u.s.e.c) at $x \in X$ if for any neighborhood V of the origin in Y there exists U, a neighborhood of the origin in X such that*

$$F_v(x) \subset F(\bar{x}) + V, \text{ for all } x \in U, v \in I.$$

If $\{F_v, v \in I\}$ is u. s. e. c at every point of X we say that it is u. s. e. c.

THEOREM 2.5. *Let X be a barrel space, Y a topological locally convex Hausdorff space, $\{F_v, v \in I\}$ a family of u. s. c. convex multivalued mappings with $\bigcup_{v \in I} F_v(x)$ bounded for each $x \in X$. Then $\{F_v, v \in I\}$ is lower semi-equicontinuous.*

Proof. Replacing F_v by $\tilde{F}_v(x) = F_v(x + \bar{x})$, $v \in I$, $x \in X$, it suffices to show that $\{F_v, v \in I\}$ is l. s. e. c at 0.

Let V be an arbitrary neighborhood of the origin in Y. We may choose a closed absolutely convex neighborhood $V' \subset V$ such that $V' + V' \subset V$. Put

$$B_v = \{x \in X / F_v(x) \cap \overline{(F_v(o) + V')} \neq \emptyset\}$$

and

$$U = \bigcap_{v \in I} B_v .$$

Since $O \in B_v$, for all $v \in I$, we have $U \neq \emptyset$.

Let $x_1, x_2 \in U$ and $\lambda \in [0,1]$ then

$$\lambda F_v(x_1) + (1-\lambda) F_v(x_2) \subset F_v(\lambda x_1 + (1-\lambda) x_2)$$

and consequently,

$$\emptyset \neq \lambda F_v(x_1) \cap \overline{(F_v(0) + V')} + (1-\lambda)(F_v(x_2) \cap \overline{(F_v(0) + V')})$$

$\subset (\lambda F_\nu(x_1) + (1-\lambda) F_\nu(x_2) \cap \overline{(F_\nu(0) + V')} \subset F_\nu(\lambda x_1 + (1-\lambda) x_2) \cap$
$(F_\nu(0) + V')\ \nu \in I$

This shows $\lambda x_1 + (1-\lambda) x_2 \in B_\nu$, for all $\nu \in$ I.

which implies that $\lambda x_1 + (1-\lambda) x_2 \in$ U and moreover U is a convex set in X

Further, by Lemma 2.1 B_ν is closed, hence U is closed.

Setting $W = U \cap (-U)$, we deduce that W is a nonempty closed absolutely convex subset of X.

We claim that W is absorbing. Indeed, let x be an arbitrary point in X. Since $\bigcup_{\nu \in I} F_\nu$ (x) and $\bigcup_{\nu \in I} F_\nu(-x)$ are bounded subsets in Y, there exists $\alpha_0 \geqslant 1$ such that

$F_\nu(x) \cup F_\nu(-x) \subset \alpha_0 V'$, for all $\nu \in I$

We can write

$F_\nu(x) \cup F_\nu(-x) \subset \alpha_0 V' + F_\nu(0) - F_\nu(0)$.

Since $\bigcup_{\nu \in I} F_\nu(0)$ is also bounded, one can find $\alpha_1 \geqslant 1$ such that

$F_\nu(x) \cup F_\nu(-x) \subseteq F_\nu(0) + \alpha_1 V'$, for all $\nu \in$ I. **(1)**

Hence it follows that

$$\frac{1}{\alpha_1} F_\nu(x) \subset \frac{1}{\alpha_1} F_\nu(0) + V'$$

and

$$\frac{1}{\alpha_1} F_\nu(x) + \left(1 - \frac{1}{\alpha_1}\right) F_\nu(0) \subset \frac{1}{\alpha_1} F_\nu(0) + \left(1 - \frac{1}{\alpha_1}\right) F_\nu(0) + V' \subset$$

$\subset F_\nu(0) + V'$ for all $\nu \in I$. (2)

On the other hand, from the convexity of F_ν we have

$$\frac{1}{\alpha_1} F_\nu(x) + \left(1 - \frac{1}{\alpha_1}\right) F_\nu(0) \subset F_\nu\left(\frac{x}{\alpha_1}\right).$$

A combination of (2) and (3)

$$F_v\left(\frac{x}{\alpha_1}\right) \cap (F_v(0) + V') \neq \emptyset, \text{ for all } v \in I$$

which proves that $\frac{x}{\alpha_1} \in U$.

By the same argument as above we can show that $-\frac{x}{\alpha_1} \in U$. Therefore

$$\frac{x}{\alpha_1} \in U \cap (-U) = W. \text{ Consequently, } W \text{ is absorbing.}$$

Now, since X is a barrel space and W is a nonempty closed absolutely convex absorbing subset of X, W is a neighborhood of the origin in X.

But, for each $v \in I$

$$\{x \in X/F_v(x) \cap (\overline{F_v(0) + V'}) \neq \emptyset\} \subset \{x \in X/F_v(x) \cap (F_v(0) + 2V') \neq \emptyset\} \subset \{x \in X/F_v(x) \cap (F_v(0) + V) \neq \emptyset\}$$

Therefore,

$$W \subset \bigcap_{v \in I} \{x \in X/F_v(x) \cap (F_v(0) \neq \emptyset\},$$

which completes the proof.

COROLLARY 2.6. *Let X be a barrel space, Y a topological locally convex Hausdorff space $F: X \to 2^Y$ a u.s.c. convex multivalued mapping with nonempty, bounded values. Then F is l. s. c.*

Proof. It suffices to show that F is l. s. c at O. Indeed, let G be an arbitrary open subset of Y and $F(0) \cap G \neq \emptyset$. Then for each $y_o \in F(0) \cap G$ there exists a neighborhood V of the origin in Y such that

$$y_o + V \subset G \tag{4}$$

By Theorem 2.5 the set

$$U = \{x \in X \setminus F(x) \cap (F(0) + V) \neq \emptyset\} \tag{5}$$

is a neighborhood of the origin in X.

Since $F(0)$ and $\{y_o\}$ are bounded in Y one can choose $\alpha \geqslant 1$ such that

$$F(o) + V \subset y_o + F(o) - y_o + V \subset y_o + \alpha V$$

Consequently,

$$\{x \in X/F(x) \cap (F(0) + V) \neq \emptyset\} \subset \{x \in X/F(x) \cap (y_o + \alpha V) \neq \emptyset\}$$

$$= \{x \in X \setminus \frac{1}{\alpha} F(x) \cap \left(\frac{1}{\alpha} y_0 + V\right) \neq \phi\}$$

$$= \{x \in X \setminus \left(\frac{1}{\alpha} F(x) + (1 - \frac{1}{\alpha}\right) y_0) \cap (y_0 + V) \neq \phi\}$$

$$\subset \{x \in X \setminus F\left(\frac{x}{\alpha} \cap\right) (y_0 + V) \neq \phi\} \subset \alpha \{x \in X \setminus F(x) \cap (y_0 + V) \neq \phi\}$$

By (5) and (6) the set $A = \{x \in X \setminus F(x) \cap y_0 + V) \neq \phi\}$ is a neighborhood of the origin. Moreover, by (4),

$$A \subset \{x \in X \setminus F(x) \cap G \neq \phi\}$$

Therefore the set $W = \{x \in X \setminus F(x) \cap G \neq \phi\}$ is a neighborhood of the orgin in X.

This completes the proof.

THEOREM 2.7. *Let X, Y be as in Theorem 2.5 and $\{F_v, v \in I\}$ a family of l.s.c concave multivalued mappings, with nonempty convex values. Suppose that $\bigcup_{v \in I} F_v(x)$ is a bounded subset for every $x \in X$. Then $\{F_v, v \in I\}$ is u.s.e.c.*

Proof. Without loss of generality, it suffices to prove that $\{F_v, v \in I\}$ is u.s.ec at the origin.

Take a closed absolutely convex neighborhood V of the origin in Y and put

$A_v = \left\{x \in X \setminus F_v(x) \subset \overline{\frac{V}{2} + F_v(0)}\right\}$, $B = \bigcap_{v \in I} A_v$. Since $0 \in A$, for all $v \in I$ we have $B \neq \varnothing$. By Lemma 2.2, A_v is a closed subset of X, $v \in I$. Therefore B is closed.

Suppose now that $x_1, x_2, \in B$ and $\lambda \in [0,1]$. Since F_v is concave, for all $v \in I$, one has

$$F_v(\lambda x_1 + (1-\lambda)x_2) \subset \lambda F_v(x_1) + (1-\lambda)F_v(x_2) \subset \lambda \left(\overline{\frac{V}{2} + F_v(0)}\right) +$$

$$+ (1-\lambda)\left(\overline{\frac{V}{2} + F_v(0)}\right) = \overline{\frac{V}{2} + F_v(0)} \quad , \quad v \in I .$$

This shows that $\lambda x_1 + (1-\lambda) x_1 \in A_v$, for all $v \in I$.

There $\lambda x_1 + (1-\lambda) x_2 \in B$ and B is a convex subset.

Setting $U = B \cap (-B)$ we claim that U is absorbing. Assume that x is an arbitrary point in X. Since $\bigcup_{v \in I} F_v(x)$, $\bigcup_{v \in I} F_v(-x)$ and $\bigcup_{v \in I} F_v(0)$ are bounded subsets, one can find a real number $\alpha \geqslant 1$ such that

$$\bigcup_{v \in I} F_v(x) \cup \bigcup_{v \in I} F_v(-x) \subset \alpha \frac{V}{2} + F_v(0) \text{, for all } v \in I.$$

We have

$$F_v\left(\frac{x}{\alpha}\right) = F_v\left(\frac{x}{\alpha} + \left(1 - \frac{1}{\alpha}\right) 0\right) \subset \frac{1}{\alpha} E_v(x) +$$

$$+ \left(1 - \frac{1}{\alpha}\right) F_v(0) \subset \frac{V}{2} + \frac{1}{\alpha} F_v(0) + \left(1 - \frac{1}{\alpha}\right) F_v(0)$$

$$\subset \frac{V}{2} + F_v(0) \quad \text{, for all } v \in I .$$

This implies that $\frac{x}{\alpha} \in B$. Similarly, one gets $-\frac{x}{\alpha} \in B$, and so $\frac{x}{\alpha} \in B \cap (-B) = U$, then U is absorbing.

Note that X is a barrel space and U is a nonempty closed absolutely convex absorbing subset of X, so U is a neighborhood of the origin in X. Further we have

$$F_v(U) \subset \overline{\frac{V}{2} + F_v(0)} \subset V + F_v(0) \qquad \text{for all } v \in I .$$

This shows that $\{F_v, v \in I\}$ is upper semi-equicontinuous at 0 completes the proof of the theorem.

COROLLARY 2.8. *Let X, Y be as in Theorem 2.5 and $F : X \to 2^Y$ a l.s.c concave multivalued mapping with nonempty convex bounded values. Then F is continuous.*

Proof. This follows immediately from Theorem 2.7.

3. STABILITY OF FIXED POINTS

From now on, $(\Omega, \mathcal{A}, \mu)$ is supposed to be a complete measurable space (cf. [2]). Let Z be a topological space, by $\mathfrak{B}(Z)$ we shall denote a σ-field of all Borel subsets of Z, $\mathcal{A} \otimes \mathfrak{B}(Z)$ the smallest σ-field containing all subsets of $\Omega \times Z$ of the form $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathfrak{B}(Z)$, $\bar{C}$ a topological closure of $C \subset Z$, R stands for the set of all real numbers.

Let X, Y be topological spaces. Consider the multivalued mapping $F: X \to 2^Y$. We recall the following definitions :

DEFINITION 3.1. (cf. [1]). *F is said to be closed if for any net $(x_\nu) \subset X$, $x_\nu \to x$, $y_\nu \in F(x_\nu)$, $y_\nu \to y$, one has $y \in F(x)$.*

DEFINITION 3. 2. (cf. [2]). *A multivalued mapping $\Gamma : \Omega \to 2^Y$ is said to be measuralbe if $\Gamma^{-1}(B) = \{t \in \Omega \setminus \Gamma(t) \cap B \neq \emptyset\} \in \mathcal{A}$ for* each $B \in \mathfrak{B}(Y)$.

In this section, we shall apply Theorem 2.5 to prove the stability of fixed points of convex multivalued mappings and of solutions of some variational inequalities.

THEOREM 3. 3. *Let X be a barrel space, Y a topological locally convex Hausdorff space, C a closed convex nonempty subset of Y and $\{F_\nu, \nu \in I\}$ a family of u.s.c convex multivalued mappings from $X \times C$ into 2^C with non-empty closed values and $F_\nu(X, C)$ in compact for each $\nu \in I$.*

Then the family $\{\Gamma_\nu, \nu \in I\}$ with $\Gamma_\nu : X \to 2^C$ defined by

$$\Gamma_\nu(x) = \{y \in C \setminus y \in F_\nu(x, y)\}$$

is l.s.e.c and for each $\nu \in I$, $x \in X$, $\Gamma_\nu(x)$ is compact convex nonempty.

Proof. Applying Himmelberg's Theorem of fixed points (Theorem 2 in [3]) we infer that $\Gamma_\nu(x) \neq \emptyset$, for each $x \in X$, $\nu \in I$.

Now, we verify that Γ_ν, $\nu \in I$ is convex. Suppose that $x_1, x_2 \in X$ and $\alpha \in [0,1]$. Taking $y_1 \in \Gamma_\nu(x_1)$, $y_2 \in F_\nu(x_2)$, we have

$$\alpha y_1 + (1-\alpha) y_2 \in \alpha \Gamma_\nu(x_1, y_1) + (1-\alpha) F_\nu(x_2, y_2) \subset$$
$$\subset F_\nu(\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_2)$$

This shows $\alpha y_1 + (1-\alpha) y_2 \in \Gamma_\nu(\alpha x_1 + (1 \alpha) x_2)$ But $y1$ and y_2 are arbitrary points of $\Gamma_\nu(x_1)$ and $\Gamma_\nu(x_2)$, therefore one has $\alpha \Gamma_\nu(x_1) + (1-\alpha)\Gamma_\nu(x_2) \subseteq \subset \Gamma_\nu(\alpha x_1 + (1-\alpha) x_2)$ and Γ_ν is a convex multivalued mapping for each $\nu \in I$. In particular, $\Gamma_\nu(x)$ is convex for all $x \in X$, $\nu \in I$.

It is easy to see that for each $v \in I$, Γ_v is a closed multivalued mapping. Furthermore $\Gamma_v(X) \subset \overline{F_v(X,C)}$, $v \in I$, is a compact subset. Therefore Γ_v is u.s.c for each $v \in I$ (cf) [1]). In view of Theorem 2.4, this completes the proof.

COROLLARY 3. 4. *Suppose that X is a topological locally convex metrizable space and Y is a Banach space and C is a closed convex nonempty subset of X and* $F : X \times C \to 2^C$ *is u.s.c convex multivalued mapping with nonempty closed values and* $\overline{F(X,C)}$ *is compact. Then there exists a continuous singte-valued mapping* $\varphi : X \to C$ *such that* $\varphi(x) \in F(x, \varphi(x))$, *for all* $x \in X$.

Proof. Since X is a topological locally convex metrizable space it follows that X is a barrel and paracompact space (cf. in [5]). By Theorem 3.3 it follows that the multivalued mapping $\Gamma : X \to 2^C$ defined by $\Gamma(x) = \{y \in C / y \in F(x,y)\}$ satisfies the conditions of the Selection Theorem due to Michael [4]. Therefore there exists a continuous single valued mapping $\varphi : x \to C$ such that $\varphi(x) \in \Gamma(x)$ for all $x \in X$. This means that $\varphi(x) \in F(x, \varphi(x))$ for all $x \in X$ which completes the proof.

THEOREM 3.5. *Let X be a barrel space, Y a separable Frechet space,* $(\Omega, \mathcal{A}, \mu)$ *a complete measurable space, and* $F : \Omega \times X \times Y \to 2^Y$ *a multivalued mapping with* $\overline{F(t, x, Y)}$ *compact,* $t \in \Omega$. *Further, assume that F satisfies:*

a) For any fixed element $(x, y) \in X \times Y$, *the multivalued mapping* $t \to F(t,x,y)$ *is measurable from* Ω *to* 2^Y

b) For any fixed element $t \in \Omega$, *the multivalued mapping* $(x, y) \to F(t, x, y)$ *is convex closed from* $X \times Y$ *to* 2^Y

Then the multivalued mapping $\wedge : \Omega \times X \to 2^Y$ *defined by*

$$\wedge(t, x) = \{y \in Y \mid y \in F(t, x, y)\}$$

has the following properties:

1) $\wedge(t,x)$ *is nonempty compact convex for all* $(t, x) \in \Omega \times X$.

2) for any fixed element $t \in \Omega$, *the multivalued mapping* $x \to \wedge(t, x)$ *is convex continuous from X to* 2^Y.

3) For any fixed element $x \in X$, *the multivalued mapping* $t \to \wedge(t, x)$ *is measurable from* Ω *to* 2^Y

Proof. Applying Theorem 3.3 yields the properties 1) and 2). It remains to prove 3). Indeed, let $x_0 \in X$ be fixed. We have

$$\wedge (t, x_0) = \{y \in Y \backslash y \in F(t, x_0, y)\} = \{y \in Y \backslash d(y,) F(t, x_0, y)) = 0\},$$

where d is a distance on Y.

Consider the subset $A = \{(t, y) \in \Omega \times Y \mid d(y, F(t, x_0, y)) = 0\}$ and put

$$h(t, y) = d(y, F(t, x_0, y)).$$

By the Debreu — Castaing's Theorem (Theorem III . 30 in [2]), for any fixed element $y_0 \in Y$ the function $t \to h(t, y_0)$ is measurable from Ω into R.

Now, take a fixed element $t_0 \in \Omega$. The multivalued mapping $G : Y \to 2^Y$ defined by $G(y) = F(t_0, x_0, y)$ is closed convex. Since $F(t_0, x_0, Y)$ is a compact subset of Y, it follows that G is a u. s. c (see [1]). Hence and by Corollary 2.6 G is continuous. Therefore the function

$$h(t_0, y) = d(y, G(y)) = \min_{z \in G(y)} d(y, z)$$

is continuous from Y into R (cf. [1]).

Further, Lemma III.14 in [2] shows that h is measurable from $\Omega \times Y$ into R. Consequently

$$A = \{(t, y) \in \Omega \times Y \mid h(t, y) = 0\} \in \mathcal{A} \otimes \mathfrak{B}(Y)$$

Note that

$$\begin{aligned} Graph \wedge (., x_0) &= \{(t, y) \in \Omega \times Y \mid y \in \wedge (t, x_0)\} \\ &= \{(t, y) \in \Omega \times Y \mid y \in F(t, x_0, y)\} = \{(t,y) \in \Omega \times Y \mid h(t, y) \\ &= 0\} = A \end{aligned}$$

Hence graph $\wedge (., x_0) \in \mathcal{A} \otimes \mathfrak{B}(Y)$. On the basis of Theorem III , 30 in [2] we again conclude that $\wedge (., x_0)$ is a measurable multivalued mapping from Ω into 2^Y.

This completes the proof.

Next, assume that X is a barrel space, Y a topological locally convex Hausdorff space, C a compact convex nonempty subset of Y. Consider the function $\varphi : X \times C \times C \to R$ with the following properties :

1, For any fixed element $y_0 \in C$, the function $\varphi_1 : X \times C \to R$ defined by $\varphi_1(x,y) = \varphi(x, y, y_0)$ is concave and for $(x_\nu) \subset X, (y_\nu) \subset C$, $x_\nu \to x$, $y_\nu \to y$ one has $\varphi_1(x, y) \geqslant \liminf_\nu \varphi_1(x_\nu, y_\nu)$.

2, For any fixed element $x \in X$, there exists a point $\widehat{y} \in C$ such that $\varphi(x, \widehat{y}, y) \geqslant 0$, for all $y \in C$.

We have:

THEOREM 3.6.

With the above assumptions, the multivalued mapping $\Sigma : X \to 2^C$ defined by

$$\Sigma(x) = \{ \widehat{y} \in C \setminus \varphi(x, \widehat{y}y) \geqslant 0, \text{ for all } y \in C \}$$

is convex continuous with $\Sigma(x)$ nonempty compact convex for all $x \in X$.

Proof. The assumption 2 shows that $\Sigma(x) \neq \phi$, for $x \in X$.
Now, let $x_1, x_2 \in X$, $\alpha \in [0, 1]$. Taking $y_1 \in x\Sigma(x_1)$, $y_2 \in \Sigma(x_2)$, one has

$$\varphi(\alpha x_1 + (1-\alpha)x_2 ; \alpha y_1 + (1-\alpha)y_2, y) \geqslant \alpha\varphi(x_1, y_1 y) +$$
$$(1-\alpha)\varphi(x_2, y_2, y) \geqslant 0 \text{ for all } y \in C.$$

This shows $\alpha y_1 + (1-\alpha)y_2 \in \Sigma(\alpha x_1 + (1-\alpha)x_2)$, and so $\alpha\Sigma(x_1) + (1-\alpha)\Sigma(x_2) \subset \Sigma(\alpha x_1 + (1-\alpha)x_2)$, which means that Σ is a convex multivalued mapping.

Let $(x_\nu) \subset X$, $x_\nu \to x$, $\widehat{y}_\nu$, $y) \geqslant \liminf \varphi(x_\nu, \widehat{y}_\nu, y) \geqslant 0$ for all $y \in C$.

Therefore, $\widehat{y} \in \Sigma(x)$, and so Σ is closed. But $\overline{\Sigma(x)} \subset C$, a compact subset of Y, therefore Σ is u.s.c.

Using Corollary 2.6 we then conclude the proof.

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REFERENCES

[1] C.Berge. *Topological Spaced*, London, 1963.

[2] C.Castaing, M. Valadier, *Convex Analysis and Measurable Multifunctions*. Lecture Notes in Math. N. 580. Springer—Verag, 1977.

[3] A. J. Himmelberg, *Fixed points of compact multifunctions* Jour. of Anal. and Appl. 38. (1972), 206−207.

[4] E. Michael, *Continuous Selections*. I. Ann. Math. 63, (1965), 361 — 382.

[5] H. Schaefer, *Topological Vector Spaces*. New York, Springer−Verlag, 1971.